Paper 268


Proceedings of the Eleventh International Conference on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

# Eurocode Stability Requirements in Optimal Shakedown Truss Design

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## Abstract

This paper considers the optimization of a perfectly elastic-plastic truss under repeated variable load. The improved mathematical model of truss volume minimization problem with strength, stiffness and stability constraints is presented. The assumptions of the calculation methods of the truss-like structures and the shakedown theory are applied. The evaluation of the stability of the elements under compression is based on EC3 requirements and related to plastic deformations in the shakedown process to correct the interpretation of the stability constraints in mathematical programming of the optimization problems. In the optimization problem, truss displacements are evaluated according to different reliability levels of the ultimate and serviceability limit states of EC. The proposed methodology is illustrated by a numerical example. The results are valid for the assumption of small displacements.

**Keywords:** optimal shakedown design, elastic-plastic truss, standards, mathematical programming.

# 1 Introduction

To design more economical structures subjected to variable as well as repeated loading, the shakedown theory may be applied [1,2,3]. This theory allows for the employment of the plastic properties of materials (particularly of steel) for reducing the design structure's volume (mass). Though the process of shakedown is explored notionally in depth [4,5], it is still in the focus of researchers' and designers' interests [6,7,8]. Practical structural design is always associated with national and international standards [9]. The Eurocode requirements [10] allow for designing the structures with plastic deformations, though the optimization in the shakedown state has not been standardized. Therefore, in order to create a practically applicable mathematical model for the problem of truss volume minimization with strength, stiffness and stability constraints, it is necessary to correctly define the physical process of the shakedown [11,12] and to assure that the structure should satisfy the requirements of the standards. The main problems associated with this task are considered in this paper. First, the problem is associated with the application of the stress-strain dependence of the perfectly elastic-plastic truss to bars under compression, which may potentially loose stability. The stability of bars is widely explored by many authors [13,14] and strictly regulated by the design standards. However, some problems of plastic state interpretation occur, when one starts implementing the algorithm of the stability check in solving a mathematical programming problem. It should be noted that the influence of bars under compression on the development of plastic deformations of a truss in the shakedown process cannot be interpreted in the same way as the influence of bars under tension (by a formal explanation of a yield condition satisfied as a strict equality). Second, the considered problem is associated with the displacements' constraints in the optimization problem of a perfectly elastic-plastic truss. Two different reliability levels for verification of the ultimate and serviceability limit states are used in the Eurocode and, in practical design, these limit states are usually evaluated separately. However, in searching for the optimal project of the structure, it is necessary to take into account both limit states' requirements in solving one problem, i.e. to combine two different reliability levels in the same mathematical model. Therefore, a method of binary displacement calculation is proposed in this paper. The improved mathematical model of truss optimization, with the included strength, stiffness and stability constraints, is created. The new mathematical programming problem is non-convex due to the combinatorial complementary slackness conditions. The results of the numerical example of cantilever truss optimization are valid, when small displacement is assumed.

# 2 General mathematical model of optimum system design

The numerical methods of structural mechanics are based on a discrete structural model. For this model, both general and particular mathematical models of problem solution (in our case, truss models) are developed. Dual relations between static (equilibrium) and kinematic (geometric) equations are taken into account, when choosing the generalized static and kinematic variables, which characterize the stress-strain state of the structure. The stress state of a discrete structure is expressed by the force vector $\boldsymbol{S} = [\boldsymbol{S}_1\ \boldsymbol{S}_2\ \ldots\ \boldsymbol{S}_\zeta]^T$, $\zeta = s \times v$, where $s$ is the number of finite elements ($k = 1, 2, \ldots, s,\ k \in K$), constituting the discrete model, $v$ is the number of nodes (design sections) of each element ($i = 1, 2, \ldots, v,\ i \in I$). The variable repeated forces $\boldsymbol{F}(t)$, acting upon the elastic-plastic structure, are characterized by time-independent upper and lower bounds $\boldsymbol{F}_{\sup}$, $\boldsymbol{F}_{\inf}$. A detailed analysis of a loading history is omitted, when the loading is described by all possible combinations $\boldsymbol{F}_j$, $\boldsymbol{F}_{\inf} \le \boldsymbol{F}_j \le \boldsymbol{F}_{\sup}$, $j = 1, 2, \ldots, p,\ j \in J$ ($p = 2^m$, where m is the number of the acting forces). The forces $\boldsymbol{S}_{ej}$ and the displacements $\boldsymbol{u}_{ej}$ of the elastic structure are determined, using the influence matrices of forces and displacements $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$: $\boldsymbol{S}_{ej} = \boldsymbol{\alpha}\boldsymbol{F}_j$, $\boldsymbol{u}_{ej} = \boldsymbol{\beta}\boldsymbol{F}_j$, $j \in J$. The limit force $S_{0k}$ ($k \in K$) is assumed to be constant over the whole finite element $k$. Then, piecewise linearized yield conditions are $\boldsymbol{\Phi}(\boldsymbol{S}_r + \boldsymbol{S}_{ej} + \boldsymbol{S}_c) \leq \boldsymbol{S}_0$, $j \in J$, while $\boldsymbol{S}_r$ denotes the unknown statically admissible residual forces. The forces $\boldsymbol{S}_c$ are resulting from constant (invariable) loading: $\boldsymbol{S}_c = \boldsymbol{\alpha F}_c$. The optimization problem of the structure is stated as follows: *for the given load variation bounds* $\boldsymbol{F}_{\sup}$, $\boldsymbol{F}_{\inf}$, *the vector of the limit forces* $\boldsymbol{S}_0$, *satisfying the optimality criterion* $\min \boldsymbol{L}^T \boldsymbol{S}_0$ *and the constraints of strength, stiffness and stability, should be found*. A general mathematical model of this problem for the bar systems reads as follows:

