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Coupled Limit Analysis and Topological Optimization for Masonry Wall Reinforcement

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Abstract

The paper focuses on the set up of an approach for designing the best distribution of the reinforcement over NT existing structures by means of combined use of tools from topology optimization (TO) and limit analysis (LA). The presented numerical results refer to the specific objective of suitably identifying the most proper placement and shaping of the reinforcement bars in a masonry wall, which can be modelled using the no-tension (NT) assumption.

Keywords: no-tension structures, reinforcement, topology optimization, limit analysis, optimal placement.

1 Introduction

The refurbishment of masonry buildings needs a deeper knowledge of the masonry behaviour and of the complexity of the fabric where the technician has to work. The approach is the same as a classical form of intervention e.g. with steel tears or about the application of new materials.

Some approximations can be assumed in the study by associating the masonry to an elastic no tension (NT) material (see for details [1-2]) and the reinforcement to an indefinitely elastic sheet, possibly with variable thickness, in such a way to model the structure as a two-dimensional NT continuum. Within this context some application of the topology optimization ([20-23]) may be attempted in order to identify the optimal distribution of the reinforcement in the interior of the region of the structure such to make the structure able to resist given loads and obeying some special constraints (e.g. the quantity of the reinforcing material involved, and/or the maximum stress/strain in the material) and aiming at optimizing some suitably defined performance index and/or design objectives.

2 Coupled Limit Analysis and Topological Optimization approach for the design of the reinforcement in masonry structures

2.1 Set up of the constrained optimization strategy

The approach for identifying the best reinforcement for the plane NT structure may be set up in different ways, for example following the LA kinetic pattern.

Let consider the domain Ω occupied by a NT material and having a contour Γ, that is subdivided in the constrained part Γ_u, where displacements are imposed and forces correspond to the reactions, and the free part Γ_p, where displacements are not constrained and forces are data. The domain Ω is subjected to surface tractions $\mathbf{p}(\mathbf{x})$ and body forces $\mathbf{f}(\mathbf{x})$, with $\mathbf{x}$ the position vector of the current point in Ω.

Let be $\mathbf{u}(\mathbf{x})$ the displacement field, $\boldsymbol{\varepsilon}(\mathbf{x})$ the total strain field, $\boldsymbol{\sigma}(\mathbf{x})$ the total stress in the reinforced body, $\rho(\mathbf{x})$ the density function of the reinforcement which can be 0 in absence of reinforcement and equal to 1 when the reinforcement is applied, such that

$$\begin{aligned} &\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_r(\mathbf{x}) \\ &\boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_{be}(\mathbf{x}) + \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \\ &\boldsymbol{\sigma}(\mathbf{x}) = \boldsymbol{\sigma}_b(\mathbf{x}) + \rho(\mathbf{x})\boldsymbol{\sigma}_r(\mathbf{x}) \end{aligned} \tag{1}$$

where the suffixes "b" and "r" denote respectively the basic structural body and the relevant material and the reinforcing material, and the suffixes “e” and “f” denote the elastic and the fracture regimen, being $\boldsymbol{\sigma}_b \leq \mathbf{0}$ or $\boldsymbol{\varepsilon}_{bf} \geq \mathbf{0}$ because of admissibility.

After the gradate application of the reinforcement, and the application of the forces, the response of the system is ruled by the usual compatibility and equilibrium equations.

A possibility for approaching the problem consists of considering the objective function as the maximum quantity of reinforcement that can be applied to the basic material volume produced by fractures in the basic body still leaving the structure prone to collapse.

