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Accurate Stresses for Homogeneous and Composite Beams

J. Aalto[1], R. Syrjä[1] and C.A. Ribas[2]
[1] Department of Civil and Structural Engineering
Aalto University, Espoo, Finland
[2] School of Industrial Engineering of Barcelona
Polytechnic University of Catalonia, Spain

Abstract

In this paper two formulations for determining the distribution of stress components in a cross-section of a composite beam under two-axial bending are studied theoretically and numerically in this paper. These formulations can be regarded as improvements of the classical formulas for determining stresses in the cross-section of a beam caused by the corresponding stress resultants. They are based on improved kinematic assumptions for the displacements, which use special warping and contraction functions. Simple two-dimensional boundary value problems for these functions are derived. It is further shown, that in connection with homogeneous and isotropic material these formulations reduce to a theory, which has been presented in classical textbooks of the theory of elasticity. In the numerical study finite element results of the two formulations are compared to those of the finite prism method. Biquadratic iso-parametric finite elements with 3×3 Gauss quadrature are used.

Keywords: beam theory, bending, stress components, composite beam, warping function, finite element.

1 Introduction

Classical formulas for determining stress components in a cross-section of a beam caused by the corresponding stress resultants are very useful, but in some cases inaccurate. Especially determining the shear stress τ_{xy} or τ_{xz} as mean value in a horizontal or vertical longitudinal cut passing through the point of interest cannot be used to plot a satisfactory stress distribution. In addition, based on the assumptions of classical beam theory, the transverse normal stresses σ_y, σ_z and the shear stress τ_{yz} vanish, although in beams composed of two or more materials such stresses exist. Classical textbooks of the theory of elasticity [1], [2], [3] consider different formulations for determining exact shear stresses τ_{xy} and τ_{xz} in a homogeneous prismatic cantilever beam under terminal load (constant shear). Finite element solution of this problem based on warping function formulation has been presented in [4]. The purpose of this paper is to derive two formulations for determining stress components in beam cross-sections, not in the special case of constant shear, but in typical engineering beam analysis context. These formulations apply also to beams composed of two or more materials, which can be orthotropic. In the *first formulation* (f1) the stress components satisfy axial equilibrium equation within the beam and axial traction boundary condition on the longitudinal faces and in the *second formulation* (f2) all three equilibrium equations and all three traction boundary condition are satisfied. In f1 two warping functions must be determined as solutions of two separate Poisson-type boundary value problems defined in the cross-section. In f2 two additional boundary value problems for four additional unknown, called here contraction functions, are needed. It is also shown in the paper, that in connection with homogeneous cross-section and isotropic material, the contraction functions can be determined analytically and the problem reduces to the warping function formulation and finite element equations, which were used in [4].

2 Assumptions

Bending of a straight uniform beam is considered. The two formulations under study are based on following assumptions: The axial displacement $u(x)$ of the beam axis is assumed to be locally linear and the transverse displacements $v(x)$ and $w(x)$ are assumed to be locally cubic functions of the axial coordinate x. Axial displacement of a generic point is assumed to be of form

$$\mathrm{u}(x,y,z) = u(x) - v'(x)y - w'(x)z - v'''\mathcal{U}_v(y,z) - w'''\mathcal{U}_w(y,z), \tag{1}$$

where $\mathcal{U}_v(y,z)$ and $\mathcal{U}_w(y,z)$ are warping functions.

In f1 following additional assumptions are made: Transverse displacements are

$$\mathrm{v}(x,y,z) = v(x),\ \mathrm{w}(x,y,z) = w(x) \tag{2}$$

and transverse normal stresses σ_y, σ_z and the shear stress τ_{yz} are zero, which is typical assumption in the beam theory. In f2 following additional assumption is made: Transverse displacements are

$$\begin{aligned}\mathrm{v}(x,y,z) &= v(x) + u'(x)\mathcal{V}_u(y,z) - v''(x)\mathcal{V}_v(y,z) - w''(x)\mathcal{V}_w(y,z),\\ \mathrm{w}(x,y,z) &= w(x) + u'(x)\mathcal{W}_u(y,z) - v''(x)\mathcal{W}_v(y,z) - w''(x)\mathcal{W}_w(y,z),\end{aligned} \tag{3}$$

where $\mathcal{V}_u(y,z)$, $\mathcal{V}_v(y,z)$, $\mathcal{V}_w(y,z)$, $\mathcal{W}_u(y,z)$, $\mathcal{W}_v(y,z)$ and $\mathcal{W}_w(y,z)$ are called here contraction functions.

3 Strains and stresses

3.1 Strains

In f1 the essential strains get the form

$$
\begin{aligned}
\varepsilon_x &\equiv \frac{\partial \mathrm{u}}{\partial x} = \varepsilon + \kappa_z y + \kappa_y z, \\
\gamma_{xy} &\equiv \frac{\partial \mathrm{v}}{\partial x} + \frac{\partial \mathrm{u}}{\partial y} = \kappa_z' \frac{\partial \mathcal{U}_v}{\partial y} + \kappa_y' \frac{\partial \mathcal{U}_w}{\partial y}, \\
\gamma_{xz} &\equiv \frac{\partial \mathrm{w}}{\partial x} + \frac{\partial \mathrm{u}}{\partial z} = \kappa_z' \frac{\partial \mathcal{U}_v}{\partial z} + \kappa_y' \frac{\partial \mathcal{U}_w}{\partial z}
\end{aligned}
\tag{4}
$$

where $\varepsilon = u'$, $\kappa_y = -w''$ and $\kappa_z = -v''$. In f2 the strains get the form

$$
\begin{aligned}
\varepsilon_x &\equiv \frac{\partial \mathrm{u}}{\partial x} = \varepsilon + \kappa_z y + \kappa_y z, \\
\varepsilon_y &\equiv \frac{\partial \mathrm{v}}{\partial y} = \varepsilon \frac{\partial \mathcal{V}_u}{\partial y} + \kappa_z \frac{\partial \mathcal{V}_v}{\partial y} + \kappa_y \frac{\partial \mathcal{V}_w}{\partial y}, \\
\varepsilon_z &\equiv \frac{\partial \mathrm{w}}{\partial z} = \varepsilon \frac{\partial \mathcal{W}_u}{\partial z} + \kappa_z \frac{\partial \mathcal{W}_v}{\partial z} + \kappa_y \frac{\partial \mathcal{W}_w}{\partial z}, \\
\gamma_{xy} &\equiv \frac{\partial \mathrm{v}}{\partial x} + \frac{\partial \mathrm{u}}{\partial y} = \kappa_z' (\mathcal{V}_v + \frac{\partial \mathcal{U}_v}{\partial y}) + \kappa_y' (\mathcal{V}_w + \frac{\partial \mathcal{U}_w}{\partial y}), \\
\gamma_{xz} &\equiv \frac{\partial \mathrm{w}}{\partial x} + \frac{\partial \mathrm{u}}{\partial z} = \kappa_z' (\mathcal{W}_v + \frac{\partial \mathcal{U}_v}{\partial z}) + \kappa_y' (\mathcal{W}_w + \frac{\partial \mathcal{U}_w}{\partial z}), \\
\gamma_{yz} &\equiv \frac{\partial \mathrm{w}}{\partial y} + \frac{\partial \mathrm{v}}{\partial z} = \varepsilon (\frac{\partial \mathcal{W}_u}{\partial y} + \frac{\partial \mathcal{V}_u}{\partial z}) + \kappa_z (\frac{\partial \mathcal{W}_v}{\partial y} + \frac{\partial \mathcal{V}_v}{\partial z}) + \kappa_y (\frac{\partial \mathcal{W}_w}{\partial y} + \frac{\partial \mathcal{V}_w}{\partial z}),
\end{aligned}
\tag{5}
$$

3.2 Stress-strain relations

The material of the beam is assumed here to be linearly elastic and orthotropic so, that the axes *x*, *y* and *z* coincide with principal directions of the material. Thus the stress-strain relations can be written as

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \tag{6}$$

where

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \tag{7}$$

and matrix $\mathbf{C}$ is of form

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{8}$$

and it's inverse is of typical form

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zx}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & -\frac{\nu_{zy}}{E_z} & 0 & 0 & 0 \\ -\frac{\nu_{xz}}{E_x} & -\frac{\nu_{yz}}{E_y} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{xy}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{xz}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{yz}} \end{bmatrix}. \tag{9}$$

Under conditions of f1 the essential stress-strain relations thus reduce to

$$\sigma_x = E_x \varepsilon_x, \; \tau_{xy} = G_{xy} \gamma_{xy}, \; \tau_{xz} = G_{xz} \gamma_{xz}. \tag{10}$$

If the material is isotropic, equations (10) corresponding to f1 are

$$\sigma_x = E \varepsilon_x, \; \tau_{xy} = G \gamma_{xy}, \; \tau_{xz} = G \gamma_{xz} \tag{11}$$

and in connection with f2 the nonzero coefficients of matrix $\mathbf{C}$ are

$$c_{11}=c_{22}=c_{33}=2G+\lambda,\ c_{12}=c_{13}=c_{23}=\lambda,\ c_{44}=c_{55}=c_{66}=G, \tag{12}$$

where

$$G=\frac{E}{2(1+\nu)},\ \lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{13}$$

