

## **Masonry Vaulted Staircases: Interpretation of Equilibrium Paths**

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### **Abstract**

The subject of the stairs organized according to a typology improperly referred to as "cantilever stairs" requires a remarkable insight, in that its equilibrium is based on a complex interaction of forces. This is displayed in a three-dimensional context and can be recognized only after some highly refined numerical analyses. In the present paper a simplified interpretation of the equilibrium mechanism of such stairs is investigated, derived from a qualitative elaboration of some numerical result available in the literature, suggesting a path for approximate structural control based on a suitable one-dimensional model of the flight of stairs.

**Keywords:** masonry, vault, cantilever stairs, roman stairs, tension stresses, equilibrium paths.

### **1 Introduction**

The relevant importance of studying ancient buildings is especially felt by the researchers of the European countries, where the monumental and historical heritage is largely diffused on the territory. In the specific the study of masonry structures is much different from the study of reinforced concrete ones where some basic rules can be also solved often by using easier softwares.

The masonry structure is a rather complex system where in many cases it is not sufficient to study the behaviour of the single structural element as in the concrete structures. The masonry shows a non conventional behaviour that must be investigated for the single cases in order to formulate the most generic model. A font of masonry complexity is often represented by the use of materials with different mechanical characteristics, coupled to a variegated organization (called "texture") of building materials in the structure.

The texture significantly influences the overall behaviour of the structure representing the major problem in the formulation of a behaviour structural model,

both univocal and complete. On the other hand the variety of textures existing in the masonry buildings pushes towards the elaboration of flexible models which can be adapted to the variety of possible situations.

In masonry buildings it is possible to recognize and isolate the basic structural elements (as e.g. vaults, arches, walls), similarly to the homolog elements of a modern structure (as e.g. floors, beams, pillars). Nevertheless the operation of the basic masonry elements is rather more complex than other civil buildings. Moreover the constituent elements' interaction is often not so evident in their real hierarchical degree and connection. That one represents a problem for setting up an efficient structural model of masonry buildings.

The present paper is a part of a much more extended research about the masonry fabrics and the problematic situation about them, also with reference to the optimal interventions of refurbishment (see e.g. from [1] to [8]).

Without entering into a general treatment of masonry fabrics, that is a complex and, under some aspects, questioned task (see e.g. [9-20]), the topic of masonry stairs is addressed, and in particular the one of the so-called "half-barrel" or "cantilever" or "Roman" stairways (Fig.1). In particular the roman stairs are widely diffused in Southern Italy, especially since the XVI century, yielding vertical connection in a large number of masonry monumental buildings (see e.g. [21,22]).