So, being $\rho(\mathbf{x})$ the local density of the reinforcement with 0-1 values, the objective function is assumed in the form

$$F(\rho) = \int_{\Omega} \rho(\mathbf{x}) dA = \max \tag{2}$$

By the limit analysis kinetic approach the constraints ruling the problem are:

a) the compatibility conditions of the strain field, which is required to be compatible with the displacements field

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}_b(\mathbf{x}) = \boldsymbol{\varepsilon}_{be}(\mathbf{x}) + \boldsymbol{\varepsilon}_{bf}(\mathbf{x}) = \nabla\mathbf{u}(\mathbf{x}) \tag{3}$$

b) the strain field must be admissible (i.e. $\boldsymbol{\varepsilon}_{bf}(\mathbf{x}) \geq 0$ in Ω) and such that a collapse mechanism can be activated

$$\begin{cases} \int_{\Gamma_p} \mathbf{p}(x)\cdot\mathbf{u}(x)ds + \int_{\Omega} \mathbf{f}(x)\cdot\mathbf{u}(x)dA \geq 1 \\ \nabla\mathbf{u}(x) \geq 0 \qquad \rightarrow \qquad \begin{cases} J_{1b}(\mathbf{x}) \geq 0 \\ J_{2b}(\mathbf{x}) \geq 0 \end{cases} \\ \rho(x)\nabla\mathbf{u}(x) = 0 \qquad \rightarrow \qquad \begin{cases} \rho(x)J_{1b}(\mathbf{x}) = 0 \\ \rho(x)J_{2b}(\mathbf{x}) = 0 \end{cases} \end{cases} \tag{4}$$

with J_{1b} and J_{2b} the first and second invariant of the displacements in the basic material

$$\begin{gathered} J_{1b} = \varepsilon_{bx} + \varepsilon_{by} \quad ; \quad J_{2b} = \varepsilon_{bx}\varepsilon_{by} - \frac{1}{4}\gamma_{bxy}^2 \qquad \text{with} \quad \varepsilon_b = \begin{bmatrix} \varepsilon_{bx} & \frac{1}{2}\gamma_{bxy} \\ \frac{1}{2}\gamma_{bxy} & \varepsilon_{by} \end{bmatrix} \\ J_{1b} = \boldsymbol{\varepsilon}_b \cdot \boldsymbol{\Delta} = \nabla\mathbf{u}\cdot\boldsymbol{\Delta} \quad ; \quad J_{2b} = \frac{1}{2}\left(\mathbf{R}^T\boldsymbol{\varepsilon}_b\mathbf{R}\right)\cdot\boldsymbol{\varepsilon}_b = \frac{1}{2}\boldsymbol{\varepsilon}_t\cdot\boldsymbol{\varepsilon}_b = \frac{1}{2}\boldsymbol{\varepsilon}_t\cdot\nabla\mathbf{u} \\ \boldsymbol{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ; \ \mathbf{R} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ; \ \boldsymbol{\varepsilon}_b = \begin{bmatrix} \varepsilon_{bx} & \frac{1}{2}\gamma_b \\ \frac{1}{2}\gamma_b & \varepsilon_{by} \end{bmatrix} ; \ \boldsymbol{\varepsilon}_t = \mathbf{R}^T\boldsymbol{\varepsilon}_b\mathbf{R} = \begin{bmatrix} \varepsilon_{by} & -\frac{1}{2}\gamma_b \\ -\frac{1}{2}\gamma_b & \varepsilon_{bx} \end{bmatrix} \end{gathered} \tag{5}$$

c) The reinforcement density $\rho(\mathbf{x})$ is invariant with respect to the static approach and is everywhere not smaller than 0 and not larger than 1

$$\rho(\mathbf{x})[\rho(\mathbf{x}) - 1] = 0 \quad \forall\mathbf{x} \in \Omega \tag{6}$$

2.2 Management of the optimal problem

The Lagrange functional of the problem set up in the previous section, with the introduction of suitable multipliers, can be written down as follows