3.3 Stresses

Based on Hooke's law in f1 the nonzero stress components can be written as

$$\begin{aligned}
&\sigma_x=\varepsilon E_x+\kappa_y E_x z+\kappa_z E_x y,\\
&\tau_{xy}=\kappa_z' G_{xy}\frac{\partial \mathcal{U}_v}{\partial y}+\kappa_y' G_{xy}\frac{\partial \mathcal{U}_w}{\partial y},\\
&\tau_{xz}=\kappa_z' G_{xz}\frac{\partial \mathcal{U}_v}{\partial z}+\kappa_y' G_{xz}\frac{\partial \mathcal{U}_w}{\partial z}
\end{aligned} \tag{14}$$

and in f2 the stress components can be written as

$$\begin{aligned}
&\sigma_x=\varepsilon(c_{11}+c_{12}\frac{\partial \mathcal{V}_u}{\partial y}+c_{13}\frac{\partial \mathcal{W}_u}{\partial z})+\kappa_y(c_{11}z+c_{12}\frac{\partial \mathcal{V}_w}{\partial y}+c_{13}\frac{\partial \mathcal{W}_w}{\partial z})\\
&\quad+\kappa_z(c_{11}y+c_{12}\frac{\partial \mathcal{V}_v}{\partial y}+c_{13}\frac{\partial \mathcal{W}_v}{\partial z}),\\
&\sigma_y=\varepsilon(c_{12}+c_{22}\frac{\partial \mathcal{V}_u}{\partial y}+c_{23}\frac{\partial \mathcal{W}_u}{\partial z})+\kappa_y(c_{12}z+c_{22}\frac{\partial \mathcal{V}_w}{\partial y}+c_{23}\frac{\partial \mathcal{W}_w}{\partial z})\\
&\quad+\kappa_z(c_{12}y+c_{22}\frac{\partial \mathcal{V}_v}{\partial y}+c_{23}\frac{\partial \mathcal{W}_v}{\partial z}),\\
&\sigma_z=\varepsilon(c_{13}+c_{23}\frac{\partial \mathcal{V}_u}{\partial y}+c_{33}\frac{\partial \mathcal{W}_u}{\partial z})+\kappa_y(c_{13}z+c_{23}\frac{\partial \mathcal{V}_w}{\partial y}+c_{33}\frac{\partial \mathcal{W}_w}{\partial z})\\
&\quad+\kappa_z(c_{13}y+c_{23}\frac{\partial \mathcal{V}_v}{\partial y}+c_{33}\frac{\partial \mathcal{W}_v}{\partial z}),\\
&\tau_{xy}=\kappa_z' c_{44}(\mathcal{V}_v+\frac{\partial \mathcal{U}_v}{\partial y})+\kappa_y' c_{44}(\mathcal{V}_w+\frac{\partial \mathcal{U}_w}{\partial y}),\\
&\tau_{xz}=\kappa_z' c_{55}(\mathcal{W}_v+\frac{\partial \mathcal{U}_v}{\partial z})+\kappa_y' c_{55}(\mathcal{W}_w+\frac{\partial \mathcal{U}_w}{\partial z}),\\
&\tau_{yz}=\varepsilon c_{66}(\frac{\partial \mathcal{W}_u}{\partial y}+\frac{\partial \mathcal{V}_u}{\partial z})+\kappa_z c_{66}(\frac{\partial \mathcal{W}_v}{\partial y}+\frac{\partial \mathcal{V}_v}{\partial z})+\kappa_y c_{66}(\frac{\partial \mathcal{W}_w}{\partial y}+\frac{\partial \mathcal{V}_w}{\partial z}).
\end{aligned} \tag{15}$$

4 Boundary value problems in the cross-section

Substituting the stress components (14) into the axial equilibrium equation and the corresponding boundary condition

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}=0 \text{ in } V, \quad n_y\tau_{xy}+n_z\tau_{xz}=0 \text{ on } S, \tag{16}$$

where V is the volume and S is the area of the longitudinal face of the beam, and using the assumptions of f1 results to two separate boundary value problems

$$\left.\begin{aligned} &\frac{\partial}{\partial y}(G_{xy}\frac{\partial \mathcal{U}_\alpha}{\partial y})+\frac{\partial}{\partial y}(G_{xz}\frac{\partial \mathcal{U}_\alpha}{\partial z})+E_x f_\alpha=0 \text{ in } A,\\ &n_y G_{xy}\frac{\partial \mathcal{U}_\alpha}{\partial y}+n_z G_{xz}\frac{\partial \mathcal{U}_\alpha}{\partial z}=0 \text{ on } s, \end{aligned}\right\} \alpha=v,w. \tag{17}$$

where A is the area of the cross-section and s is its boundary line, for the warping functions $\mathcal{U}_v(y,z)$ and $\mathcal{U}_w(y,z)$, respectively. Because of space limitation, here and in the following, three new functions $f_u=1$, $f_v=y$ and $f_w=z$ have been adopted.

Substituting the stress components (15) into the equilibrium equations and the corresponding boundary conditions

$$\left.\begin{aligned} &\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}=0\\ &\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}=0\\ &\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}=0 \end{aligned}\right\} \text{ in } V, \quad \left.\begin{aligned} &n_y\tau_{xy}+n_z\tau_{xz}=0\\ &n_y\sigma_y+n_z\tau_{yz}=0\\ &n_y\tau_{yz}+n_z\sigma_z=0 \end{aligned}\right\} \text{ on } S \tag{18}$$

and using the assumptions of f2 results to the following boundary value problems for the contraction functions $\mathcal{V}_\alpha(y,z)$, $\mathcal{W}_\alpha(y,z)$ $\alpha=u,v,w$

$$\left.\begin{aligned} &\left.\begin{aligned} &\frac{\partial}{\partial y}(c_{22}\frac{\partial \mathcal{V}_\alpha}{\partial y}+c_{23}\frac{\partial \mathcal{W}_\alpha}{\partial z})+\frac{\partial}{\partial z}(c_{66}\frac{\partial \mathcal{W}_\alpha}{\partial y}+c_{66}\frac{\partial \mathcal{V}_\alpha}{\partial z})+\frac{\partial}{\partial y}(c_{12}f_\alpha)=0\\ &\frac{\partial}{\partial y}(c_{66}\frac{\partial \mathcal{W}_\alpha}{\partial y}+c_{66}\frac{\partial \mathcal{V}_\alpha}{\partial z})+\frac{\partial}{\partial z}(c_{23}\frac{\partial \mathcal{V}_\alpha}{\partial y}+c_{33}\frac{\partial \mathcal{W}_\alpha}{\partial z})+\frac{\partial}{\partial z}(c_{13}f_\alpha)=0 \end{aligned}\right\} \text{ in } A,\\ &\left.\begin{aligned} &n_y(c_{22}\frac{\partial \mathcal{V}_\alpha}{\partial y}+c_{23}\frac{\partial \mathcal{W}_\alpha}{\partial z})+n_z(c_{66}\frac{\partial \mathcal{W}_\alpha}{\partial y}+c_{66}\frac{\partial \mathcal{V}_\alpha}{\partial z})+n_y c_{12}f_\alpha=0\\ &n_y(c_{66}\frac{\partial \mathcal{W}_\alpha}{\partial y}+c_{66}\frac{\partial \mathcal{V}_\alpha}{\partial z})+n_z(c_{23}\frac{\partial \mathcal{V}_\alpha}{\partial y}+c_{33}\frac{\partial \mathcal{W}_\alpha}{\partial z})+n_z c_{13}f_\alpha=0 \end{aligned}\right\} \text{ on } s, \end{aligned}\right\} \alpha=u,v,w \tag{19}$$

and the following boundary value problems for the warping functions $\mathcal{U}_\alpha(y,z)$ $\alpha = v, w$

$$\left.\begin{aligned}
&\frac{\partial}{\partial y}(c_{44}\frac{\partial \mathcal{U}_\alpha}{\partial y}) + \frac{\partial}{\partial z}(c_{55}\frac{\partial \mathcal{U}_\alpha}{\partial z}) + (c_{12}+c_{44})\frac{\partial \mathcal{V}_\alpha}{\partial y} \\
&+(c_{13}+c_{55})\frac{\partial \mathcal{W}_\alpha}{\partial z} + \mathcal{V}_\alpha\frac{\partial c_{44}}{\partial y} + \mathcal{W}_\alpha\frac{\partial c_{55}}{\partial z} + c_{11} f_\alpha = 0 \text{ in } A, \\
&n_y c_{44}\frac{\partial \mathcal{U}_\alpha}{\partial y} + n_z c_{55}\frac{\partial \mathcal{U}_\alpha}{\partial z} + n_y c_{44}\mathcal{V}_\alpha + n_z c_{55}\mathcal{W}_\alpha = 0 \text{ on } s,
\end{aligned}\right\} \alpha = v, w. \tag{20}$$