find

$$\min \ \boldsymbol{L}^T \boldsymbol{S}_0, \tag{1}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{\Gamma S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_c\right) \geq \boldsymbol{0}, \tag{2}$$

$$\boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0, \ \boldsymbol{\lambda}_j \geq \boldsymbol{0}, \ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j, \tag{3}$$

$$\boldsymbol{S}_{0,\inf} \leq \boldsymbol{S}_0 \leq \boldsymbol{S}_{0,\sup}, \tag{4}$$

$$\boldsymbol{u}_{\inf} \leq \left(\boldsymbol{H\lambda} + \boldsymbol{u}_{ej} + \boldsymbol{u}_c\right) \leq \boldsymbol{u}_{\sup}, j = 1, 2, ..., p, j \in J. \tag{5}$$

The generalized force $S$ includes the bending moment $M$ and the axial force $N$, if the model (1)–(5) is applied to the frames or the Virendeel type of a truss. The objective function (1) expresses the optimal distribution of the limit forces $\boldsymbol{S}_0 = [S_{01}\ S_{02}\ ...\ S_{0\zeta}]^{\mathrm{T}}$, while $\boldsymbol{L}$ is the vector of weight coefficients. The objective function can implicitly express the minimum cost or the volume of the structure (actually, it is not a volume minimization problem). The constraints (2) determine the vector of the statically admissible residual forces $\boldsymbol{S}_r = \boldsymbol{G\lambda}$, ensuring the shakedown of the elastic-plastic system under the given variable repeated load ($\boldsymbol{G}$ is the influence matrix of residual forces). The conditions (2), supplemented with the complementary slackness conditions of mathematical programming (3), ensure that the principle of minimum deformation energy of the unloaded system will be satisfied. Then, the components of the vector $\boldsymbol{\lambda}$ obtain the physical meaning of plastic multipliers in the optimization model of an elastic-plastic structure. The configuration matrix $\boldsymbol{\Gamma}$ defines the limit force attribution to the elements of the discrete model. The shape of the linearized yield conditions' matrix $\boldsymbol{\Phi}$ depends on the chosen yield conditions' model [9, 15]. The displacements in the stiffness conditions (5) are as follows: the residual $\boldsymbol{u}_r = \boldsymbol{H\lambda}$, the elastic $\boldsymbol{u}_{ej}$ and $\boldsymbol{u}_c$, resulting from the invariable loading ($\boldsymbol{F}_c$). The limits of the displacements of the structure $\boldsymbol{u}_{\inf}$ and $\boldsymbol{u}_{\sup}$ are determined according to the Eurocode requirements. The stability check of the elements in this model is performed by changing the admissible bounds of the limit forces $\boldsymbol{S}_{0,\inf}$ and $\boldsymbol{S}_{0,\sup}$ in the conditions (4). The vector of the limit forces $\boldsymbol{S}_0$ and the vector of plastic multipliers $\boldsymbol{\lambda}$ are the unknowns in the problem (1)–(5). This optimization problem is non-convex due to the combinatorial complementary slackness conditions.

