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An Exact Elastic Plastic Solution for a Thin Disc subject to Thermal Loading

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Abstract

This paper is concerned with a thin axisymmetric disc of variable thickness subject to thermal loading. It is assumed that the state of stress is two-dimensional. The material model is elastic perfectly/plastic. The Mises yield criterion and its associated flow rule are accepted. An exact analytical elastic/plastic solution is found. An illustrative numerical example to demonstrate the effect of the initial thickness of the disc on the magnitude of temperature corresponding to the initiation of plastic yielding and the magnitude of temperature at plastic collapse is provided.

Keywords: thermal loading, thin disc, variable thickness, plastic collapse, analytical solution.

1 Introduction

Thin plates and discs with holes and embedded inclusions have many structural applications. A significant amount of analytical research has been carried out in the area of stress and strain analysis of such structures subject to various types of loading, see [1 - 4] among many others. Further references can be found in these papers. Various aspects of the influence of temperature on stress and strain distributions in thin plates and discs have been studied in [5 - 10] among others. The assumptions made regarding yield criterion, strain hardening and unloading have significant effects on the predicted response and residual stress and strain fields [11]. Even though closed-form solutions involve more assumptions than numerical solutions, the former are very useful for studying such effects. In particular, it is known that elastic-plastic solutions for thin discs and plates are very sensitive to loading parameters [9, 10, 12, 13]. Moreover, plastic collapse of such structures may occur due to strain localization along a certain radius (in the case of axisymmetric problems) [2]. These features of solutions may cause some difficulties with numerical analysis of similar structures. In fact, it has been mentioned in [14] that the application of computational models to plane stress problems leads to specific difficulties non-existent in other formulations.

In the present paper, an analytical solution for a thin annular disc of variable thickness subject to thermal loading is given. The solution is a generalization of the solution proposed in [10]. It is shown that the variation of the initial thickness of the disc has a significant effect on the magnitude of temperature at which plastic deformation initiates and on the magnitude of temperature at plastic collapse when the entire disc becomes plastic.

2 Statement of the problem

Consider a thin disc of radius b with a central circular hole of radius a, which is inserted into a rigid container of radius a (Figure 1). It is convenient to introduce a cylindrical coordinate system (r, θ, z) with its z-axis coinciding with the axis of symmetry of the disc. The initial thickness of the disc varies according to the equation [15]

$$h = h_0 \left(\frac{r}{a} \right)^n \tag{1}$$

where h_0 is the thickness at the edge of the hole and n is a constant. The disc is subject to thermal loading by a uniform temperature field varying with the time. The disc has no stress at the initial temperature. Its inner radius is stress free during the process of deformation. The outer radius is fixed to the container. It is evident that the problem is axisymmetric. In particular, the solution is independent of θ. Moreover, the normal stresses in the cylindrical coordinates, σ_r, σ_θ and σ_z, are the principal stresses. It is also assumed that the state of stress is two-dimensional, $\sigma_z = 0$. Thermal expansion caused by a rise of temperature and the constraints imposed on the disc affect the initial zero-stress state. It is assumed that the rise of

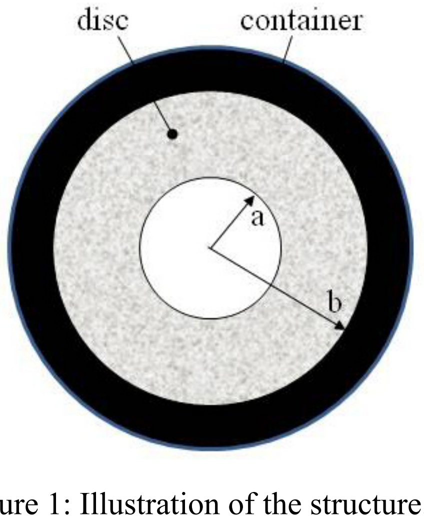

Figure 1: Illustration of the structure.

temperature above the reference state, T, is a monotonically increasing function of the time, t. The boundary conditions are

$$\sigma_r = 0 \tag{2}$$

at $r = a$ and

$$u = 0 \tag{3}$$

at $r = b$. Here u is the radial displacement. The circumferential displacement vanishes everywhere.