$$\begin{aligned} &\mathcal{L}[\rho(\mathbf{x}), \mathbf{u}(\mathbf{x}); \eta(\mathbf{x}), \chi(\mathbf{x}), \nu(\mathbf{x}), \upsilon(\mathbf{x}), \omega(\mathbf{x})] = \\ &= \int_{\Omega} \rho(\mathbf{x})dA + k\left[\int_{\Gamma_p} \mathbf{p}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})ds + \int_{\Omega} \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dA - 1\right] + \\ &\quad + \int_{\Omega} \eta(\mathbf{x})J_{1b}(\mathbf{x})dA + \int_{\Omega} \chi(\mathbf{x})J_{2b}(\mathbf{x})dA + \\ &\quad + \int_{\Omega} \nu(\mathbf{x})\rho(\mathbf{x})J_{1b}(\mathbf{x})dA + \int_{\Omega} \upsilon(\mathbf{x})\rho(\mathbf{x})J_{2b}(\mathbf{x})dA + \int_{\Omega} \omega(\mathbf{x})\rho(\mathbf{x})[\rho(\mathbf{x}) - 1]dA \end{aligned} \tag{7}$$

where it is intended that $\nu(\mathbf{x})$ and $\upsilon(\mathbf{x})$ and $\omega(\mathbf{x})$ are Lagrange multipliers without any constraint on their sign, whilst k is a non-negative constant, and $\eta(\mathbf{x})$ and $\chi(\mathbf{x})$ are scalar non-negative multipliers, since they are related to the inequality constraints of the problem in Eqs (2) to (5).

Therefore one gets

$$
\begin{gathered}
k\left[\int_{\Gamma_p} \mathbf{p}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})ds + \int_{\Omega} \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dA\right] = 0 \\
\int_{\Gamma_p} \mathbf{p}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})ds + \int_{\Omega} \mathbf{f}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})dA - 1 \geq 0 \\
\left.\begin{array}{c} \eta(\mathbf{x})J_{1b}(\mathbf{x}) = 0 \\ \chi(\mathbf{x})J_{2b}(\mathbf{x}) = 0 \\ k \geq \mathbf{0} \\ \eta(\mathbf{x}) \geq 0 \\ \chi(\mathbf{x}) \geq 0 \end{array}\right\} \qquad \left.\begin{array}{c} J_{1b}(\mathbf{x}) \geq 0 \\ J_{2b}(\mathbf{x}) \geq 0 \\ \rho(\mathbf{x})(\rho(\mathbf{x})-1) = 0 \\ \rho(\mathbf{x})J_{1b}(\mathbf{x}) = 0 \\ \rho(\mathbf{x})J_{2b}(\mathbf{x}) = 0 \end{array}\right\}
\end{gathered}
\tag{8}
$$

Moreover, the variational conditions must be fulfilled in solution.

<u>1) For independent variation of the displacement field $\mathbf{u}(\mathbf{x})$:</u>

$$
\delta_{\mathbf{u}}\mathscr{L}[\rho(\mathbf{x}),\mathbf{u}(\mathbf{x});\eta(\mathbf{x}),\chi(\mathbf{x}),\nu(\mathbf{x}),\upsilon(\mathbf{x}),\omega(\mathbf{x})] = 0 \qquad \forall\delta\mathbf{u}(\mathbf{x}) \tag{9}
$$

After some algebraic developments and by the conditions in Eq, (18), one gets

$$
k\left[\int_{\Gamma_p} \mathbf{p}(\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{x})ds + \int_{\Omega} \mathbf{f}(\mathbf{x})\cdot\delta\mathbf{u}(\mathbf{x})dA\right] = \int_{\Omega} \{\rho(\mathbf{x})S_r + S_b(\mathbf{x})\}\cdot\nabla\delta\mathbf{u}(\mathbf{x})dA \quad \forall\delta\mathbf{u}(\mathbf{x}) \tag{10}
$$

with

$$
\begin{gathered}
\mathbf{S}(\mathbf{x}) = \rho(\mathbf{x})\mathbf{S}_r(\mathbf{x}) + \mathbf{S}_b(\mathbf{x}) \\
\mathbf{S}_r(\mathbf{x}) = -[\nu(\mathbf{x})\boldsymbol{\Delta} + \upsilon(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})] \;; \qquad \mathbf{S}_b(\mathbf{x}) = -[\eta(\mathbf{x})\boldsymbol{\Delta} + \chi(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})]
\end{gathered}
\tag{11}
$$

where $S_b(\mathbf{x})$ represents a NT stress field, because of the non-negative character of $\eta(\mathbf{x})$ and $\chi(\mathbf{x})$ and of the tensors $\boldsymbol{\Delta}$ and $\boldsymbol{\varepsilon}_t(\mathbf{x})$.