# 3 Truss volume minimization

The uniaxial state of the stress, generated by the axial forces $\boldsymbol{N}$, is often used in truss design. In this case, the generalized force $S$ becomes the axial force $N$ in the model (1)–(5). The objective function of the truss optimization model expresses the volume $V = \boldsymbol{L}^T\boldsymbol{A}$, where $\boldsymbol{L}$ is the vector of the element length, $\boldsymbol{A}$ is the vector of the element's cross-sectional areas. The yield conditions of the elements (2) can be written by implementing the vectors of the maximal and minimal values of the elastic axial forces $\boldsymbol{N}_{e,\max}$, $\boldsymbol{N}_{e,\min}$, such that $\boldsymbol{N}_{e,\min} \le \boldsymbol{N}_{\mathrm{e},j} = \boldsymbol{\alpha F}_j \le \boldsymbol{N}_{e,\max}$, $j = 1, 2, ..., p,\ j \in J$. The mathematical model of the truss volume minimization problem reads as follows:

find

$$\min\ \boldsymbol{L}^T\boldsymbol{A}, \tag{6}$$

subject to

$$\boldsymbol{\varphi}_{\max} = \boldsymbol{N}_0 - \boldsymbol{N}_r - \boldsymbol{N}_{e,\max} - \boldsymbol{N}_c \ge \boldsymbol{0}, \tag{7}$$

$$\boldsymbol{\varphi}_{\min} = \boldsymbol{N}_{0,\mathrm{cr}} + \boldsymbol{N}_r + \boldsymbol{N}_{e,\min} + \boldsymbol{N}_c \ge \boldsymbol{0}, \tag{8}$$

$$\boldsymbol{\lambda}_{\max}^T\boldsymbol{\varphi}_{\max} = 0,\ \boldsymbol{\lambda}_{\mathrm{cr}}^T\boldsymbol{\varphi}_{\min} = 0, \boldsymbol{\lambda} = [\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{\mathrm{cr}}] \ge \boldsymbol{0}, \tag{9}$$

$$\boldsymbol{A} \ge \boldsymbol{A}_{\min}, \tag{10}$$

$$\boldsymbol{u}_{\inf} \le (\boldsymbol{H\lambda} + \boldsymbol{u}_{ej} + \boldsymbol{u}_c) \le \boldsymbol{u}_{\sup},\ j = 1, 2, ..., p,\ j \in J, \tag{11}$$

where $\boldsymbol{\varphi}_{\max}$ and $\boldsymbol{\varphi}_{\min}$ are the vectors of the yield condition values of the elements under tension and compression, respectively. These vectors are calculated according to the vectors of the limit axial forces $\boldsymbol{N}_0$ and $\boldsymbol{N}_{0,cr}$, the vectors of the residual axial forces $\boldsymbol{N}_r$, the vectors of the maximal and minimal values of the elastic axial forces determined by the variable repeated loads $\boldsymbol{N}_{e,\max}$, $\boldsymbol{N}_{e,\min}$, and, in the general case, the vectors of the elastic axial forces determined by the invariable load $\boldsymbol{N}_c$. The limit axial force of the $k$-th element under tension is calculated basically as the product of the cross-sectional area and yield stress: $N_{0,k} = A_k \cdot f_{y,k}$, whereas the limit axial force of the element under tension must be reduced because of a possible loss of stability. In order to find the solutions to more practical problems, the requirements of particular standards are often implemented in mathematical models. The Eurocode methodology of reducing the limit axial force of an element under compression by a coefficient $\chi$ will be used in this paper. It means that for the $k$-th discrete element $N_{0,cr,k} = N_{0,k} \cdot \chi_k$, while the reduction coefficient $\chi$ is the function of the element's geometrical and physical characteristics. The residual forces in the conditions (7)–(8) are determined in the shakedown process according to the plastic multipliers $\boldsymbol{\lambda}$: $\boldsymbol{N}_\mathrm{r} = \boldsymbol{G\lambda}$. The influence matrix of the residual forces $\boldsymbol{G}$ depends on the equilibrium and stiffness matrices of the structure. The described shakedown design method, with the stability evaluation and the assumption that $\boldsymbol{N}_c = \boldsymbol{0}$ and $\boldsymbol{u}_c = \boldsymbol{0}$, was used in the earlier work [14]. The optimal solution of the problem (6)–(11) includes the vector of the element's cross-sectional areas $\boldsymbol{A}^*$ and the vector of plastic multipliers $\boldsymbol{\lambda}^*$ (here and further, an asterisk marks the optimal solution). Taking into account that the problem conditions depend on the unknowns, the solution algorithm is iterative. The abovementioned research has shown that it does not cause any difficulties for relatively small problems and the convergence is easily achieved.

## 3.1 Plastic deformations under stability conditions

When mathematical programming is used for optimal shakedown truss design, the complementary slackness conditions of mathematical programming (3), (9) are written down alongside strength conditions. The multipliers $\boldsymbol{\lambda} = [\boldsymbol{\lambda}_{max}, \boldsymbol{\lambda}_{cr}]$ obtain the physical meaning of plastic multipliers for the elements under tension and compression, respectively. In designing the elastic-plastic bar structures, the stress-strain state is usually simplified, using the so-called Prandtl diagram. It is further used to explain the emergence of plastic deformations in the shakedown process (see Fig. 1).