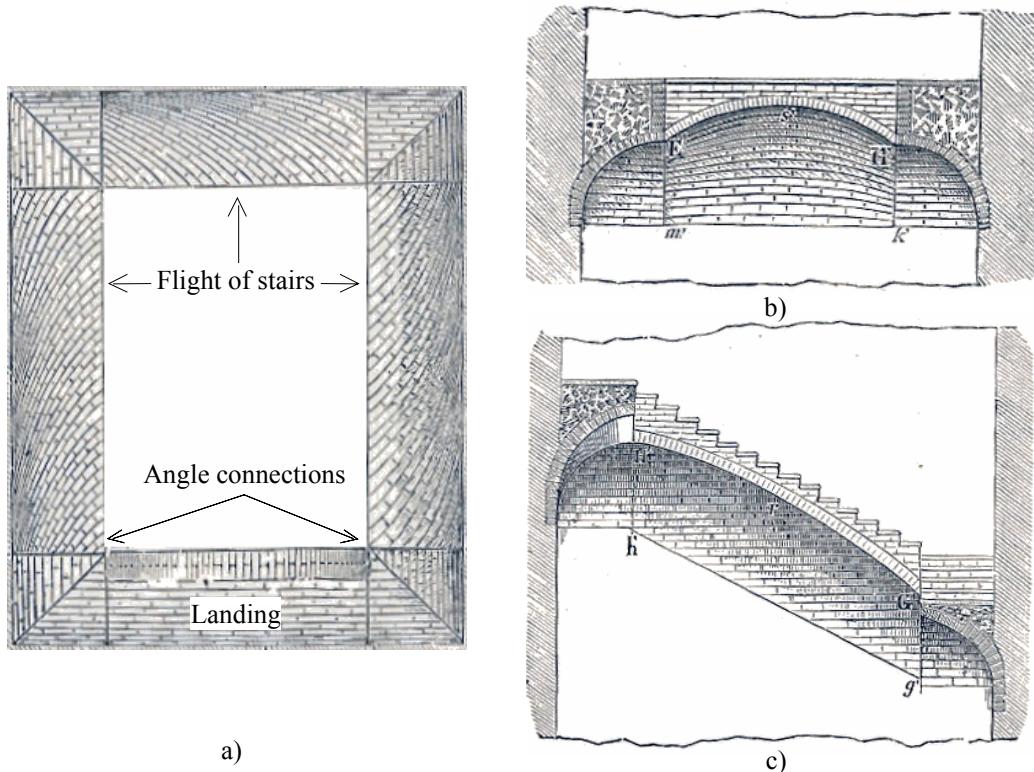


Figure 1. Roman stairways: a) plan of stairs; b) landing's section; b) section of a flight of stairs.

At first approach the structural behaviour of the roman stairway appears similar to that of a masonry vault where the elements must work by contrast between them.

In this circumstance since the masonry is not effective in sustaining tension stresses and bending, and the geometry is similar to an incomplete vault, the equilibrium state cannot be searched for in the two-dimensional pattern (Fig. 2 a, b and c).

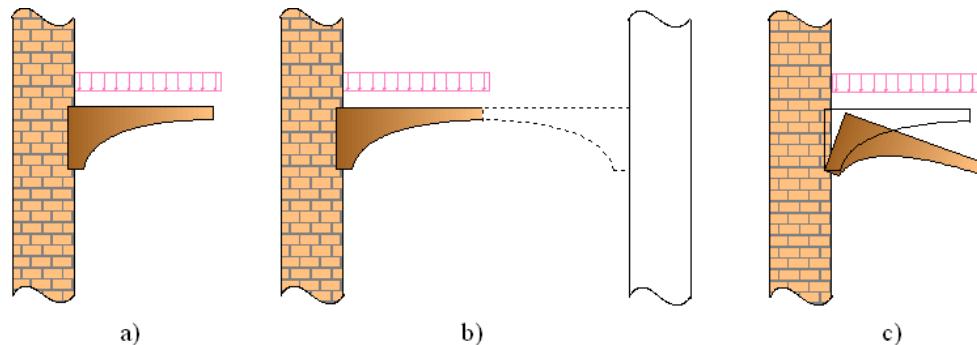


Figure 2. a) Section of a flight of stairs, b) improbable behaviour like an half vault, c) absence of equilibrium conditions in two-dimensional regimen.

The investigation of three-dimensional equilibrium paths must be made accounting for the space articulation of such structural organisms and searching for stress fields in equilibrium with the applied loads and compatible with the resistant skill of masonry.

After identifying the basic internal force distributions through which the stairs can equilibrate their own weight and live loads, and the correlation that was intended by the original builders between statics and masonry tissue, it is possible to design the reinforcement of the vaults, that should be conceived in way to sustain the possible equilibrium paths.

Apart from complex analyses, the professional practice also demands clear and sharp models for structural analysis and calculation. Simplified approaches often help the interpretation of the structural behaviour and suggest procedures for decision and design in order to possibly certify with immediacy the reliability of the structural system and/or assess static emergency, without being forced to resort to sophisticated, demanding, time-wasting -and sometimes uncertain-, software.

Simple and more or less intuitive approaches are anyway mandatory to conceive and design reinforcement works, especially when highly specialized techniques like FRP are to be used.

The paper aims to illustrate a course of action for the analysis and check, based on the classic "beam model", that can be traced after a procedure of understanding of the resistant mechanisms is set up, as a result of a elaboration of the results of complex analyses carried on according to the modern numerical techniques.

First of all a basic rule in the equilibrium of a vaulted stair is recognized by looking at the texture of the masonry apparatus for the punctual effect of the friction between a block and the other also for the joints stagger.

## 2 Masonry vaulted stairs

### 2.1 The equilibrium path

In the static analysis of a vaulted staircase like Figure 1 a simplified model can be considered as the composition of some basic components: the landing, the angle connection, and the flight of stairs (two or three depending on the structure morphology). The structure is globally supported by the outside walls system which represent the stair box.