It is assumed that the thickness of the disc is everywhere sufficiently small for the stresses to be averaged through the thickness. In this case the only non-trivial equilibrium equation becomes

$$\frac{\partial}{\partial r}\left(hr\sigma_r\right) = h\sigma_\theta . \tag{4}$$

Substituting Equation (1) into Equation (4) gives

$$r\frac{\partial \sigma_r}{\partial r} + \left(1+n\right)\sigma_r = \sigma_\theta . \tag{5}$$

The total radial, ε_r, and circumferential, ε_θ, strains are defined by

$$\varepsilon_r = \varepsilon_r^T + \varepsilon_r^e + \varepsilon_r^p, \quad \varepsilon_\theta = \varepsilon_\theta^T + \varepsilon_\theta^e + \varepsilon_\theta^p \tag{6}$$

where the superscript T denotes the thermal portions of the total strains, the superscript e the elastic portions of the total strains and the superscript p the plastic portions of the total strains. It follows from Hooke's law that

$$\varepsilon_r^e = \frac{\sigma_r - \nu\sigma_\theta}{E}, \quad \varepsilon_\theta^e = \frac{\sigma_\theta - \nu\sigma_r}{E} \tag{7}$$

where E is Young's modulus and ν is Poisson's ratio. The thermal portions of the total strains are given by

$$\varepsilon_r^T = \varepsilon_\theta^T = \alpha T \tag{8}$$

where α is the thermal coefficient of linear expansion. In the plastic range, Mises' yield criterion is adopted. For the problem under consideration this criterion reduces to

$$s_r^2 + \sigma^2 - \sigma s_r = k^2 \tag{9}$$

where σ is the hydrostatic stress, k is the shear yield stress, a material constant for perfectly plastic materials, and s_r is the deviatoric radial stress, $s_r = \sigma_r - \sigma$. Also, s_θ will stand for the deviatoric circumferential stress, $s_\theta = \sigma_\theta - \sigma$. Since $\sigma_z = 0$, the hydrostatic stress is

$$3\sigma = \sigma_r + \sigma_\theta . \tag{10}$$

The normality rule is written in terms of the strain rate components. A consequence of this rule is

$$\frac{\xi_r^p}{\xi_\theta^p} = \frac{s_r}{s_\theta} \tag{11}$$

where ξ_r^p and ξ_θ^p are the plastic portions of the total radial and circumferential strain rates. Another essential equation following from the normality rule expresses plastic incompressibility, $\xi_r^p + \xi_\theta^p + \xi_z^p = 0$, where ξ_z^p is the plastic portion of the total axial strain rate. This equation serves to determine ξ_z^p and is not important for the present solution. At small strains,

$$\xi_r^p = \frac{\partial \varepsilon_r^p}{\partial t}, \quad \xi_\theta^p = \frac{\partial \varepsilon_\theta^p}{\partial t}. \tag{12}$$

3 Thermo-elastic solution

At the beginning of the process the entire disc is elastic. At this stage,

$$\frac{\partial u}{\partial r} = \varepsilon_r^T + \varepsilon_r^e, \quad \frac{u}{r} = \varepsilon_\theta^T + \varepsilon_\theta^e. \tag{13}$$

Eliminating u between these two equations, excluding the strain components by means of Equations (7) and (8), and taking into account that T is independent of r yield

$$r\left(\frac{\partial \sigma_\theta}{\partial r} - \nu \frac{\partial \sigma_r}{\partial r}\right) + (1+\nu)(\sigma_\theta - \sigma_r) = 0. \tag{14}$$

Eliminating the stress σ_θ between Equations (5) and (14) gives

$$r^2 \frac{\partial^2 \sigma_r}{\partial r^2} + (3+n) r \frac{\partial \sigma_r}{\partial r} + n(1+\nu)\sigma_r = 0. \tag{15}$$

The general solution to this equation is

$$\frac{\sigma_r}{k} = A\left(\frac{r}{a}\right)^{\gamma_1} + B\left(\frac{r}{a}\right)^{\gamma_2}, \quad \frac{\sigma_\theta}{k} = A\left(1+n+\gamma_1\right)\left(\frac{r}{a}\right)^{\gamma_1} + B\left(1+n+\gamma_2\right)\left(\frac{r}{a}\right)^{\gamma_2} \tag{16}$$

where A and B are constants of integration and

$$\gamma_1 = -\left(1+\frac{n}{2}\right) - \frac{1}{2}\sqrt{\left(2-n\right)^2 + 4n\left(1-\nu\right)}, \quad \gamma_2 = -\left(1+\frac{n}{2}\right) + \frac{1}{2}\sqrt{\left(2-n\right)^2 + 4n\left(1-\nu\right)}.$$