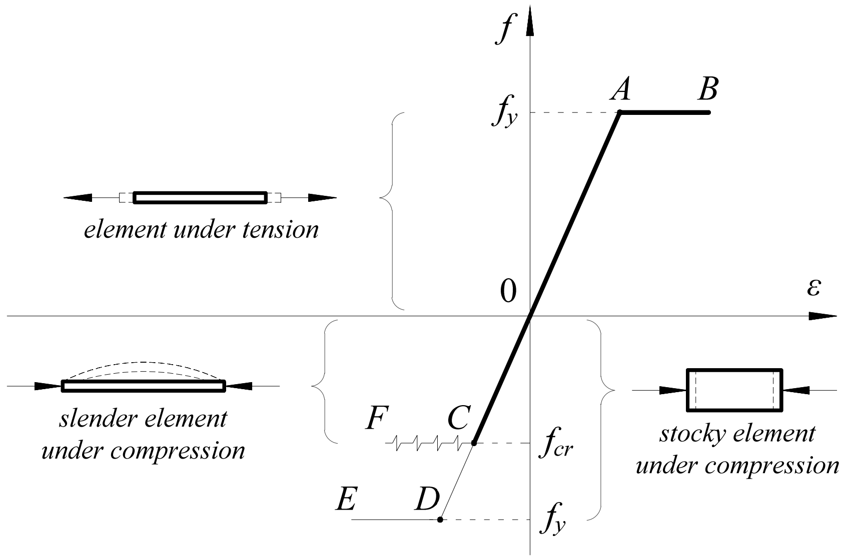


Figure 1. Stress-strain graph of perfectly elastic-plastic material

When a positive side of the graph (positive stress $f$ and strain $\varepsilon$) referring to the elements under tension is considered, it is evident that plastic deformations occur only when the elastic state (the section 0-A) is over, when the stress reaches the yield stress value, i.e. $f = f_y$ (the section A-B). A more complicated case is found, when the elements under compression are examined. In the simplest case, when the element's buckling is not considered, a negative side of the graph is symmetric about the positive one, i.e. the element is deformed according to the curve 0-D-E. The same case is found, when the critical stress reaches the yield stress $f_{cr} = f_y$. According to the Eurocode, the above-mentioned case refers to the elements, having very small non-dimensional slenderness: $\bar{\lambda} \le 0{,}2$ (it should not be confused with plastic multipliers $\boldsymbol{\lambda}$). When stability verification is implemented in the mathematical programming problem, it is found that, in the general case, the deformations emerge according to the curve 0-C-F. However, contrary to the case of tension, the plastic deformations of the elements under compression (when the limit state is reached, i.e. after the loss of stability) are not defined in the EC and cannot be evaluated. Therefore, a true deformation curve of the element under compression is only elastic – 0-C, if C ≠ D, or elastic-plastic – 0-D-E, when D = C and E = F. Thus, the solution algorithm of the mathematical programming problem comes into conflict with the Eurocode requirements. Therefore, the above-mentioned complementary slackness conditions (9) are inadequate for ensuring the shakedown of a truss. This inaccuracy is eliminated by introducing a new condition in the mathematical model, which ensures that plastic multipliers (i.e. plastic deformations) can emerge only due to the limit stress of the elements under tension or in very stocky elements (small non-dimensional slenderness) under compression:

$$\lambda_{\mathrm{cr},k}\left(N_{0,k}-N_{0,\mathrm{cr},k}\right)=0\ ,\ k=1,\ 2,\ \ldots,s\ ,\ k\in K\ . \tag{12}$$

This condition ensures that slender elements under compression (when $\boldsymbol{N}_{0,cr} < \boldsymbol{N}_0$, $\chi < 1$) cannon cause the occurrence of nonzero plastic multipliers. The correct determination of the plastic multipliers $\boldsymbol{\lambda}$ is an essential task because they are used in the same problem for calculating the residual forces and displacements.