5 Weak forms

Weak forms of the boundary value problems (17) for the warping functions can be written as

$$\int_A (G_{xy}\frac{\partial \delta\mathcal{U}_\alpha}{\partial y}\frac{\partial \mathcal{U}_\alpha}{\partial y} + G_{xz}\frac{\partial \delta\mathcal{U}_\alpha}{\partial z}\frac{\partial \mathcal{U}_\alpha}{\partial z})dA = \int_A \delta\mathcal{U}_\alpha E_x f_\alpha dA \quad \alpha = v, w. \tag{21}$$

Weak forms of the boundary value problems (19) for the contraction functions can be written as

$$\begin{aligned}
&\int_A (c_{22}\frac{\partial \delta\mathcal{V}_\alpha}{\partial y}\frac{\partial \mathcal{V}_\alpha}{\partial y} + c_{66}\frac{\partial \delta\mathcal{V}_\alpha}{\partial z}\frac{\partial \mathcal{V}_\alpha}{\partial z} + c_{23}\frac{\partial \delta\mathcal{V}_\alpha}{\partial y}\frac{\partial \mathcal{W}_\alpha}{\partial z} + c_{66}\frac{\partial \delta\mathcal{V}_\alpha}{\partial z}\frac{\partial \mathcal{W}_\alpha}{\partial y})dA \\
&+\int_A (c_{66}\frac{\partial \delta\mathcal{W}_\alpha}{\partial y}\frac{\partial \mathcal{V}_\alpha}{\partial z} + c_{23}\frac{\partial \delta\mathcal{W}_\alpha}{\partial z}\frac{\partial \mathcal{V}_\alpha}{\partial y} + c_{66}\frac{\partial \delta\mathcal{W}_\alpha}{\partial y}\frac{\partial \mathcal{W}_\alpha}{\partial y} + c_{33}\frac{\partial \delta\mathcal{W}_\alpha}{\partial z}\frac{\partial \mathcal{W}_\alpha}{\partial z})dA \\
&+\int_A (c_{12}\frac{\partial \delta\mathcal{V}_\alpha}{\partial y} f_\alpha + c_{13}\frac{\partial \delta\mathcal{W}_\alpha}{\partial z} f_\alpha)dA = 0 \quad \alpha = u, v, w.
\end{aligned} \tag{22}$$

Weak forms of the boundary value problems (20) for the warping functions can be written as

$$\begin{aligned}
\int_A (c_{44}\frac{\partial \delta\mathcal{U}_\alpha}{\partial y}\frac{\partial \mathcal{U}_\alpha}{\partial y} + c_{55}\frac{\partial \delta\mathcal{U}_\alpha}{\partial z}\frac{\partial \mathcal{U}_\alpha}{\partial z})dA = &\int_A [\delta\mathcal{U}_\alpha(c_{11} f_\alpha + c_{12}\frac{\partial \mathcal{V}_\alpha}{\partial y} + c_{13}\frac{\partial \mathcal{W}_\alpha}{\partial z}) \\
&- c_{44}\frac{\partial \delta\mathcal{U}_\alpha}{\partial y}\mathcal{V}_\alpha - c_{55}\frac{\partial \delta\mathcal{U}_\alpha}{\partial z}\mathcal{W}_\alpha]dA \quad \alpha = v, w.
\end{aligned} \tag{23}$$

6 Finite element equations

Using finite element approximations

$$\hat{\mathcal{U}}(y,z)=\sum_{i=1}^{m}N_i^e(y,z)a_{ui}^e,\ \hat{\mathcal{V}}(y,z)=\sum_{i=1}^{m}N_i^e(y,z)a_{vi}^e,\ \hat{\mathcal{W}}(y,z)=\sum_{i=1}^{m}N_i^e(y,z)a_{wi}^e \tag{24}$$

for the warping and contraction functions, where $N_i^e(y,z)$ are element shape functions, a_{ui}^e, a_{vi}^e and a_{wi}^e are element nodal values and m is the number of element nodes, the boundary value problems can be discretized. Typical elements of the element matrix $\mathbf{K}^e$ and element vectors $\mathbf{R}_v^e$ and $\mathbf{R}_w^e$ corresponding to the weak forms (21) of f1 are

$$K_{ij}^e=\int_{A^e}(G_{xy}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}+G_{xz}\frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial z})dA,\ R_{\alpha i}^e=\int_A N_i^e E_x f_\alpha dA\ \ \alpha=v,w. \tag{25}$$

Typical elements of the element matrix $\mathbf{K}^e$ and element vectors $\mathbf{R}_u^e$, $\mathbf{R}_v^e$ and $\mathbf{R}_w^e$ corresponding to the weak forms (22) of f2 are

$$\begin{aligned}
K_{2i-1,2j-1}^e &= \int_{A^e}(c_{22}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}+c_{66}\frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial z})dA,\\
K_{2i-1,2j}^e &= \int_{A^e}(c_{23}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial z}+c_{66}\frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial y})dA,\\
K_{2i,2j-1}^e &= \int_{A^e}(c_{66}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial z}+c_{23}\frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial y})dA,\\
K_{2i,2j}^e &= \int_{A^e}(c_{66}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}+c_{33}\frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial z})dA,\\
R_{\alpha 2i-1}^e &= -\int_A c_{12}\frac{\partial N_i^e}{\partial y}f_\alpha dA,\ \ R_{\alpha 2i}^e=-\int_A c_{13}\frac{\partial N_i^e}{\partial z}f_\alpha dA\quad \alpha=u,v,w.
\end{aligned} \tag{26}$$

Typical elements of the element matrix $\mathbf{K}^e$ and element vectors $\mathbf{R}_v^e$ and $\mathbf{R}_w^e$ corresponding to the to the weak forms (23) of f2 are

$$\begin{aligned}
K_{ij}^e &= \int_{A^e}(c_{44}\frac{\partial N_i^e}{\partial y}\frac{\partial N_j^e}{\partial y}+c_{55}\frac{\partial N_i^e}{\partial z}\frac{\partial N_j^e}{\partial z})dA,\\
R_{\alpha i}^e &= \int_A(N_i^e c_{11}f_\alpha+c_{12}N_i^e\frac{\partial\hat{\mathcal{V}}_\alpha}{\partial y}+c_{13}N_i^e\frac{\partial\hat{\mathcal{W}}_\alpha}{\partial z}-c_{44}\frac{\partial N_i^e}{\partial y}\hat{\mathcal{V}}_\alpha-c_{55}\frac{\partial N_i^e}{\partial z}\hat{\mathcal{W}}_\alpha)dA\ \ \alpha=v,w.
\end{aligned} \tag{27}$$

7 Stress resultants and generalized strains

7.1 Stress resultants in terms of generalized strains

For the normal force and the bending moments one gets

$$
\begin{aligned}
N &\equiv \int_A \sigma_x dA = \overline{EA}\varepsilon + \overline{ES}_y\kappa_y + \overline{ES}_z\kappa_z, \\
M_y &\equiv \int_A \sigma_x z dA = \overline{ES}_y\varepsilon + \overline{EI}_y\kappa_y + \overline{EI}_{yz}\kappa_z, \\
M_z &\equiv \int_A \sigma_x y dA = \overline{ES}_z\varepsilon + \overline{EI}_{yz}\kappa_y + \overline{EI}_z\kappa_z.
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
&\overline{EA} = \int_A \overline{E}_x dA,\ \overline{ES}_y = \int_A \overline{E}_x z dA,\ \overline{ES}_z = \int_A \overline{E}_x y dA, \\
&\overline{EI}_y = \int_A \overline{E}_x z^2 dA,\ \overline{EI}_z = \int_A \overline{E}_x y^2 dA,\ \overline{EI}_{yz} = \int_A \overline{E}_x yz dA
\end{aligned} \tag{29}
$$

where in f1

$$
\overline{E}_x = E_x \tag{30}
$$

and in f2

$$
\overline{E}_x = c_{11} + c_{12}\frac{\partial \mathcal{V}_u}{\partial y} + c_{13}\frac{\partial \mathcal{W}_u}{\partial z}. \tag{31}
$$

7.2 Generalized strains in terms of stress resultants

The center of cross-section, which defines the origin of the $y,z-$coordinates and the beam $x-$axis, is reasonable to choose so, that the weighted first moments $\overline{ES}_y$ and $\overline{ES}_z$ vanish. In this case equations (28) can easily be solved for the generalized strains resulting to

$$
\varepsilon = \frac{N}{\overline{EA}},\ \kappa_y = \frac{\overline{EI}_y M_y - \overline{EI}_{yz} M_z}{\overline{EI}_y \overline{EI}_z - \overline{EI}_{yz}^2},\ \kappa_z = \frac{\overline{EI}_y M_z - \overline{EI}_{yz} M_y}{\overline{EI}_y \overline{EI}_z - \overline{EI}_{yz}^2}. \tag{32}
$$

In f1 these equations are those of the classical beam theory.

7.3 Center of cross-section

The location of the center C of the cross-section in $y,z-$coordinate system, whose origin is chosen arbitrarily, can be obtained using equations

$$y_{\mathrm{C}} = \frac{\overline{ES}_z}{\overline{EA}},\ z_{\mathrm{C}} = \frac{\overline{ES}_y}{\overline{EA}}, \tag{33}$$

where the $\overline{EA}$, $\overline{ES}_y$ and $\overline{ES}_z$ have been determined using equations (29) in this coordinate system. In f2, before they can be determined, the contraction functions $\mathcal{V}_u$ and $\mathcal{W}_u$ must be known. Therefore the first boundary value problem (19) must first be solved in the original coordinate system. Thereafter the center of the cross-section can be determined.