Equation (5) has been used to find σ_θ after solving Equation (15). Substituting the solution (16) into Equation (7) determines ε_θ^e. Then, using this expression for ε_θ^e and Equation (8) the radial displacement can be found from Equation (13). As a result,

$$\frac{u}{rq} = A\left(1+n-\nu+\gamma_1\right)\left(\frac{r}{a}\right)^{\gamma_1} + B\left(1+n-\nu+\gamma_2\right)\left(\frac{r}{a}\right)^{\gamma_2} + \tau \tag{17}$$

where $q = k/E$ and $\tau = \alpha T/q$. Substituting the boundary conditions (2) and (3) into Equations (16) and (17) leads to

$$\begin{aligned} A &= \tau A^e, \quad A^e = \frac{1}{\left(1+n-\nu+\gamma_2\right)\left(b/a\right)^{\gamma_2} - \left(1+n-\nu+\gamma_1\right)\left(b/a\right)^{\gamma_1}}, \\ B &= \tau B^e, \quad B^e = \frac{1}{\left(1+n-\nu+\gamma_1\right)\left(b/a\right)^{\gamma_1} - \left(1+n-\nu+\gamma_2\right)\left(b/a\right)^{\gamma_2}} \end{aligned}. \tag{18}$$

The validity of the solution presented in this section is controlled by the yield criterion (9). Substituting Equation (16) into Equation (10), using the definition for the deviatoric radial stress and excluding A and B by means of Equation (18) yield

$$\begin{aligned} \frac{\sigma}{k} &= \frac{\tau}{3}\left[A^e\left(2+n+\gamma_1\right)\left(\frac{r}{a}\right)^{\gamma_1} + B^e\left(2+n+\gamma_2\right)\left(\frac{r}{a}\right)^{\gamma_2}\right], \\ \frac{s_r}{k} &= \frac{\tau}{3}\left[A^e\left(1-n-\gamma_1\right)\left(\frac{r}{a}\right)^{\gamma_1} + B^e\left(1-n-\gamma_2\right)\left(\frac{r}{a}\right)^{\gamma_2}\right] \end{aligned}. \tag{19}$$

Substituting Equation (19) at $r = a$ into Equation (9) determines the value of $\tau = \tau_e$ at which the plastic zone starts to develop. The variation of τ_e with a/b at several values on n and $\nu = 0.3$ is depicted in Figure 2 (curve 1 corresponds to $n = 0$, curve 2 to $n = 0.5$, curve 3 to $n = 1$, curve 4 to $n = 1.5$, and curve 5 to $n = 2$). It is evident from Equation (1) that the solution for $n = 0$ corresponds to the disc of constant thickness.

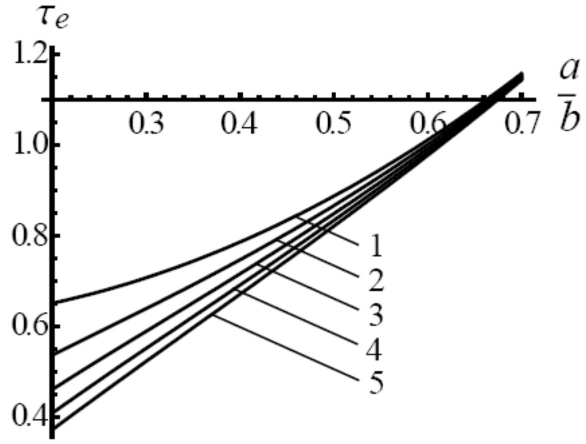

Figure 2: Effect of the ratio a/b and n on the magnitude of temperature corresponding to the initiation of plastic yielding at $r = a$.

4 Elastic plastic solution: stress analysis

When $\tau > \tau_e$ a plastic zone exists in the interval $a \le r \le c$ where c is the radius of the elastic plastic boundary. In a limit case $c = b$ and plastic collapse occurs (the entire disc becomes plastic). The yield criterion (9) is valid in the plastic zone. This criterion is satisfied by the following standard substitution

$$s_r = \frac{2k\sin\varphi}{\sqrt{3}}, \quad \sigma = \frac{k\left(\sin\varphi + \sqrt{3}\cos\varphi\right)}{\sqrt{3}} \tag{20}$$

where φ is a function of r and τ. Using Equation (10) and the definition for the deviatoric radial stress the radial and circumferential stresses as well as the deviatoric circumferential stress can be obtained in the form

$$\sigma_r = k\left(\cos\varphi + \sqrt{3}\sin\varphi\right), \quad \sigma_\theta = 2k\cos\varphi, \quad s_\theta = k\frac{\left(\sqrt{3}\cos\varphi - \sin\varphi\right)}{\sqrt{3}}. \tag{21}$$