## 3.2 Displacement constraints according to the Eurocode

In the Eurocode standards, all design calculations are divided into two groups and are aimed at verifying the ultimate and serviceability limit states. Two different reliability levels are used for these limit states. In using the partial factor method, these levels are achieved by applying the respective representative values of the action. When the strength (7) and stability (8) conditions of the mathematical model are there for the ultimate limit state verification, the serviceability limit state for the structure must be secured as well. A structure can be reliable only if none of the limit states is exceeded. Therefore, stiffness conditions (11) (displacement constraints of the truss nodes) must be introduced into the model [16]. Assuming that $\boldsymbol{u}_c = \boldsymbol{0}$, they can be expressed as follows:

$$\boldsymbol{u}_{\mathrm{inf}}\leq\left(\boldsymbol{u}_r+\bar{\boldsymbol{u}}_{ej}\right)\leq\boldsymbol{u}_{\mathrm{sup}}\ ,\ j=1,\ 2,\ \ldots,p\ ,\ j\in J\ , \tag{13}$$

where $\boldsymbol{u}_{\mathrm{sup}}$ and $\boldsymbol{u}_{\mathrm{inf}}$ are the known vectors of the upper and lower admissible bounds of displacement variation. The displacement of a perfectly elastic-plastic truss consists of two components: the residual $\boldsymbol{u}_r = \boldsymbol{H\lambda}$ and pseudo-elastic $\bar{\boldsymbol{u}}_{ej}$. The residual component is obtained from the shakedown process (by using the plastic multipliers $\boldsymbol{\lambda}$ and the influence matrix $\boldsymbol{H}$), therefore, it is determined by the ultimate state with a high reliability level. The vector $\boldsymbol{u}_r$ is calculated in the optimization process of a structure subjected to variable repeated loading of design values. Therefore, for all possible combinations of loading $j$, $\boldsymbol{F}_{jd} = \gamma_E \cdot \boldsymbol{F}_{jk}$ (where index $d$ means the *design value*, $k$ is the *characteristic value* and $\gamma_E$ is a partial factor for the action). The pseudo-elastic component is calculated, using the Hooke's law, and determined by the serviceability limit state with a lower reliability level. It is calculated, using all possible combinations of the characteristic loading values:

$$\bar{\boldsymbol{u}}_{ej} = \boldsymbol{\beta F}_{jk}\,,\; j = 1, 2, ..., p,\; j \in J\,. \tag{14}$$

This approach, based on the dual reliability level, allows for designing a more economical structure, compared to the earlier presented models because lower reliability is used for the displacement constraints than the strength and stability conditions.

## 3.3 The improved model of truss volume optimization

The improved mathematical model of the problem of volume minimization of a perfectly elastic-plastic truss with new displacement constraints and plastic deformation conditions can be expressed as follows:

find

$$\min \boldsymbol{L}^{\boldsymbol{T}} \boldsymbol{A}\,, \tag{15}$$

subject to

$$\boldsymbol{\varphi}_{\max} = \boldsymbol{N}_0 - \boldsymbol{N}_r - \boldsymbol{N}_{e,\max} \geq \boldsymbol{0}\,, \tag{16}$$

$$\boldsymbol{\varphi}_{\min} = \boldsymbol{N}_{0,\mathrm{cr}} + \boldsymbol{N}_r + \boldsymbol{N}_{e,\min} \geq \boldsymbol{0}\,, \tag{17}$$

$$\boldsymbol{\lambda}_{\max}^T \boldsymbol{\varphi}_{\max} = 0\,,\; \boldsymbol{\lambda}_{\mathrm{cr}}^T \boldsymbol{\varphi}_{\min} = 0\,, \boldsymbol{\lambda} = \left[\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{\mathrm{cr}}\right] \geq \boldsymbol{0}\,, \tag{18}$$

$$\lambda_{\mathrm{cr},k}\left(N_{0,k} - N_{0,\mathrm{cr},k}\right) = 0\,,\; k = 1, 2, ..., s\,,\; k \in K\,, \tag{19}$$

$$\boldsymbol{A} \geq \boldsymbol{A}_{\min}\,, \tag{20}$$

$$\boldsymbol{u}_{\inf} \leq \left(\boldsymbol{u}_r + \bar{\boldsymbol{u}}_{ej}\right) \leq \boldsymbol{u}_{\sup}\,,\; j = 1,\; 2,\; ..., p\,,\; j \in J\,. \tag{21}$$

The conditions of the model consist of the yield conditions (16)–(17), the complementary slackness conditions of mathematical programming (18), the complementary conditions for plastic multipliers (19), the construction regulation constraints (20) and the displacement constraints (21). $\boldsymbol{A}_{\min}$ is the vector of the minimum values of cross-sectional areas of all elements (usually, it is determined by joint construction or other design requirements). The unknowns of the problem (15)–(21) are the vector of the element's cross-sectional areas $\boldsymbol{A}$ and the vector of plastic multipliers $\boldsymbol{\lambda}$. The influential matrices of the elastic forces, elastic displacements, residual forces and residual displacements $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$, used for calculations, are dependent on the variable cross-sectional areas $\boldsymbol{A}$. Therefore, the solution algorithm is performed in an iterative manner (Fig. 2). This model suits for either – discrete or continuous – optimization. The discrete optimization is more practical, but, the continuous optimization can also be used. For example, using section properties, obtained from the continuous optimization, it is possible to choose nearest fitting discrete cross-section from assortment.