8 Homogeneous and isotropic beam and formulation f2

If the cross-section is homogeneous and isotropic, it can be shown by substitution into the boundary value problems (19), that the contraction functions are

$$\mathcal{V}_u = -\nu y,\ \mathcal{W}_u = -\nu z,\ \mathcal{V}_v = -\frac{\nu}{2}(y^2 - z^2),\ \mathcal{W}_v = \mathcal{V}_w = -\nu yz,\ \mathcal{W}_w = -\frac{\nu}{2}(z^2 - y^2). \tag{34}$$

In this case the stresses (15) reduce to

$$\begin{aligned}
&\sigma_x = \varepsilon E + \kappa_y E z + \kappa_z E y,\ \sigma_y = 0,\ \sigma_z = 0,\\
&\tau_{xy} = \kappa_z' G[\frac{\partial \mathcal{U}_v}{\partial y} - \frac{\nu}{2}(y^2 - z^2)] + \kappa_y' G(\frac{\partial \mathcal{U}_w}{\partial y} - \nu yz),\\
&\tau_{xz} = \kappa_z' G(\frac{\partial \mathcal{U}_v}{\partial z} - \nu yz) + \kappa_y' G[\frac{\partial \mathcal{U}_w}{\partial z} - \frac{\nu}{2}(z^2 - y^2)],\ \tau_{yz} = 0.
\end{aligned} \tag{35}$$

and the boundary value problems (20) for the warping functions reduce to

$$\begin{aligned}
&\frac{\partial^2 \mathcal{U}_v}{\partial y^2} + \frac{\partial^2 \mathcal{U}_v}{\partial z^2} + \frac{2-2\nu}{1-2\nu} y = 0 \text{ in } A,\ n_y \frac{\partial \mathcal{U}_v}{\partial y} + n_z \frac{\partial \mathcal{U}_v}{\partial z} = n_y \frac{\nu}{2}(y^2 - z^2) + n_z \nu yz \text{ on } s,\\
&\frac{\partial^2 \mathcal{U}_w}{\partial y^2} + \frac{\partial^2 \mathcal{U}_w}{\partial z^2} + \frac{2-2\nu}{1-2\nu} z = 0 \text{ in } A,\ n_y \frac{\partial \mathcal{U}_w}{\partial y} + n_z \frac{\partial \mathcal{U}_w}{\partial z} = n_y \nu yz + n_z \frac{\nu}{2}(z^2 - y^2) \text{ on } s.
\end{aligned} \tag{36}$$

Finite element discretization of the weak forms corresponding to these boundary value problems results to finite element equations, which are identical to those used in [4].

9 Setting constraints to the contraction functions

The contraction functions $\mathcal{V}(y,z)$ and $\mathcal{W}(y,z)$ must satisfy the following constraints at the center C of the cross section

$$\mathcal{V}(y_{\mathrm{C}},z_{\mathrm{C}})=0,\ \mathcal{W}(y_{\mathrm{C}},z_{\mathrm{C}})=0,\ \frac{\partial\mathcal{W}}{\partial y}(y_{\mathrm{C}},z_{\mathrm{C}})-\frac{\partial\mathcal{V}}{\partial z}(y_{\mathrm{C}},z_{\mathrm{C}})=0\,. \tag{37}$$

The subscript u, v or w has been omitted here for convenience. These constraints must be discretized and added as three additional equations to the system equations for solving the finite approximations $\hat{\mathcal{V}}(y,z)$ and $\hat{\mathcal{W}}(y,z)$ of the contraction functions. In order to be able to perform this discretization, one has first to find the specific element e (or elements) within which the center C of the cross-section is located. Second one hast to determine the natural co-ordinates $\xi_{\mathrm{C}},\eta_{\mathrm{C}}$ of the center C in this element. This latter procedure consists of solving a pair of nonlinear equations

$$\begin{cases} y(\xi,\eta)=y_{\mathrm{C}}, \\ z(\xi,\eta)=z_{\mathrm{C}} \end{cases} \tag{38}$$

where

$$y(\xi,\eta)=\sum_{i=1}^{9}N_i^e(\xi,\eta)y_i^e,\ \mathrm{z}(\xi,\eta)=\sum_{i=1}^{9}N_i^e(\xi,\eta)z_i^e \tag{39}$$

in element e. This solution is easily performed numerically. After the natural co-ordinates $\xi_{\mathrm{C}},\eta_{\mathrm{C}}$ are known, the constraints (37) can be discretized as follows. Within element e they are written as

$$\begin{aligned} &\hat{\mathcal{V}}(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\equiv\sum_{i=1}^{9}N_i^e(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\mathcal{V}_i^e=0,\quad \hat{\mathcal{W}}(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\equiv\sum_{i=1}^{9}N_i^e(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\mathcal{W}_i^e=0, \\ &\frac{\partial\hat{\mathcal{W}}}{\partial y}(\xi_{\mathrm{C}},\eta_{\mathrm{C}})-\frac{\partial\hat{\mathcal{V}}}{\partial z}(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\equiv-\sum_{i=1}^{9}\frac{\partial N_i^e}{\partial z}(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\mathcal{V}_i^e+\sum_{i=1}^{9}\frac{\partial N_i^e}{\partial y}(\xi_{\mathrm{C}},\eta_{\mathrm{C}})\mathcal{W}_i^e=0 \end{aligned} \tag{40}$$

or

$$\widehat{\mathbf{K}}^e\mathbf{a}^e=\mathbf{0}\,, \tag{41}$$

where

$$\underset{3\times18}{\widehat{\mathbf{K}}^e}=\begin{bmatrix} N_1^e(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) & 0 & & N_9^e(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) & 0 \\ 0 & N_1^e(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) & \cdots & 0 & N_9^e(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) \\ -\frac{\partial N_1^e}{\partial z}(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) & \frac{\partial N_1^e}{\partial y}(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) & & -\frac{\partial N_9^e}{\partial z}(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) & \frac{\partial N_9^e}{\partial y}(\xi_{\mathrm{C}},\eta_{\mathrm{C}}) \end{bmatrix},\ \underset{18\times1}{\mathbf{a}^e}=\begin{Bmatrix} \mathcal{V}_1^e \\ \mathcal{W}_1^e \\ \vdots \\ \mathcal{V}_9^e \\ \mathcal{W}_9^e \end{Bmatrix}. \tag{42}$$

Equations (41) are discretized element equations, which express contribution of element e to the discretized constraint equations in terms of the element degrees of freedom. Corresponding three additional system equations are of form

$$\widehat{\mathbf{K}}\mathbf{a} = \mathbf{0}, \tag{43}$$

where the system matrix $\widehat{\mathbf{K}}$ is assembled from the elements of the element matrix $\widehat{\mathbf{K}}^e$ (or matrices) using standard procedure.

10 Numerical examples

10.1 Model problem

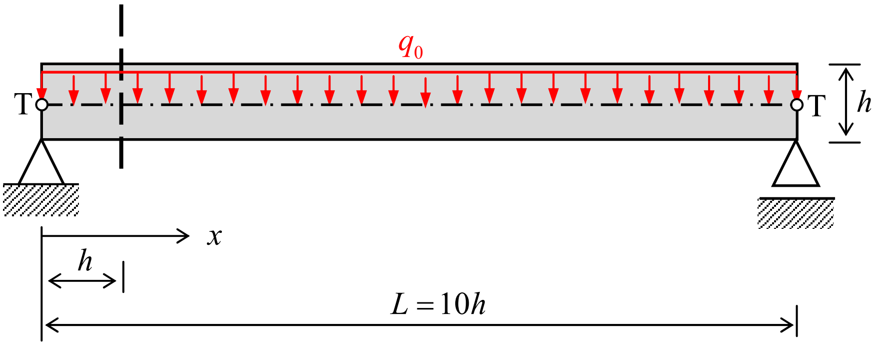

Figure 1: Simply supported beam under uniform load

A simply supported beam of Fig. 1 under uniform load q_0 is used as a model problem in the numerical examples. The cross-section at distance h (height of the beam) from the left support is considered. The non-zero stress resultants at this cross-section can be easily determined and they are $Q_y = 4q_0h$ and $M_z = 4.5q_0h^2$. Because only of the effect of bending is studies, the load is assumed to act to axis $\mathrm{T}-\mathrm{T}$, which is defined by the centres of twist of the cross-sections (see Fig. 1).

10.2 Example beams and discretization of the cross-section

Three beams with different cross-sections were analysed. Each beam is composed of two isotropic materials 1 and 2. Moduli of elasticity of these materials are related by $E_2 = 8E_1$ and Poisson's ratios are $\nu_1 = 0$ and $\nu_2 = 0.3$. Such material constants represent typical values of a concrete-steel composite beam. The cross-sections of the three problems are square, circle and J-shaped as shown in Figs. 2a, 3a and 4a, respectively. The cross-sections were discretized using bi-quadratic iso-parametric finite elements with 9×9 Gauss numerical integration. Figs. 2b, 3b and 4b show the grids used.