Substituting Equation (21) into Equation (5) results in

$$r\frac{\partial\varphi}{\partial r} = \frac{(1-n)\cos\varphi - \sqrt{3}(1+n)\sin\varphi}{\sqrt{3}\cos\varphi - \sin\varphi}. \tag{22}$$

Using Equation (21) the boundary condition (2) transforms to $\cos\varphi + \sqrt{3}\sin\varphi = 0$ at $r = a$. The unique solution to this equation is determined with the use of the inequality $\sigma_\theta - \sigma_r < 0$ at $r = a$. Then, it follows from Equation (21) that $\cos\varphi < 0$ and, therefore,

$$\varphi = \frac{5\pi}{6} \tag{23}$$

at $r = a$. Integrating Equation (22) and using the boundary condition (23) yield

$$\frac{r}{a} = 3^{m/2}\left[(n-1)\cos\varphi + \sqrt{3}(n+1)\sin\varphi\right]^{-m} \exp\left[\frac{\sqrt{3}(\varphi - 5\pi/6)}{2(1+n+n^2)}\right] \tag{24}$$

where $m = (n+1/2)/(1+n+n^2)$. It is evident from this equation that φ is independent of τ. Let φ_c be the value of φ at the elastic-plastic boundary. Then, it follows from Equation (24) that

$$\frac{c}{a} = 3^{m/2}\left[(n-1)\cos\varphi_c + \sqrt{3}(n+1)\sin\varphi_c\right]^{-m} \exp\left[\frac{\sqrt{3}(\varphi_c - 5\pi/6)}{2(1+n+n^2)}\right]. \tag{25}$$

The general solution (16) and (17) is valid in the elastic zone, $c \le r \le b$, but *A* and *B* are not determined by Equation (18). The radial displacement from Equation (17) should satisfy the boundary condition (3). Therefore,

$$A(1+n-\nu+\gamma_1)\left(\frac{b}{a}\right)^{\gamma_1} + B(1+n-\nu+\gamma_2)\left(\frac{b}{a}\right)^{\gamma_2} + \tau = 0. \tag{26}$$

As usual, it is assumed that all stresses are continuous across the elastic plastic boundary. Then, it follows from Equations (16) and (21) that

$$\begin{aligned} &A\left(\frac{c}{a}\right)^{\gamma_1} + B\left(\frac{c}{a}\right)^{\gamma_2} = \left(\cos\varphi_c + \sqrt{3}\sin\varphi_c\right), \\ &A(1+n+\gamma_1)\left(\frac{c}{a}\right)^{\gamma_1} + B(1+n+\gamma_2)\left(\frac{c}{a}\right)^{\gamma_2} = 2\cos\varphi_c \end{aligned} . \tag{27}$$

Solving these equations for *A* and *B* gives

$$\begin{aligned} A &= \frac{\left[\left(\cos\varphi_c + \sqrt{3}\sin\varphi_c\right)(1+n+\gamma_2) - 2\cos\varphi_c\right]}{\gamma_2 - \gamma_1}\left(\frac{c}{a}\right)^{-\gamma_1}, \\ B &= \frac{\left[2\cos\varphi_c - \left(\cos\varphi_c + \sqrt{3}\sin\varphi_c\right)(1+n+\gamma_1)\right]}{\gamma_2 - \gamma_1}\left(\frac{c}{a}\right)^{-\gamma_2} \end{aligned} . \tag{28}$$

Using Equation (28) to exclude *A* and *B* in Equation (26) leads to

$$\tau = \frac{\left(1+n-\nu+\gamma_1\right)\left[\left(\cos\varphi_c+\sqrt{3}\sin\varphi_c\right)\left(1+n+\gamma_2\right)-2\cos\varphi_c\right]}{\gamma_1-\gamma_2}\left(\frac{b}{c}\right)^{\gamma_1}+ \\ +\frac{\left(1+n-\nu+\gamma_2\right)\left[2\cos\varphi_c-\left(\cos\varphi_c+\sqrt{3}\sin\varphi_c\right)\left(1+n+\gamma_1\right)\right]}{\gamma_1-\gamma_2}\left(\frac{b}{c}\right)^{\gamma_2}. \quad (29)$$