The landing can be easily modelled by means of the arch model. Analogously the angle connection can be analyzed as a quarter of a trough-vault, which is stable provided that the converging flights and/or landing are stable. So the resulting forces are transmitted to the flight of the stairs, which becomes the central problem in the equilibrium. The drift of the flight with respect to the landing yields a longitudinal compression which is associated to an arch effect in the flight thickness helping its equilibrium. Nevertheless the profile of the flight is usually rectilinear, that doesn't help the arch effect in the longitudinal direction, and the vertical reaction of the flight is concentrated at the connection point of the flight to the outside wall.

After all the flight of stairs can be outlined like a horizontally wide and tilted beam, which is equipped with a lateral continuous support at one edge of the cross section, subject to a vertical load. The vertical load is given by the sum of the permanent load plus the accidental ones. It is applied in an eccentric position regarding the line of support and is statically equivalent to a load distributed on the middle line of the section (called "generatrix" line) at a distance "d" from the constraint of the outside wall (Fig. 3.a).

The applied load activates the reaction  $r$  which acts to support the flight of stairs; the reaction can be assumed equal and opposite to the load  $f$ , but it is applied with an eccentric position with regards to the last one. Therefore a torsional load acts as  $m_t = -f \cdot d$ , which results in the total torsional internal force  $M_t$  and may be distributed on the flight's length as the diagrams in Fig. 3b, 3c or 3d.

The force  $M_t$  is balanced by the two torsional moments  $\mathcal{M}_t$  and  $\mathcal{M}_{t'}$ , which can eventually be assumed equal between them like in Fig. 3.d, where they are applied to the bases of the flights and are balanced from the angle connections.

If the flight of stairs is able to completely absorb the torsion activated, the equilibrium can exist also without resorting to other mechanisms of maintenance, like e.g. the arch effect in the lengthwise direction. Another way to balance the "*cantilever*" could be given by an horizontal shear reaction  $T$  (Fig. 4.a).

Nevertheless this condition would translate in a flexional regime inside the flight of stairs and in the subsequent cracking being the masonry not resisting to tension stresses. In order to adapt itself to one more favourable distribution of tensions the classic fracture in the middle line of the flight of stairs can be originated (Fig. 4.b). After cracking the flight can activate a arch-type effect represented by a stress curve included into the profile of the flight, but which has a non-conventional shape hardly analyzable with elementary methods.