By the PVW, Eq. (11) expresses the condition that the external forces are in equilibrium with a stress field that is non-negative semi-definite where no reinforcement is applied, i.e. $\rho(\mathbf{x}) = 0$.

2) For independent variation of the reinforcement distribution:

$$\delta_\rho \mathcal{L}[\rho(\mathbf{x}),\mathbf{u}(\mathbf{x});\eta(\mathbf{x}),\chi(\mathbf{x}),\nu(\mathbf{x}),\upsilon(\mathbf{x}),\omega(\mathbf{x})]=0 \qquad \forall\delta\rho(\mathbf{x})$$

$$\int_\Omega \delta\rho(\mathbf{x})dA+\int_\Omega \nu(\mathbf{x})\delta\rho(\mathbf{x})J_{1b}(\mathbf{x})dA+\int_\Omega \upsilon(\mathbf{x})\delta\rho(\mathbf{x})J_{2b}(\mathbf{x})dA+\int_\Omega \omega(\mathbf{x})[2\rho(\mathbf{x})-1]\delta\rho(\mathbf{x})dA=0 \quad (12)$$

and after some algebra

$$\int_\Omega \left\{1+\omega(\mathbf{x})[2\rho(\mathbf{x})-1]+\left[\nu(\mathbf{x})\mathbf{\Delta}+\frac{1}{2}\upsilon(\mathbf{x})\boldsymbol{\varepsilon}_t(\mathbf{x})\right]\cdot\nabla\mathbf{u}(\mathbf{x})\right\}\delta\rho(\mathbf{x})dA=0 \qquad (13)$$

The third addend in the coefficient of $\delta\rho(\mathbf{x})$ in the integrals is always positive.

Therefore $\omega(\mathbf{x})$ shall be positive where $\rho(\mathbf{x}) = 0$ and we tend to add the reinforcement, while it shall be negative where the reinforcement exists and we want to drop it out.

3 Numerical investigation for a NT wall

Results from numerical investigation developed with reference to the reinforcement of the masonry wall with a central opening shown in Figure 1(a) are reported in the following. The panel is complemented by a horizontal girder on the top and by a steel beam on the hole. The load pattern is given by the self-weight of the panel and by a horizontal force at the left end of the girder, that cannot be resisted by the simple unreinforced system.

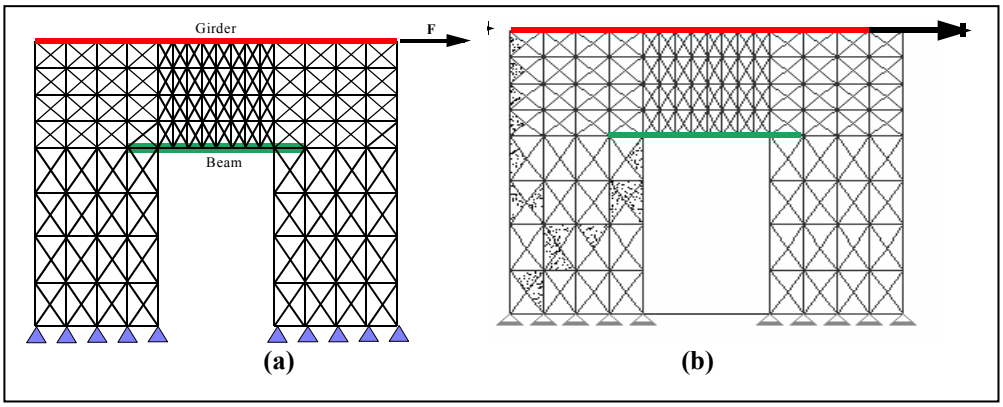

Figure 1: (a) The FEM model of the considered panel;
(b) Optimal reinforcement distribution (signed by the black points).

The procedure described in the previous Sect.2 has been implemented in an appropriate calculus code set up by the authors for the specific problem.