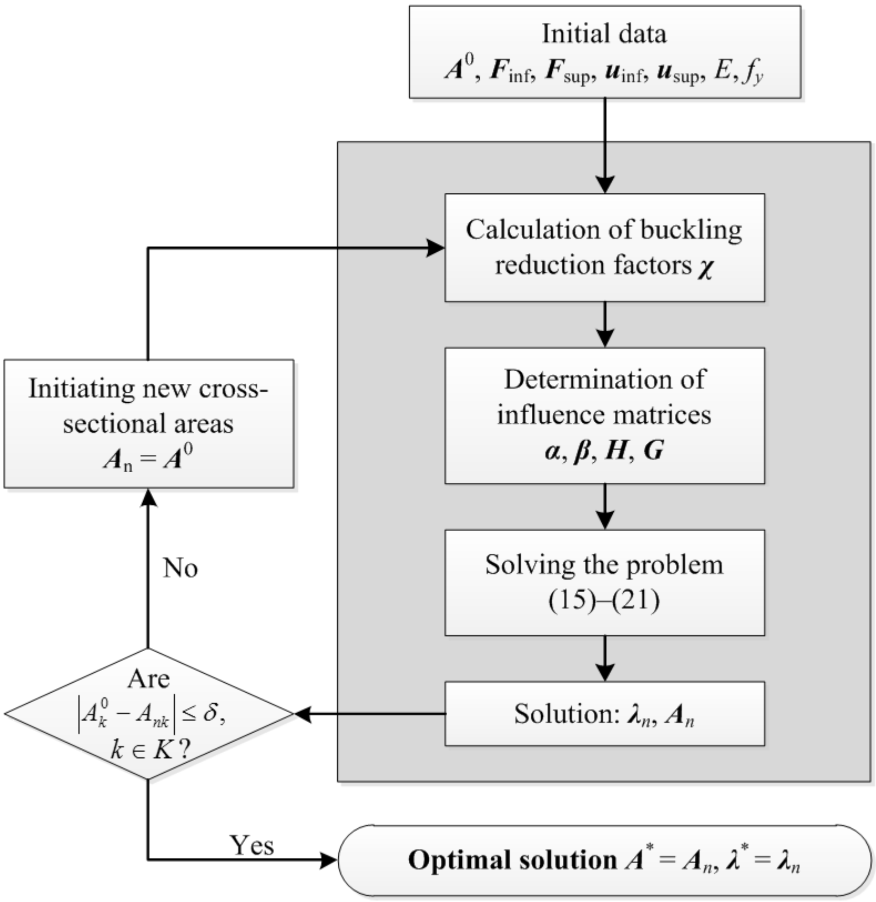


Figure 2. Flowchart of the proposed solution algorithm

# 4 Numerical example

The minimum volume problem of the nine-bar truss, shown in Fig. 3, is considered. This kind of truss geometry was earlier analysed by several authors [17,18]. The truss is subjected to the action of two independent loads, the vertical forces $F_1$ and $F_2$. The characteristic and design loading domains (used for evaluating the particular limit states) are also shown in Fig. 3 (values in kN; the partial factor $\gamma_E = 1.5$). The main task is to solve the problem (15)–(21), i.e. to determine the cross-sectional areas $\boldsymbol{A}$ of the elements and the volume of the whole truss $V$. The bars of the truss are grouped into three groups (only three different sections are designed): $A_{g1} = A_1 = A_2 = A_3 = A_4 = A_9$; $A_{g2} = A_5 = A_6$; $A_{g3} = A_7 = A_8$. The elasticity modulus of the material is $E$ = 21000 kN/cm$^2$, while the yield stress $f_y$ = 20 kN/cm$^2$. The prescribed minimum values of the cross-sectional areas of the truss bars are as follows: $A_{g1,\min}$ = 8 cm$^2$, $A_{g2,\min}$ = 5 cm$^2$, $A_{g3,\min}$ = 5 cm$^2$. For checking the stability, it is assumed that the sections of the bars are circular tubes with the thickness of $t_{g1}$ = 10 mm, $t_{g2} = t_{g3}$ = 5 mm. These values can be changed according to the optimization results. The diameter of the tubes is calculated according to the cross-sectional areas – a

continuous optimization is applied. In order to evaluate the changes made in the new mathematical model, several different cases of truss optimization are considered:
Case C1 – truss volume optimization using the model (6)–(11);
Case C2 – truss volume optimization using the improved model with new complementary conditions for plastic multipliers (15)–(21).
The method of binary displacement calculation is used in both cases. The following total displacement constraints are imposed: $u_{v1} \leq 0.015$ m, $u_{v2} \leq 0.01$ m (see Fig. 3).