10.3 Analysis methods

In each beam problem the cross-section of interest was analysed using the presented formulations f1 and f2. For comparison the beam was further analysed using finite prism method [6]. The semi-analytical discretization of the method was performed using 21 terms of trigonometric series in axial direction and identical grids as in f1 and f2 to discretize the cross-section of the beam. The finite prism analysis gives numerical solution of a prismatic three-dimensional body, whose end cross-sections

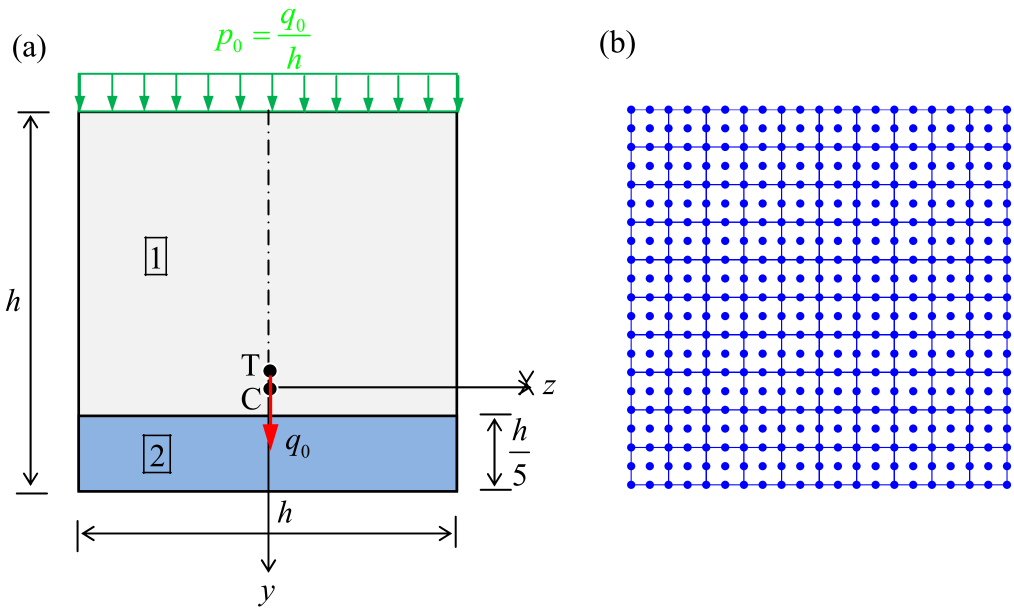

Figure 2: Square cross-section (a) geometry and (b) finite element grid

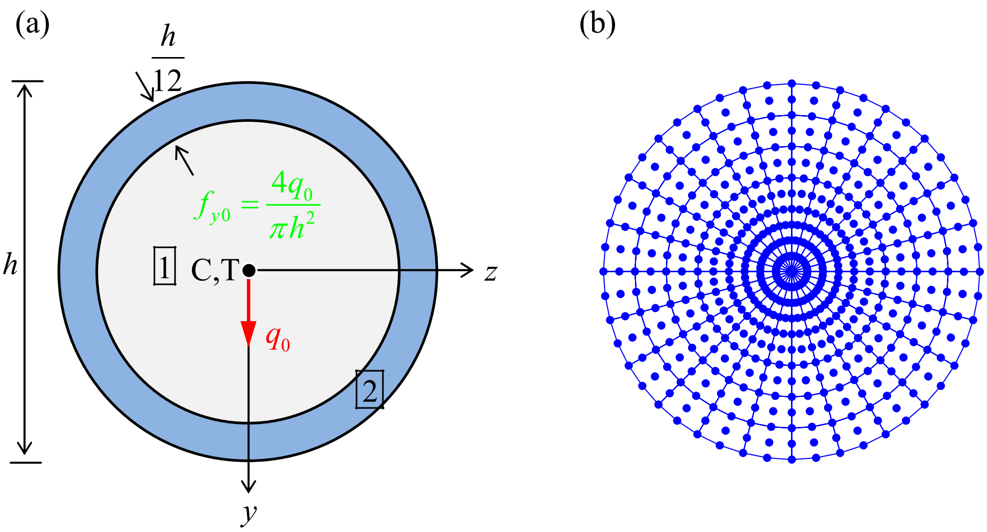

Figure 3: Circular cross-section (a) geometry and (b) finite element grid

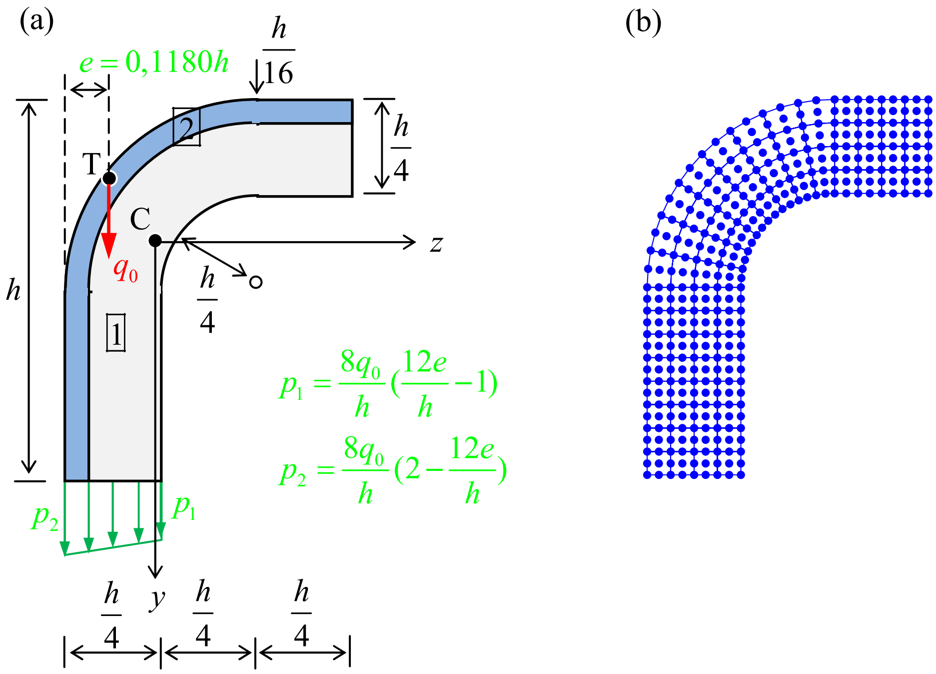

Figure 4: J-shaped cross-section (a) geometry and (b) finite element grid

are rigid in their own plane but free to warp in the axial direction. The finite prism solution is quite accurate and is regarded as a reference solution here. Formulations f1 and f2 can be regarded as techniques, by which the cross-sectional stress distributions of the classical beam theory can be improved. Only the resulting uniform load q_0 per unit length of the beam has effect to the stress resultants Q_y and M_z and further to the stress distributions, not its detailed distribution inside and on the faces of the beam. In the finite prism analysis, however, more detailed distribution of tractions or volume forces, which are equivalent to q_0, is required. The chosen more detailed loading in the square cross-section beam is uniform surface load p_0 on the upper face (Fig. 2a), in the circular cross-section beam uniform volume force f_{y0} (Fig. 3a) and in the J-shaped cross-section beam linear surface load with end values p_1 and p_2 on the lower face of the web (Fig. 4a). The length $e = 0{,}1180h$ in figure 4a defines the horizontal location of the centre of twist. It has been determined by solving warping function based torsion analysis of the J-shaped cross-section using finite elements [5] and the grid of Fig. 4b.

10.4 Results of analysis

Figs. 5-6, 7-8 and 9-10, show the results of the square, circular and J-shaped cross-section beams, respectively. Formulation f1 determines only the typical stress components σ_x, τ_{xy} and τ_{xz} of beam theory, which we call here primary stresses, but formulation f2 can also determine the remaining, which we call here secondary.

10.4.1 Results of the square cross-section

The results of f2 for all stress components but σ_y, seem to be close to the results of the finite prism method, and thus quite accurate. The reason for the difference in σ_y is, that the finite prism method takes into account the surface load p_0 correctly, but f2 takes into account its effect through the resultant q_0. The results of f1 for the stress components σ_x and τ_{xy} are slightly smaller than those of f2 and the shear stress τ_{xz} vanishes. It is shown in Appendix A, that in uniaxial bending of a beam with layered rectangular cross section, the results of f1 should coincide with those of classical beam theory. Therefore the finite element results of f1 should converge to the results of classical theory. Using this one easily gets for the normal stress on the upper and lower faces $\sigma_{x1} = -2475/36 q_0/h \approx -18.2 q_0/h$ and $\sigma_{x2} = 900/17 q_0/h \approx 52.9 q_0/h$, respectively. These coincide well with the numerical results of Fig. 5c. For the maximum shear stress, the classical beam theory gives $\tau_{xy,\max} = 605/102 q_0/h \approx 5.93 q_0/h$ along the $z-$axis passing through the center C. Finite element solution of f1 gives $\tau_{xy,\max} \approx 6.03 q_0/h$ on the upper face of material 2 (see Fig. 6c). If, however, the grid spacing is halved, the finite element result is $\tau_{xy,\max} \approx 5.93 q_0/h$ and it is located close to $z-$axis.