In this equation, c can be excluded by means of Equation (25). The resulting equation determines φ_c as a function of τ in implicit form. Then, c can be found as a function of τ from Equation (25). Finally, A and B can be found as functions of τ from Equation (28). The distribution of the stresses σ_r and σ_θ in the region $a \le r \le c$ is determined from Equations (21) and (24) in parametric form with φ being the parameter. Having A and B as functions of τ, the distribution of the stresses σ_r and σ_θ in the region $c \le r \le b$ is determined from Equation (16). Having the dependence of c on τ it is possible to find the value of τ at which the entire disc becomes plastic from the equation $c(\tau) = b$. The corresponding value of τ is denoted by τ_p. The variation of τ_p with a/b at several values of n and $\nu = 0.3$ is depicted in Figure 3 (curve 1 corresponds to $n = 0$, curve 2 to $n = 0.5$, curve 3 to $n = 1$, curve 4 to $n = 1.5$, and curve 5 to $n = 2$). It is also of interest to find a relative increase in temperature from its magnitude corresponding to the initiation of plastic deformation to the value at which the entire disc becomes plastic. An appropriate measure of this quantity is $\tau_\Delta = \left(\tau_p - \tau_e\right)/\tau_e$. The variation of τ_Δ with a/b at several values of n and $\nu = 0.3$ is shown in Figure 4 (curve 1 corresponds to $n = 0$, curve 2 to $n = 0.5$, curve 3 to $n = 1$, curve 4 to $n = 1.5$, and curve 5 to $n = 2$).

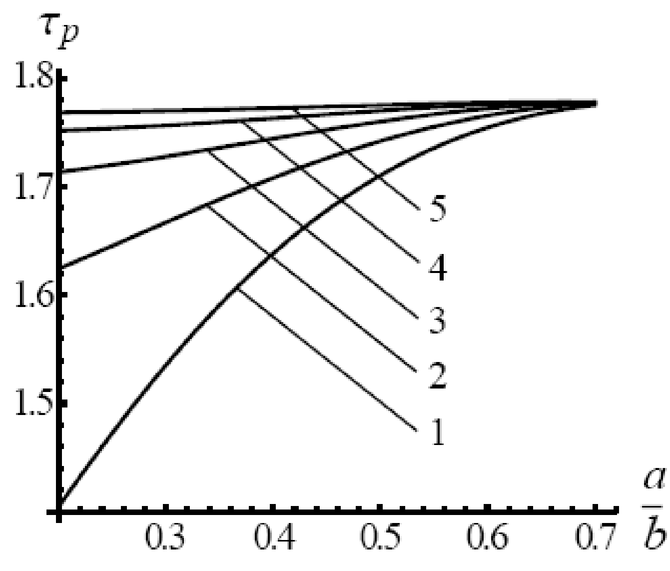

Figure 3: Effect of the ratio *a*/*b* and *n* on the magnitude of temperature corresponding to plastic collapse.

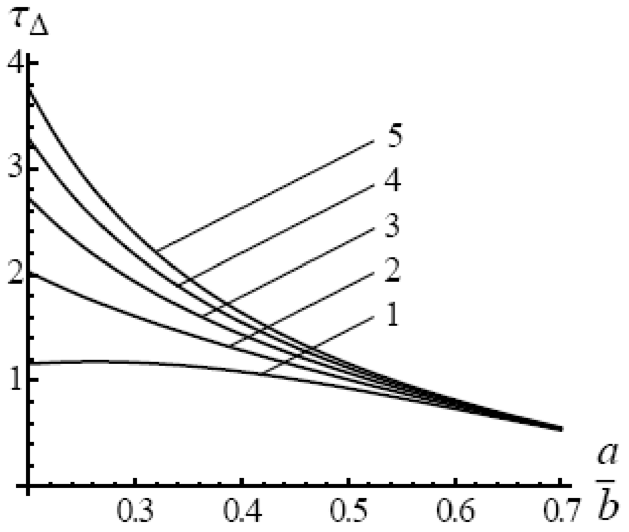

Figure 4: Effect of the ratio a/b and n on the increase in temperature from its magnitude at the initiation of plastic yielding to the magnitude at plastic collapse.