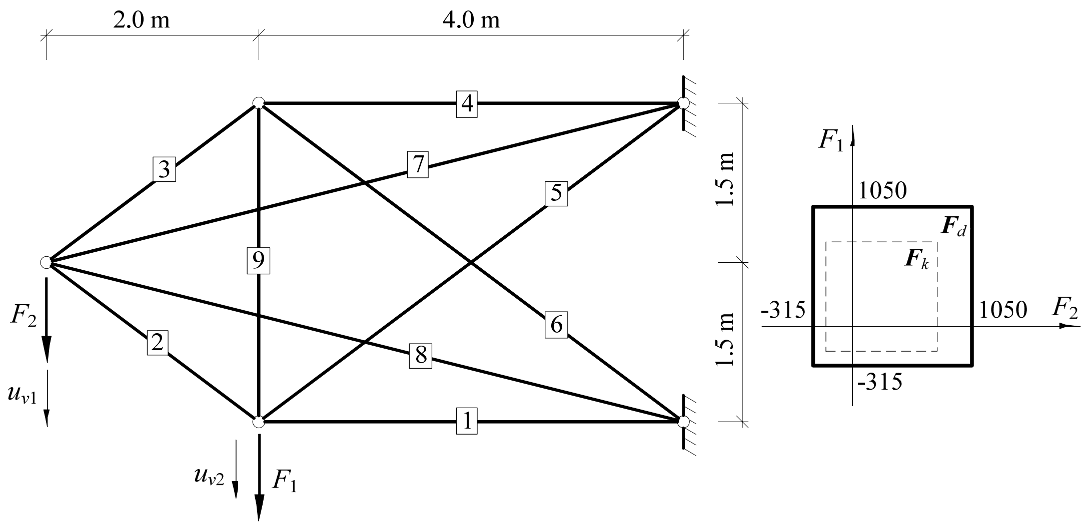


Figure 3. Truss geometry and loading

The results of the numerical optimization calculations are shown in Table 1. Truss plastic deformations in the particular optimization cases are illustrated in Fig. 4. Optimal truss volume values were found: C1 – $V^*$= 0.2599 m$^3$, C2 – $V^*$ = 0.2651 m$^3$.

| Bar No | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Plastic multipliers of tension $\boldsymbol{\lambda}_{max} \cdot 10^3$ | | | | | | | | | |
| C1 | 0.000 | 0.000 | 0.000 | 0.450 | 0.084 | 0.000 | 0.000 | 0.000 | 0.000 |
| C2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.438 | 0.000 | 0.000 | 0.000 | 0.000 |
| Plastic multipliers of compression $\boldsymbol{\lambda}_{cr} \cdot 10^3$ | | | | | | | | | |
| C1 | 1.096 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| C2 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Residual axial forces $\boldsymbol{N}_r$, kN | | | | | | | | | |
| C1 | 105.50 | 55.197 | -1.397 | 14.954 | -76.683 | -20.090 | 47.820 | -92.185 | 12.892 |
| C2 | 37.988 | -2.281 | -9.656 | 26.188 | -49.766 | -42.391 | 14.044 | -4.201 | 31.228 |
| Optimal cross-sectional areas $\boldsymbol{A}$, cm$^2$ | | | | | | | | | |
| C1 | 76.684 | | | | 69.206 | | 54.974 | | 76.684 |
| C2 | 89.090 | | | | 76.171 | | 37.485 | | 89.090 |

Table 1. Calculated results for the volume minimization problems

The complementary conditions for plastic multipliers (19) in case C2 did not allow to emerge plastic deformations in element under compression as it was in the case C1 (element No 1). Thus different stress state, distribution of residual forces and plastic deformations are found. Bigger value of the truss volume is found in the case C2. The distribution of the cross-sectional areas also differs. The cross-sectional area of the third group of elements (elements No 7 and 8) is smaller, meanwhile cross-sectional areas of the first and second group are bigger in the case C2. Initial constraints of the truss nodes displacements are the same for both cases of optimization, yet results are different: in the first case these conditions were satisfied as strict equalities, while in the second case they were satisfied with some reserve. Therefore, due to reduced possibility of plastic deformations emergence, stiffer structure is designed using the model (15)–(21).