10.4.2 Results of the circular cross-section

The results of both f2 and f1 for the primary stresses seem to be close to those of the finite prism method, and thus quite accurate. The results of f2 for the secondary stresses are mainly lower than those of the finite prism method. One reason for this might be the distance of the studied cross-section from the left support. Therefore a check was made by determining the maximum stress σ_z at distances h and $2h$ from the left support, respectively. At distances h and $2h$ the results of f2 were $77,2\%$ and $86,4\%$ of the finite prism results, respectively. Results of f2, however, describe distribution of the secondary stresses quite well.

10.43 Results of the J-shaped cross-section

The results of both f2 and f1 for the primary stresses seem to be mainly close to those of the finite prism method, and thus quite accurate. The results of f2 for the secondary stresses are mainly slightly lower than those of the finite prism method. They, however, describe the stress distributions quite well.

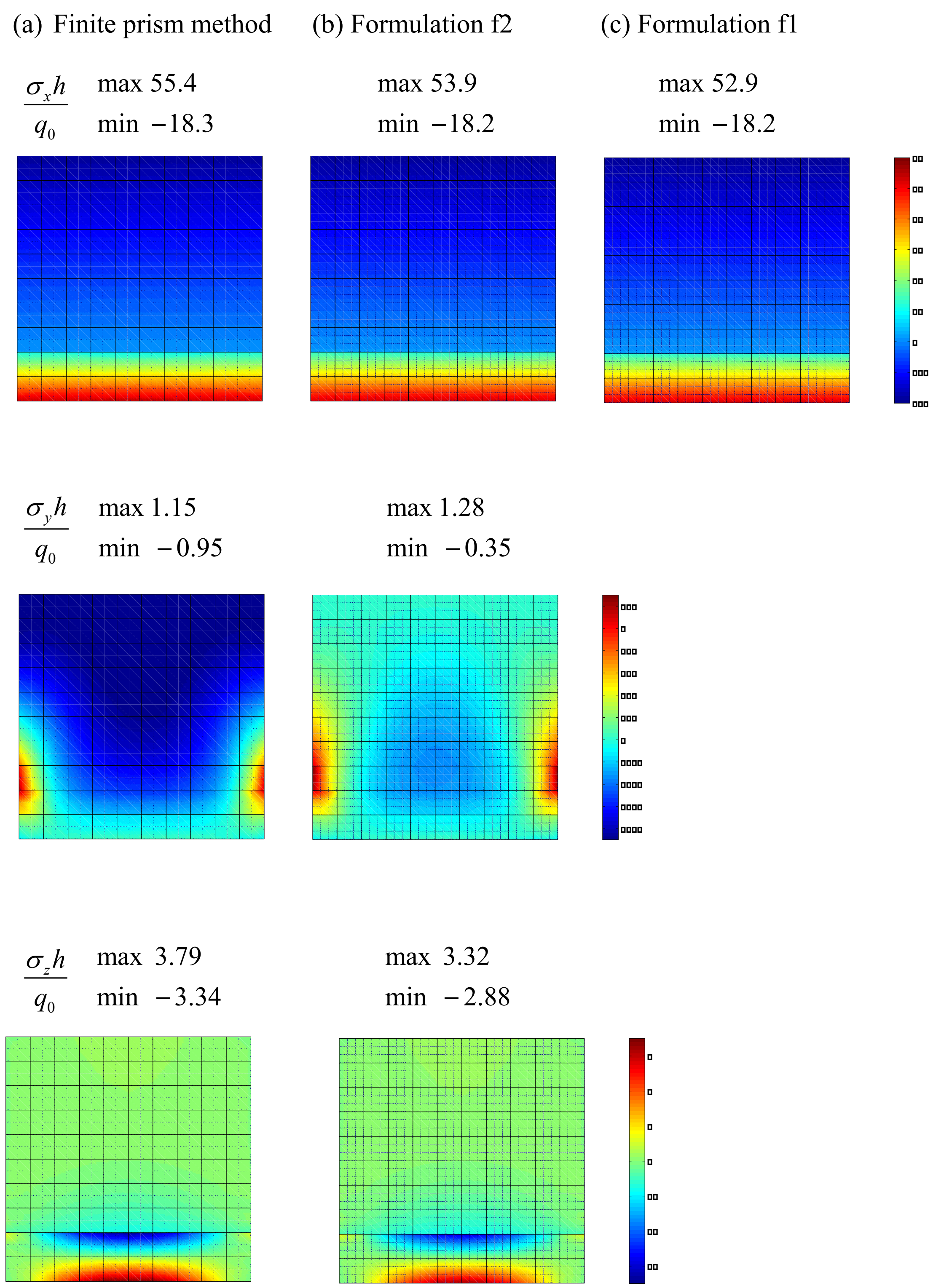

Figure 5: Normal stresses with finite prism method and formulations f2 and f1; Square cross-section

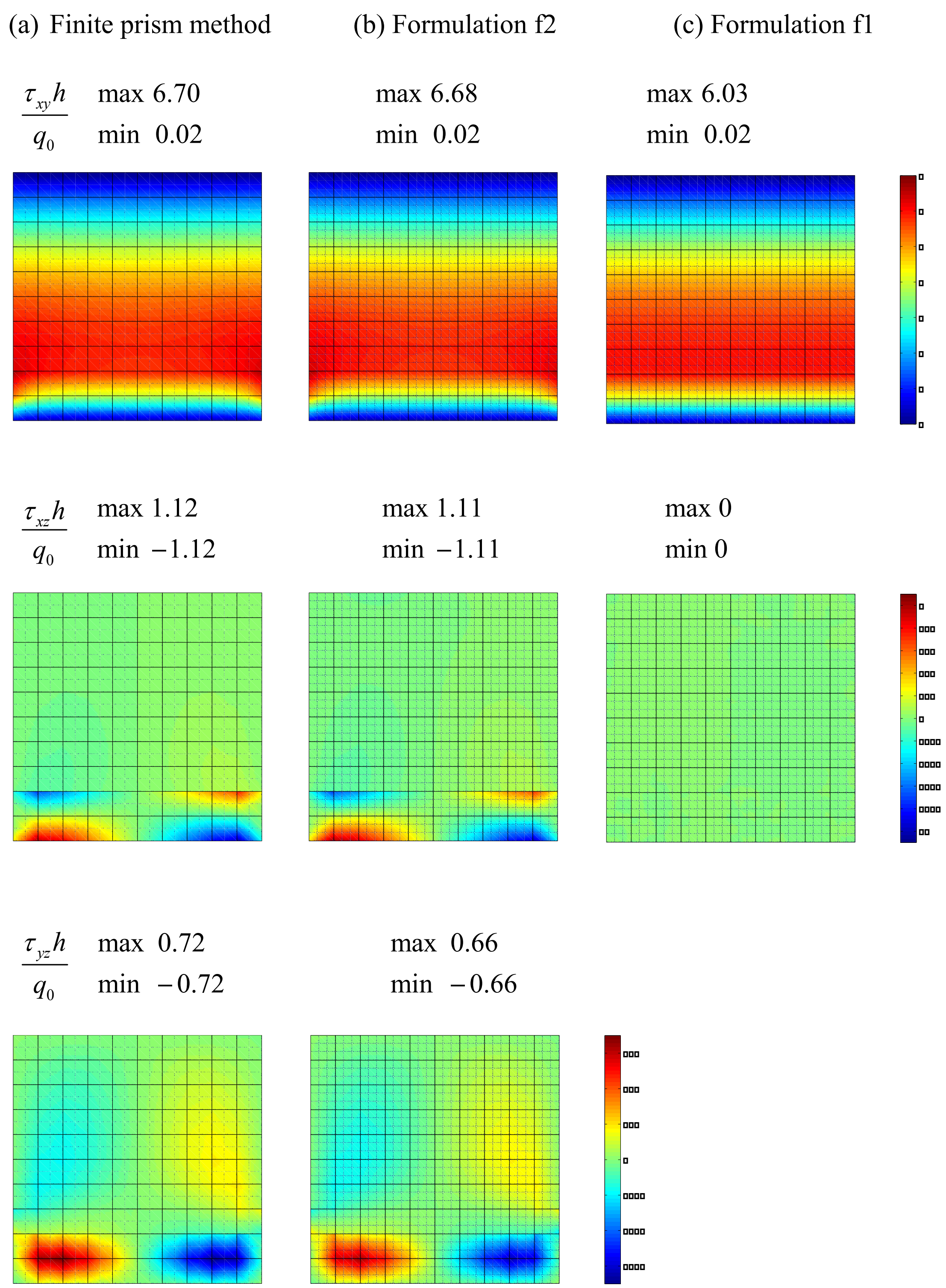

Figure 6: Shear stresses with finite prism method and formulations f2 and f1; Square cross-section