5 Elastic plastic solution: kinematic analysis

The distribution of the strains ε_r^e and ε_θ^e in the plastic zone is obtained from Equations (7) and (21) in the form

$$\varepsilon_r^e = q\left[(1-2\nu)\cos\varphi + \sqrt{3}\sin\varphi\right], \quad \varepsilon_\theta^e = q\left[(2-\nu)\cos\varphi - \nu\sqrt{3}\sin\varphi\right]. \tag{30}$$

Using Equation (12) and taking into account that τ is a monotonically increasing function of t it is possible to rewrite Equation (11) as

$$\frac{\partial \varepsilon_r^p}{\partial \tau} = \frac{s_r}{s_\theta}\frac{\partial \varepsilon_\theta^p}{\partial \tau}. \tag{31}$$

Since φ is independent of τ, it follows from Equations (20) and (21) that s_r and s_θ are also independent of τ. Therefore, Equation (31) can be immediately integrated with the use of the condition $\varepsilon_r^p = 0$ at $\varepsilon_\theta^p = 0$ to give

$$\varepsilon_r^p = \frac{2\sin\varphi}{\left(\sqrt{3}\cos\varphi - \sin\varphi\right)}\varepsilon_\theta^p \tag{32}$$

where s_r and s_θ have been excluded by means of Equations (20) and (21). Using Equations (8), (30) and (32) the total radial and circumferential strains in the plastic zone can be determined as

$$\varepsilon_r = \alpha T + q\left[\left(1-2\nu\right)\cos\varphi + \sqrt{3}\sin\varphi\right] + \frac{2\sin\varphi}{\left(\sqrt{3}\cos\varphi - \sin\varphi\right)}\varepsilon_\theta^p,$$
$$\varepsilon_\theta = \alpha T + q\left[\left(2-\nu\right)\cos\varphi - \nu\sqrt{3}\sin\varphi\right] + \varepsilon_\theta^p \quad . \qquad (33)$$

In the case under consideration, the equation of strain compatibility is $r\partial\varepsilon_\theta/\partial r = \varepsilon_r - \varepsilon_\theta$. Replacing differentiation with respect to r with differentiation with respect to φ leads to

$$r\frac{\partial\varepsilon_\theta}{\partial\varphi}\frac{\partial\varphi}{\partial r} = \varepsilon_r - \varepsilon_\theta . \qquad (34)$$

Using Equations (22) and (33) Equation (34) can be transformed to

$$\frac{\partial\varepsilon_\theta^p}{\partial\varphi} = q\left[\left(2-\nu\right)\sin\varphi + \nu\sqrt{3}\cos\varphi\right] + \frac{q\left(\sqrt{3}\cos\varphi - \sin\varphi\right)\left[\sqrt{3}\left(1+\nu\right)\sin\varphi - \left(1+\nu\right)\cos\varphi\right] + \sqrt{3}\left(\sqrt{3}\sin\varphi - \cos\varphi\right)\varepsilon_\theta^p}{\left[\left(1-n\right)\cos\varphi - \sqrt{3}\left(1+n\right)\sin\varphi\right]} . \qquad (35)$$

This is a linear differential equation of first order for ε_θ^p. Therefore, its general solution can be found with no difficulty. The radial displacement and, therefore, the total circumferential strain must be continuous across the elastic/plastic boundary. Since A and B involved in Equation (17) have been determined in the previous section, the distribution of the total circumferential strain in the elastic zone is known. This distribution along with Equation (33) provides the boundary condition for Equation (35).

6 Conclusions

A new exact analytical solution for thin axisymmetric disc of variable thickness subject to thermal loading has been found. Numerical methods have been only adopted to solve transcendental equations. A remarkable feature of the solution obtained is that the ratio s_r/s_θ is independent on temperature (or time) in the plastic zone. This feature of the solution enables Equation (31) to be immediately integrated to give a relation between the plastic portions of the total radial and circumferential strains instead of the original relation between strain rate components. The illustrative example demonstrates that the initial thickness may have a significant effect on response of the disk to thermal loading. This effect can be used to design discs according to selected design criteria. The solution may also serve as a benchmark problem to verify numerical codes developed for solving plane stress problems. In particular, the distribution of stresses and strains has been obtained in closed form. The only exception is Equation (35) for the plastic portion of the total circumferential strain. However, the solution to this linear differential equation can be reduced to numerical evaluation of ordinary integrals with no difficulty.

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