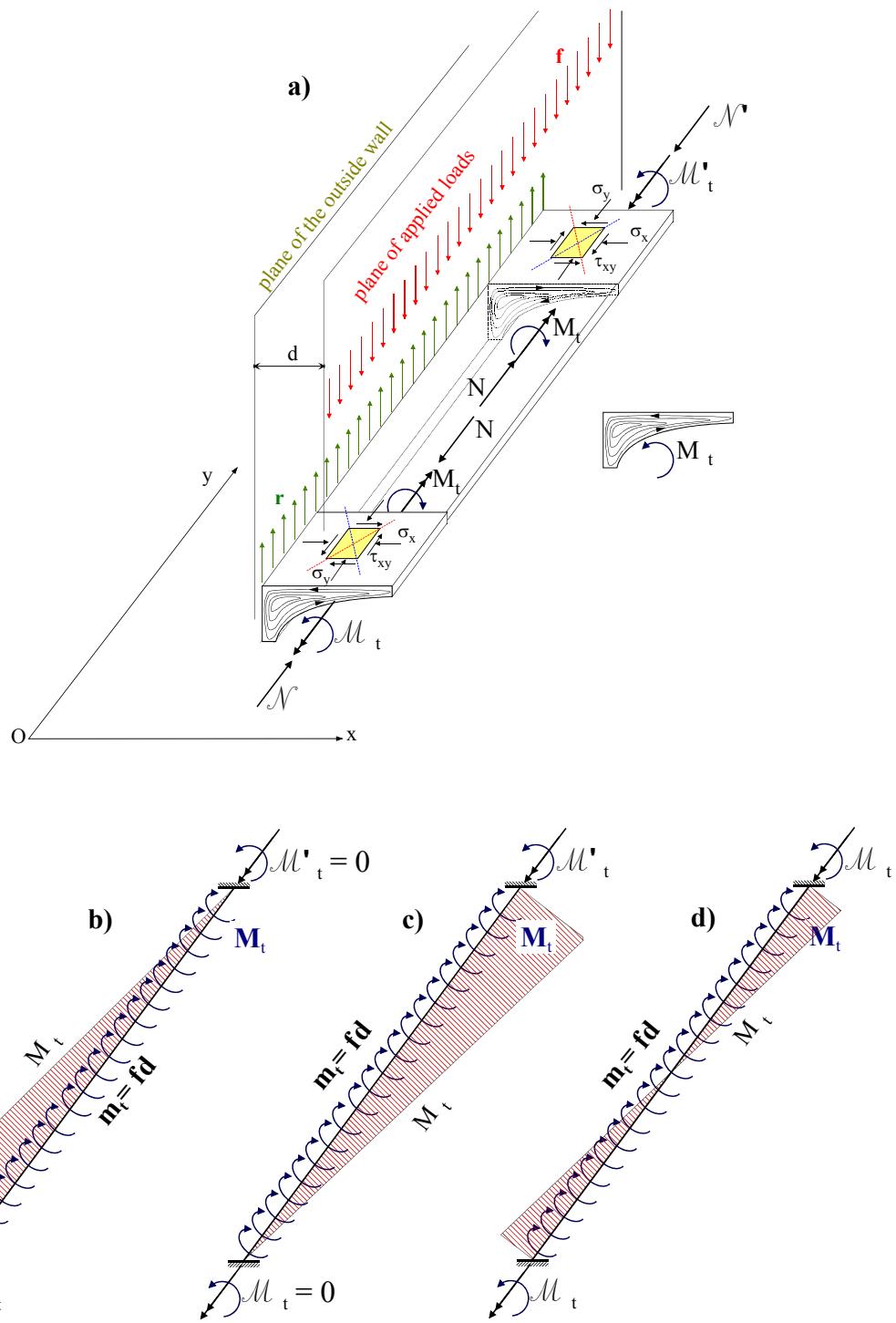


Figure 3. Torsion regime in the flight of stairs: a) outline of active load and reactions; b), c), d) mono-dimensional model.

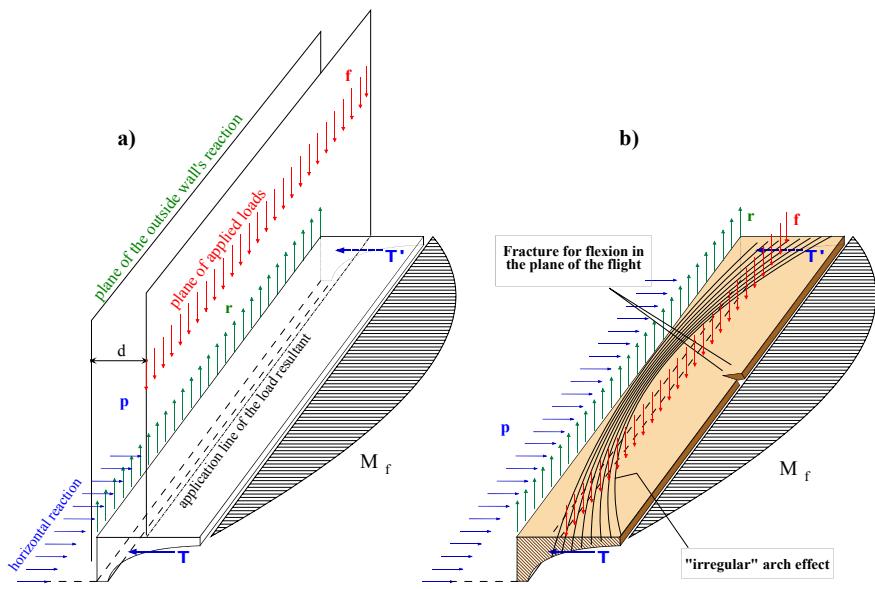


Figure 4. Bending regime in the flight of stairs: a) absorption of the overturning moment through shear and bending in the flight surface; b) fracture from bending and transit to other stress regime.