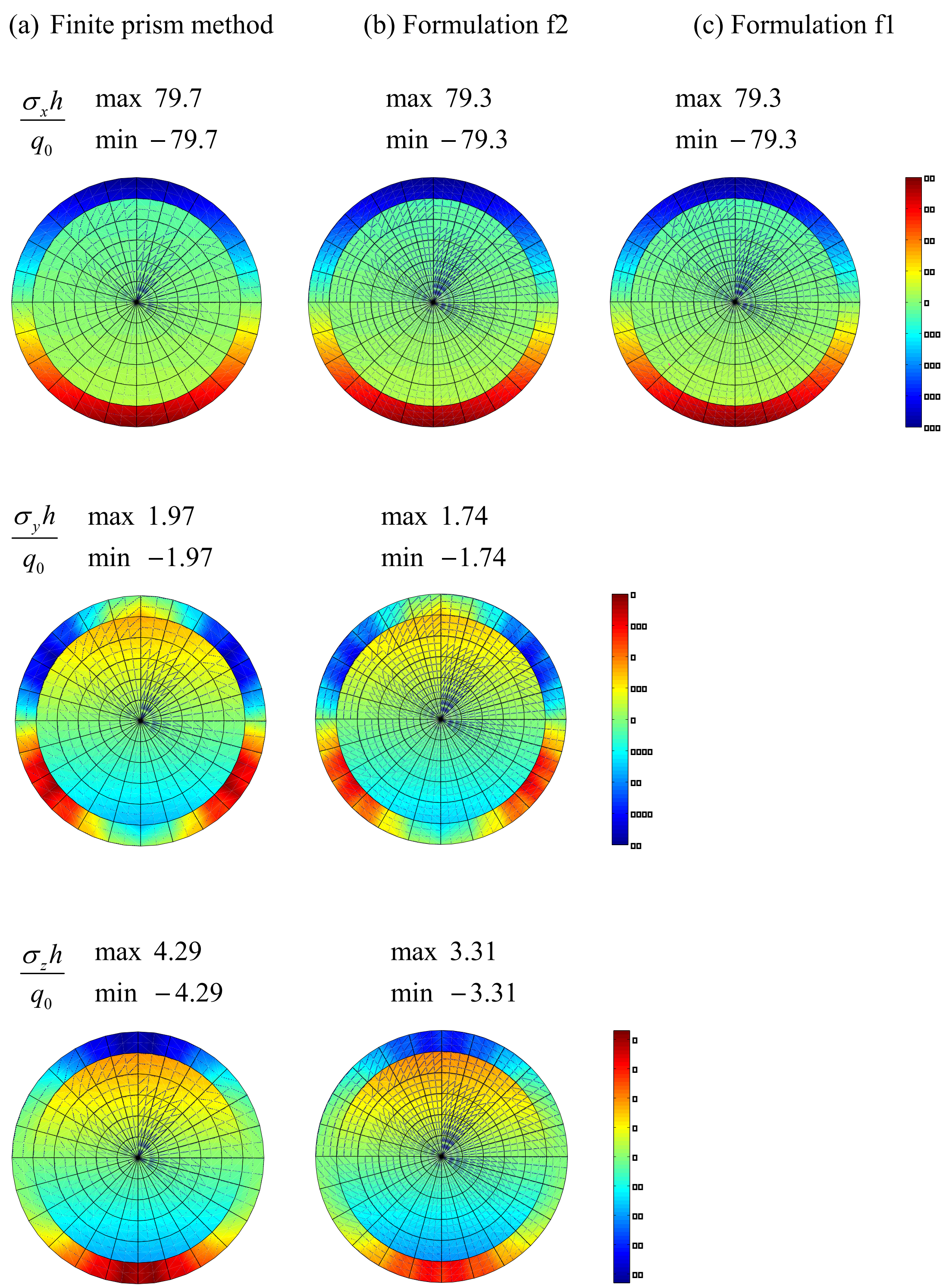

Figure 7: Normal stresses with finite prism method and formulations f2 and f1; Circular cross-section

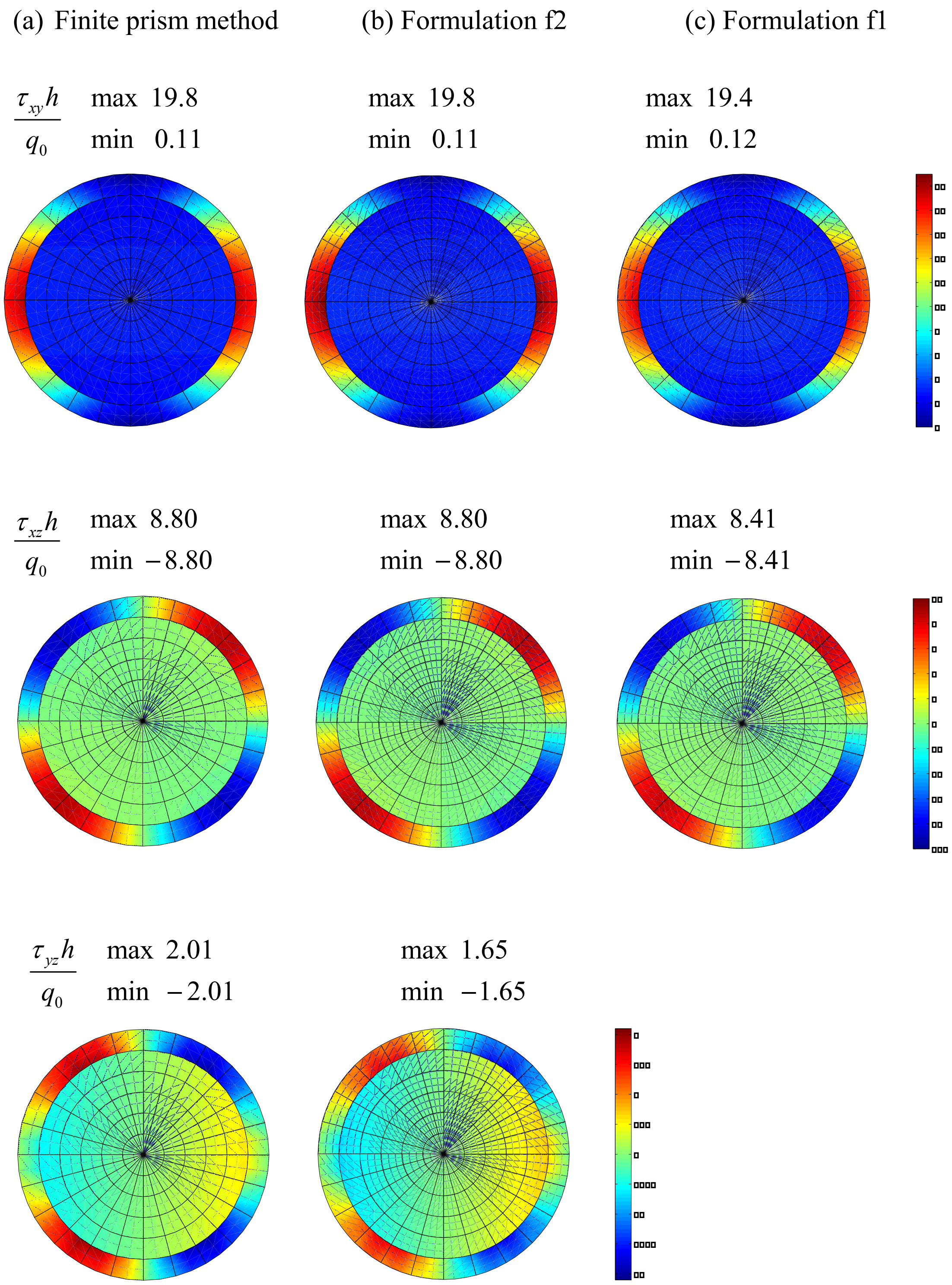

Figure 8: Shear stresses with finite prism method and formulations f2 and f1; Circular cross-section