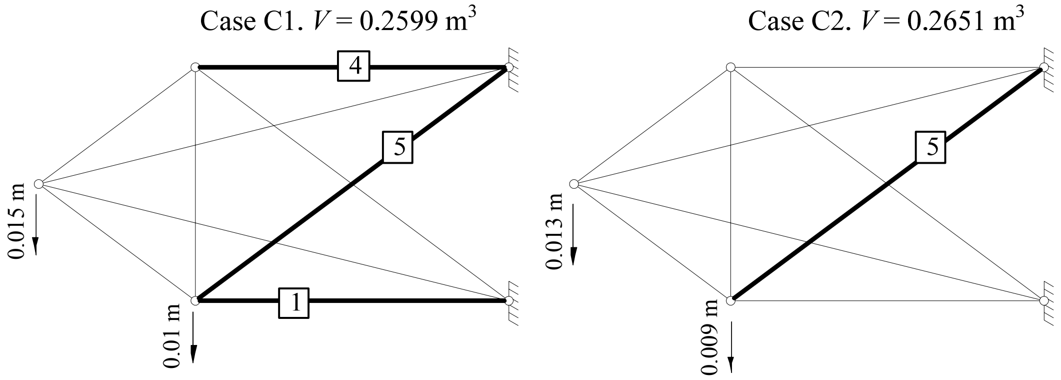


Figure 4. Truss plastic deformations and nodal displacements

# 5 Conclusion

Practical implementation of optimal shakedown design should not be based only on theoretical improvements, but should take into the account the existing design standards. When the stability requirements are implemented in a mathematical programming problem, some difficulties in evaluating plastic multipliers arise. The complementary slackness conditions are not adequate to ensure the shakedown of structures. The improved mathematical model of truss volume minimization problem with strength, stiffness and stability constraints is presented. Different reliability levels are also used for the verification of the ultimate and serviceability limit states of trusses in the suggested mathematical model. Thus, the shakedown theory acquires the potentiality to be used in actual standardized truss design.

# References

[1] J.A. König, "Shakedown of elastic-plastic structures", Elsevier, Amsterdam, 1978.

[2] J. Atkočiūnas, "Optimal shakedown design of elastic-plastic structures", Technika, Vilnius, 2011.

[3] E. Stein, G. Zhang, R. Mahnken, “Shakedown analysis for perfectly plastic and kinematic hardening materials”, in “CISM. Progress in computational analysis of inelastic structures”, Wien, New York, Springer-Verlag, 175–224, 1993.
[4] D. Weichert, G. Maier, “Inelastic analysis of structures under variable repeated loads”, Kluwert Academic Publishers, Dordrecht, 2000.
[5] D. Weichert, A.R.S. Ponter, “Limit states of materials and structures”, Springer, Wien, 2009.
[6] D.K. Vu, M. Staat, I.T. Tran, “Analysis of pressure equipment by application of the primal-dual theory of shakedown”, in “Communications in Numerical Methods in Engineering”, 23(3):213–225, 2007.
[7] F. Giambanco, L. Palizzolo, C. Polizzotto, “Optimal shakedown design of beam structures”, in “Structural optimization”, 8:156–167, 1994.
[8] A. Makrodimopoulos, “Computational formulation of shakedown analysis as a conic quadratic optimization problem”, in “Mechanics Research Communications”, 33:72–83, 2006.
[9] J. Atkočiūnas, A. Venskus, “Optimal shakedown shakedown design of frames under stability conditions according to standards”, in “Computers and Structures” 89, 435–443, 2011.
[10] EN 1993–1–1, Design of steel structures Part 1-1: General rules and rules for buildings, 2005.
[11] J-W. Simon, D. Weichert, “Numerical lower bound shakedown analysis of engineering structures”, “Computer Methods in Applied Mechanics and Engineering”, 200:2828–2839, 2011.
[12] S. Benfratello, L. Cirone, F. Giambanco, “A multicriterion design of steel frames with shakedown constraints”, in “Computers and Structures”, 84:269–282, 2006.
[13] P.A. Kirby, D.A. Nethercot, “Design for structural stability”, Wiley, New York, 1979.
[14] D. Merkevičiūtė, J. Atkočiūnas, “Optimal shakedown design of metal structures under stiffness and stability constrains”, in “Journal of Constructional Steel Research”, 62, 1270–1275, 2006.
[15] N. Zouain, L. Borges, J. Silvera, “An algorithm for shakedown analysis with nonlinear yield conditions”, in “Computer Methods in Applied Mechanics and Engineering”, 191:2463–2481, 2002.
[16] J. Atkočiūnas, A. Borkowski, J.A. König, “Improved bounds for displacements at shakedown”, in “Computer Methods in Applied Mechanics and Engineering”, 28(3):365–376, 1981.
[17] S. Kaliszky, J. Lógó, “Plastic behavior and stability constrains in the shakedown analysis and optimal design of trusses”, in “Structural and Multidisciplinary Optimization”, 24: 118–124, 2002.
[18] F. Tin-Loi, “Optimum shakedown design under residual displacement constrains”, in “Structural and Multidisciplinary Optimization”, 19: 130–139, 2000.