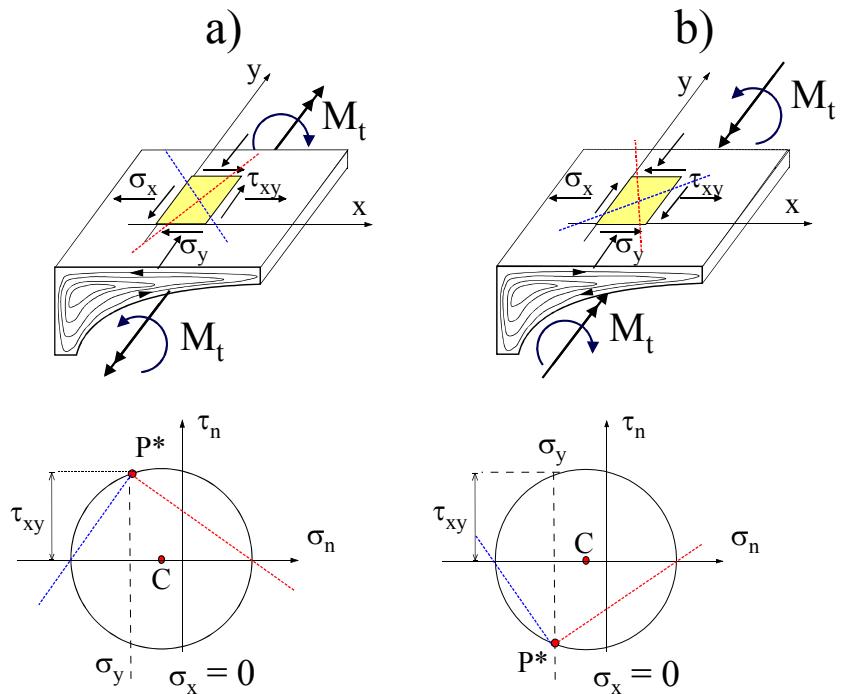


Figure 5. The tensional stresses and the main directions into the lower part a) and the upper part b) of the flight of stairs.

The torsion stress would provoke at the various points of flight a quasi purely shear-stress state (Fig. 5) with the principal directions inclined by an angle of about  $45^\circ$  with respect to the axis of the flight. Nevertheless because of the inclination, the flight is subjected also to some longitudinal reactions  $\mathcal{N}$  and  $\mathcal{N}'$  in correspondence of the angle connections. These reactions are translated in a normal compressive stress “ $N$ ”, that, since it is generally modest, can be assumed easily constant and equal to its medium value, and that however takes a part to advantage of the jam improving the resistance ability of the mortar joint - rows.

So if the normal stress  $\sigma_y$  of the section of the flight is roughly constant, the torsional moment  $M_t$  varies because of the torsional load (Fig. 3.b, c, d) and produces the dependency of the main directions from the position of the considered transversal section (Fig. 6).

The trend of the isostatic lines as shown in Fig. 6 is about according to the texture of the malt's joints in Fig. 1.b.

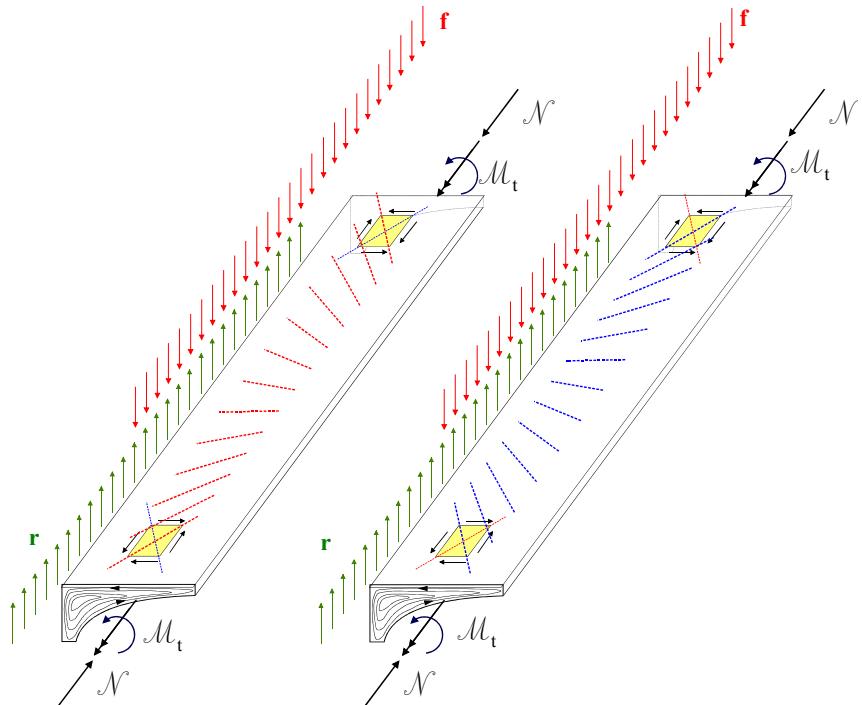


Figure 6. The trend of the tension isostatic lines to varying of the generic point's position on the flight at. a) extrados and b) intrados.