The sketch captured after the running of the code for achieving the optimal design of the reinforcement, representing the requested distribution is shown in Figure 1(b).

The procedure yields a resistant mechanism, able to neutralize the collapsing action of the horizontal force as made clear in Figure 2, where the reinforcement acts as a tie-rod. This interpretation is confirmed also looking at the results of the FEM-NT reinforced panel, and in particular to the isostatic compression lines reported in Figure 3.

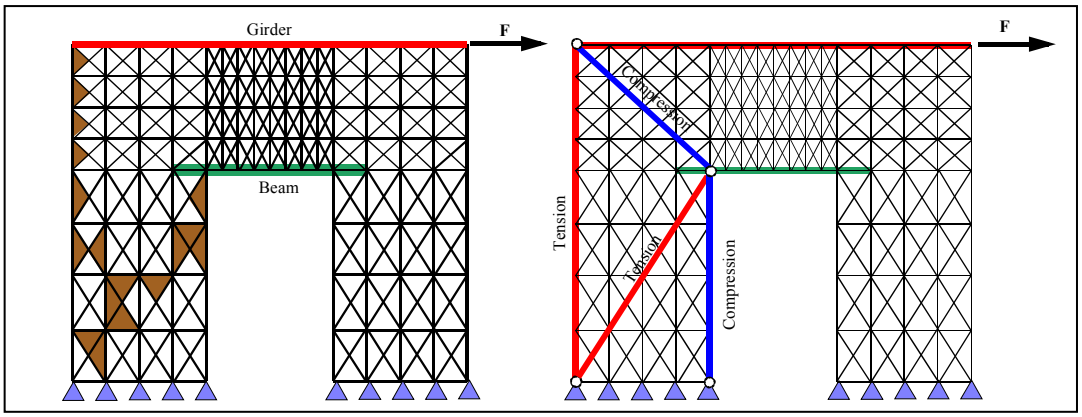

Figure 2: Resistant mechanism after the reinforcement

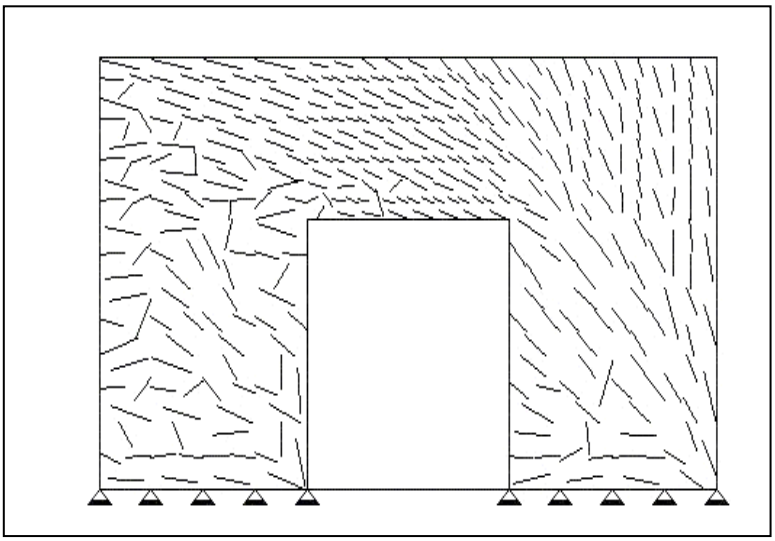

Figure 4: Results from FEM analysis of the reinforced panel.
Isostatic lines in the basic NT material

4 Conclusions

In the paper an original approach based on the coupling of LA and structural TO for the design of the reinforcement in masonry structures is presented. By using the topology optimization approach for structures conceived as an assemblage of a number of connected components, the problem is set with the objective to identify the number, the dimensions and the arrangement of the members.

For continuum systems, essentially the shape of the reinforcement that make the structure able to resist the loads is optimised, and the solution consists of deciding whether any point in the domain where the structure must be included is filled with material or not.

The numerical investigation developed in the paper applied to a masonry wall shows the potential and effectiveness of the approach.

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