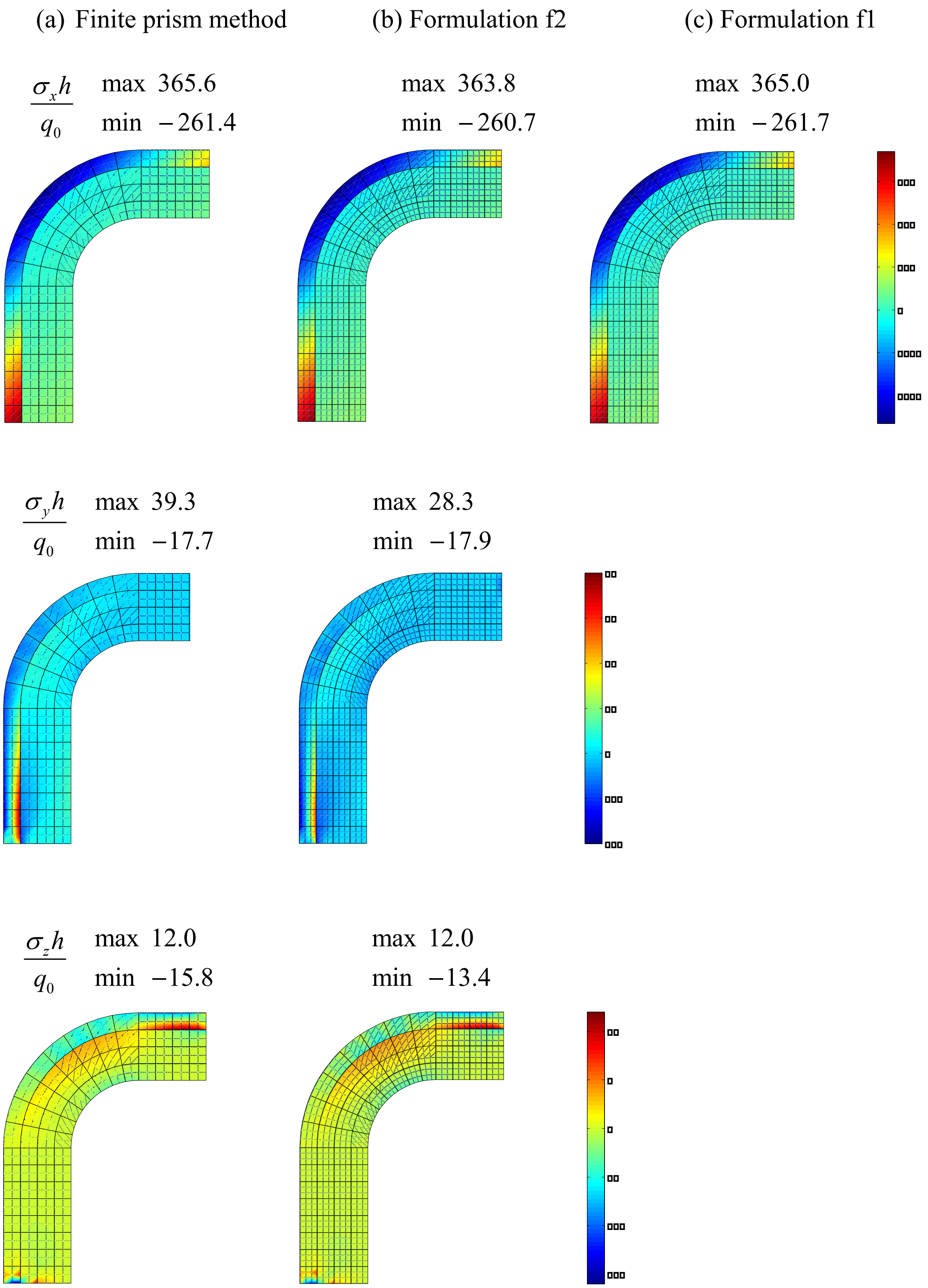

Figure 9: Normal stresses with finite prism method and formulations f2 and f1; J-shaped cross-section

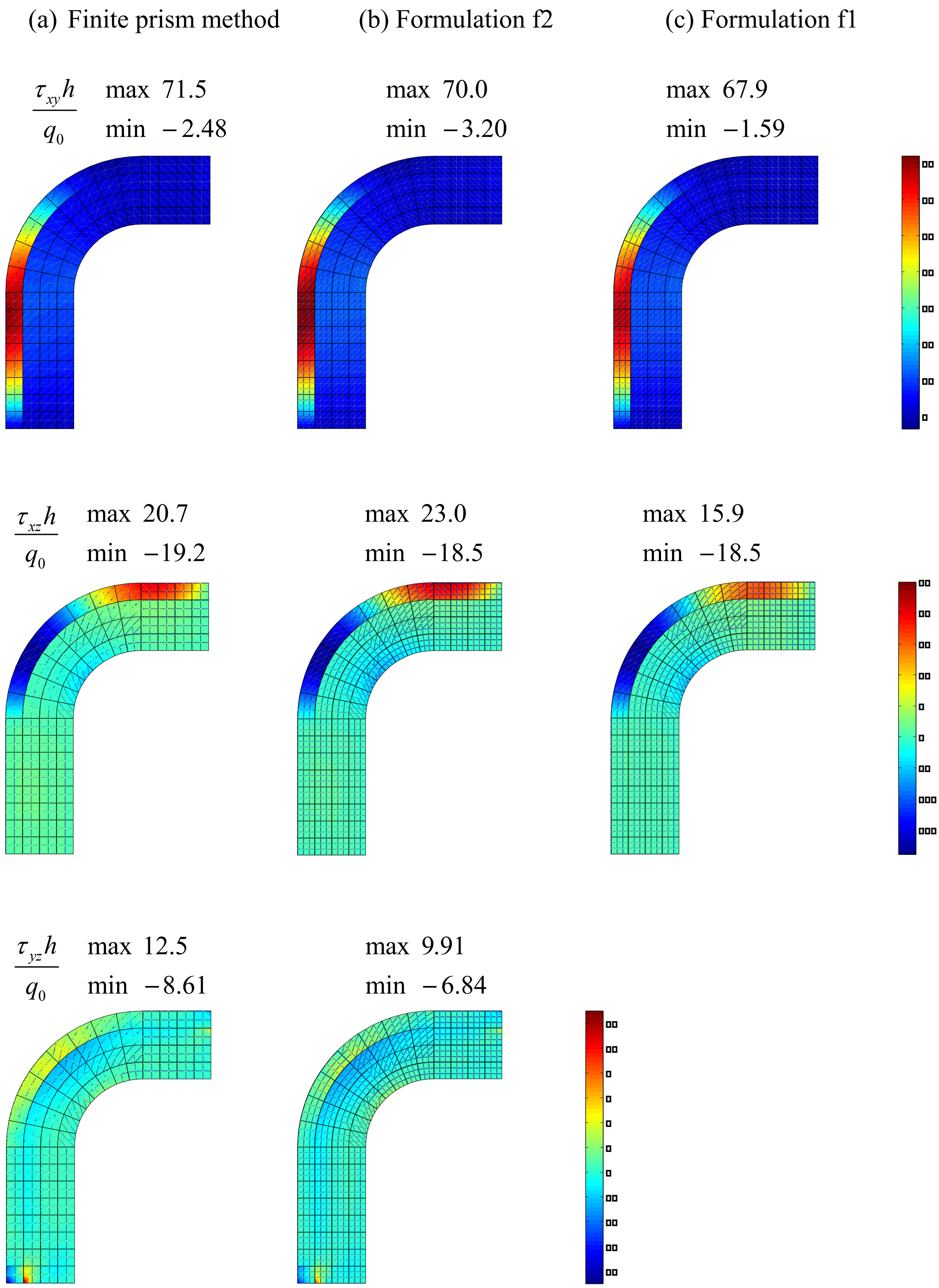

Fig 10: Shear stresses with finite prism method and formulations f2 and f1; J-shaped cross-section

11 Conclusions

Two formulations for determining distribution of the stress components in a cross-section of a composite beam under two-axial bending were studied. Compared to the classical formulas of beam theory, the first formulation gives identical result for the normal stress σ_x, but improves the shear stresses τ_{xy} and τ_{xz} so, that the axial equilibrium equation and the corresponding boundary condition are satisfied. The second formulation improves all the stress components σ_x, σ_y, σ_z, τ_{xy}, τ_{xz} and τ_{yz} so, that all the equilibrium equations and the corresponding boundary conditions are satisfied. Compared to the numerical results obtained with finite prism method, both formulations give results, which describe the primary stresses quite accurately. The results of the second formulation for the secondary stresses are also satisfactory, but slightly lower than those of the finite prism method.

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Appendix A: Stress components of formulation f1 in uniaxial bending of a layered rectangular cross-section

In uniaxial bending in the $xy-$plane $Q_z = 0$ and $M_y = 0$. Because the cross-section is symmetric $\overline{EI}_{yz} = 0$. Thus equations (32) and (14) give

$$\sigma_x = \frac{M_z E_x}{\overline{EI}_z} y, \; \tau_{xy} = \frac{Q_y G_{xy}}{\overline{EI}_z} \frac{\partial \mathcal{U}_v}{\partial y}, \; \tau_{xy} = \frac{Q_y G_{xz}}{\overline{EI}_z} \frac{\partial \mathcal{U}_v}{\partial z}. \tag{A1}$$

If we assume that $\mathcal{U}_v = \mathcal{U}_v(y)$, the boundary condition (17b) on the vertical sides of the cross-section is automatically satisfied. Now equations (A1) get the form

$$\sigma_x = \frac{M_z E_x}{\overline{EI}_z} y,\ \tau_{xy} = \frac{Q_y G_{xy}}{\overline{EI}_z} \frac{d\mathcal{U}_v}{dy},\ \tau_{xy} = 0, \tag{A2}$$

and boundary value problem (17) gets the form

$$\frac{d}{dy}(G_{xy} \frac{d\mathcal{U}_v}{dy}) + E_x y = 0\ y_1 \le y \le y_2,\quad \frac{d\mathcal{U}_v}{dy} = 0\ y = y_1 \text{ and } y = y_2. \tag{A3}$$

where y_1 and y_2 are the coordinates of the top and bottom faces of the cross-section. Integrating equation (A3a) once and using the first boundary condition (A3b) results to

$$G_{xy} \frac{d\mathcal{U}_v}{dy} = -\int_{y_1}^{y} E_x y dy\,. \tag{A4}$$

Substituting this result into equation (A2) results to

$$\sigma_x = \frac{M_z E_x}{\overline{EI}_z} y,\ \tau_{xy} = -\frac{Q_y}{\overline{EI}_z} \int_{y_1}^{y} E_x y dy,\ \tau_{xz} = 0. \tag{A5}$$

These are expressions of the stress components of classical beam theory.