## 2.2 The shear stresses

In the following two different hypotheses of constitution of the flight are considered. In the first case the whole section of the flight is a bearing element (Fig. 7.a); in the second case the resistant section is represented only by the vault of the flight (Fig. 7.b).

Assuming in the first case that the transversal section of the flight is compact and it integrally contributes to contrast the torsional solicitation, the torsional shear stress can be solved through an approximate interpretation of the flow function.

The function is considered with reference to the Fig. 7.a

$$F(x, y) = K \left( x^2 - b^2 \right) \left( y^2 - h^2 \right) \left[ (x - x_c)^2 + (y - y_c)^2 - R^2 \right] \quad (1)$$

where  $b$  is the half of the flight's width (including the part embedded in the outside wall),  $h$  is the half of the constraint's height,  $x_c$  and  $y_c$  are the coordinates of the centre of the intrados' curvature,  $R$  is the radius of curvature and  $K$  is one constant dependent from the entity of the applied torsional moment.

The function  $F(x, y)$  is negative on the total area of the section and is identically zero on the boundary (Fig. 7), therefore it responds to the condition of limit equilibrium. Moreover, if one assumes

$$\tau_{zx} = \frac{\partial F}{\partial x} ; \quad \tau_{zy} = - \frac{\partial F}{\partial y} \quad (2)$$

the indefinite equations of equilibrium remain identically satisfied, and the isosists of the function  $F(x, y)$  describe the flow lines of an equilibrium shear stress field.

Nevertheless the selected function  $F(x, y)$  does not verify the condition  $\Delta_2 F(x, y) = \text{const}$ , since generally the stresses in Eq. (2) do not respect the internal congruence condition. The function  $F(x, y)$  can be so called "pseudo-function of flow".

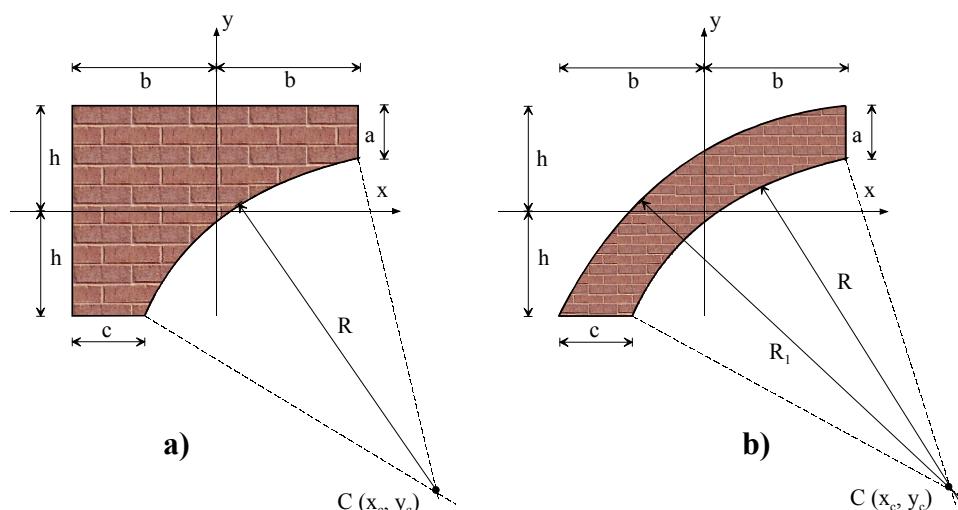


Figure 7. Geometry of the flight in the hypothesis of: a) integrally resistant section, and b) partially resistant section.

By considering a physiological and partial internal arrangement of the masonry, and a solution equilibrated at any point, the Eq. (1) can be credited sub condition of total equilibrium that concurs to estimate the constant K as

$$K = \frac{\pi t}{2V} ; V = \int_A F(x,y) dx dy \quad (3)$$

where V represents the volume of the function F(x,y).

In Fig. 8.a) the flow lines of the shear stress are shown; and in Fig. 8.b) the map of the values of the shear stress is shown for a full resistant section with the following geometry

$$b = h = 0.7m ; a = c = 0.3m ; R = 1.5m ; x_c = 1.097m ; y_c = -1.097m \quad (4)$$

and a torsional moment  $M_t = 1,600 \text{ Kg m}$  which is calculated with reference to a flight with light of 4,50 m. The anomaly in the position of the maximum tangential stress is obvious consequence of the violation of the congruence condition from the pseudo-function of flow in Eq. (1).

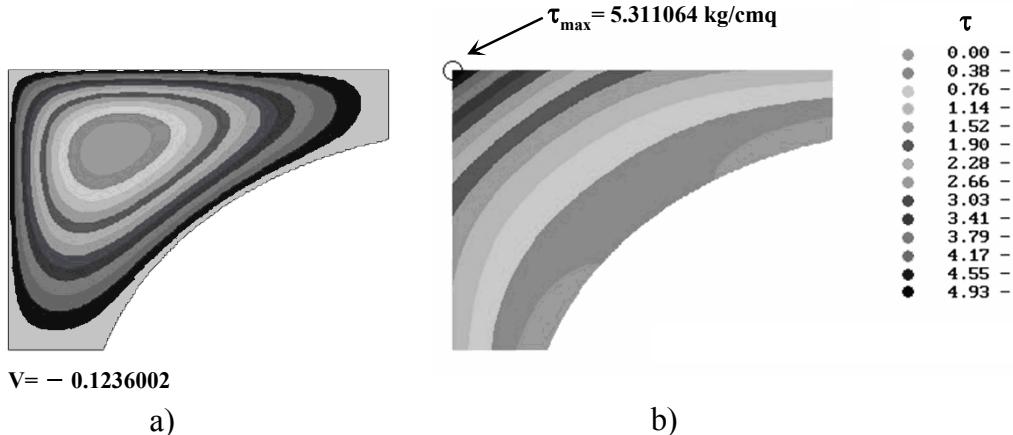


Figure 8. Hypothesis of integrally resistant section: a) flow lines of tension, and b) map values of the tangential stress.

On the contrary if only a partial section of the vault of the flight resists (Fig. 7.b), the pseudo-function of flow assumes the shape

$$F(x,y) = K \left[ x^2 - b^2 \right] \left[ y^2 - h^2 \right] \left[ (x - x_c)^2 + (y - y_c)^2 - R^2 \right] \left[ (x - x_c)^2 + (y - y_c)^2 - R_1^2 \right] \quad (5)$$

In Fig. 9.a) and b) the flow lines of the tangential stress and the map of the values of the tangential stress are respectively shown with reference to a partially resistant section as in Fig. 7.b with the same geometry of the previous case.

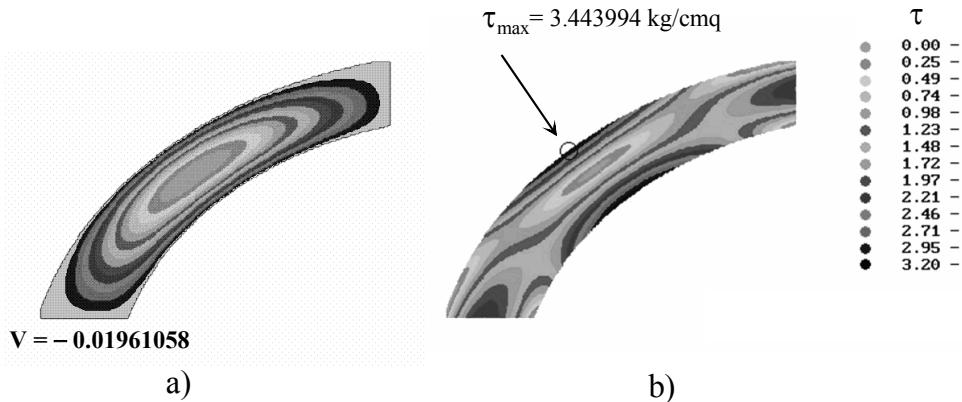


Figure 9. Hypothesis of a partially resistant section: a) flow lines of tension, and b) map values of the tangential stress.

### 3 Conclusions

The “cantilever” or “Roman” stairs represent one frequent typology encountered in historical masonry buildings, in particular in the Neapolitan area. Their static interpretation demands some complex analyses, which in many cases cannot be summarized in a controllable behaviour for certifying its static suitability. In the present paper some considerations are explained for a simplified approximate check of a cantilever stair in order to facilitate the analysis and the identification of particularly complex and delicate cases for which it is indispensable to resort to more sophisticated instruments of calculation.

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