Abstract

This paper is devoted to introduce curved fibre with high order approximations in a continuum media via tailoring process demonstrating that, if the order of fibre approximation is at least the same as the solid finite element, the coupling is conforming and the deficiency of the technique disappears, that is, does not guarantee the continuity among the continuum and the fibre material between nodes. Moreover, the spreading strategy that makes it possible to make a complete analysis of randomly fibre reinforced materials without increasing the number of variables. Applications for two dimensional large deformation analysis of elastic bodies shows the potential of the proposed formulation.

Keywords: finite element method, geometrical non-linearity, fibre-reinforced solids, curved fibre finite elements, tailoring, conform coupling fibre-matrix.

1 Introduction

Fibre reinforced solids are usually analyzed by homogeneous analog which makes difficult to identify the contact stresses between fibres and matrix. Alternatively, interesting techniques proposing the fibre-matrix coupling are present in literature. Some of them introduce fibres into continuum by direct mathematical considerations generating solid finite elements that consider fibres properties [1-3]. Others introduce fibres as a point to point bar discretization which leads to a difficult mesh generation process. Another strategy is to write fibre nodes coordinates as functions of solid finite elements nodes by means of shape functions and introduce the strain energy of fibres into the solution process [4].

The last strategy makes possible to consider fibres at any place of the continuum without increasing the amount of unknowns in the solution procedure. However, in general, it does not guaranty the continuity among the continuum and fibre material between nodes and, as a consequence, this technique is called tailoring process [1].
In this paper we develop a curved fibre finite element with high order approximations based on tailoring description that ensures the conform coupling between fibre-matrix. The nodal parameters are positions, not displacements. The formulation is classified as total Lagrangian and the Saint-Venant-Kirchhoff constitutive law [5, 6] is chosen to model the material behavior.

To solve geometrical nonlinear problems we adopt the Principle of Minimum Total Potential Energy [7] and the Newton-Raphson iterative procedure [8] to solve the nonlinear system.

The paper is organized as follows. Section 2 describes the general nonlinear solution process. Section 3 describes the procedure used to model the two-dimensional continuum, plate element. Section 4 presents the any order fibre finite elements and the spreading strategy that makes possible a complete analysis of any order fibres into high order plate finite elements without increasing the number of degrees of freedom. Section 5 presents the numerical examples validating the proposed formulation. Finally, conclusions are presented in Section 6.

2 The non-linear solution

In this section, the strategy adopted to solve the reinforced plate geometrically nonlinear equilibrium is described.

The non-linear analysis starts writing the total potential energy as follows:

\[ \Pi(Y) = U(Y) - \Omega(Y) \]  

where \( \Pi \) is the total potential energy of the system, \( U \) is the strain energy including matrix and fibre contributions written regarding plate nodal positions and \( \Omega \) is the potential energy of external conservative applied forces given by:

\[ \Omega = F_j Y_j \]  

where \( F_j \) is the vector of external forces and \( Y_j \) is the current position vector.

The Principle of Minimum Total Potential Energy [7] is applied writing the equilibrium equation as the derivative of total energy regarding nodal positions (plate for instance), as:

\[ g_j = \frac{\partial \Pi}{\partial Y_j} = \frac{\partial U}{\partial Y_j} - F_j = F_j^{\text{int}} - F_j = 0 \]

where \( F_j^{\text{int}} \) is the internal force vector or strain energy gradient vector calculated regarding plate nodal positions. The nodal current positions are the unknowns of the problem, so, when adopting a trial position in Equation (3), \( g_j \) is not null and becomes the unbalanced force vector of the Newton-Raphson [8] strategy for
solving nonlinear systems. Expanding the unbalanced force vector around the trial solution \( Y_i^{0} \), one has:

\[
g_j(Y_i) = g_j(Y_i^{0}) + \left. \frac{\partial g_j}{\partial Y_k} \right|_{Y_i^{0}} \Delta Y_k + O_j^2 = 0
\]  

which can be rewritten, neglecting higher order terms as:

\[
\Delta Y_k = -\left. \frac{\partial g_j}{\partial Y_k} \right|_{Y_i^{0}} g_j(Y_i^{0}) = -\left. \frac{\partial^2 U}{\partial Y_k \partial Y_j} \right|_{Y_i^{0}} g_j(Y_i^{0})
\]  

where \( \Delta Y_k \) is the correction of position and \( \left. \frac{\partial^2 U}{\partial Y_k \partial Y_j} \right|_{Y_i^{0}} \) is the Hessian matrix or tangent stiffness matrix.

The trial solution is improved by:

\[
Y_i = Y_i^{0} + \Delta Y_k
\]  

until \( \Delta Y_k \) or \( g_j \) become sufficiently small [8].

### 3 Isoparametric plate finite element

In this section the necessary expressions to consider the continuum part of a general composite in the solution process are presented.

#### 3.1 Kinematical approximation and positional mapping

Figure 1 shows the matrix mapping from the initial configuration \( B_0 \) to its current configuration \( B \). This mapping is done by means of a dimensionless auxiliary configuration \( B_j \).

The initial configuration \( B_0 \) whose points have coordinates \( x_i \) is mapped from the dimensionless space \( B_j \) with coordinates \( \xi \) using shape functions of any order, \( \phi_l(\xi, \xi_2) \), and by the coordinates of the nodes \( l \) in the initial configuration, \( X^l_i \), such as:

\[
x_i = f_i^0 = \phi_l(\xi, \xi_2) X^l_i
\]
Similarly, the current configuration $B$ is mapped from the dimensionless space $B_i$ by the expression:

$$y_i = f_i^l = \phi_i(\xi_1, \xi_2)Y_i^l$$  \hspace{1cm} (8)

where $y_i$ are coordinates of points in the current configuration, $Y_i^l$ are the current node positions, $l = I, ..., N$ are nodes and $i = 1, 2$ correspond to nodal degrees of freedom.

![Figure 1: Mapping of the initial and current configurations.](image)

The deformation function $f$ that maps initial configuration $B_0$ to the current configuration $B$ can be written as a composition of mappings $f^0$ and $f^l$ as:

$$f = f^l \circ \left(f^0\right)^{-l}$$  \hspace{1cm} (9)

The deformation gradient $A$ can be derived directly from $A^0$ and $A^l$ as [9,10]:

$$A = A^l \cdot (A^0)^{-l}, \text{ with } A^0_i = \frac{\partial f^0}{\partial \xi_j}, A^l_i = \frac{\partial f^l}{\partial \xi_j}$$  \hspace{1cm} (10)

Equation (10) can be understood as a numerical chain rule because the initial mapping gradient $A^0$ is a known numerical quantity.
3.2 Continuum strain energy

To simulate the continuum portion of the composite (matrix) we adopted the Saint-Venant-Kirchhoff specific strain energy function [5, 6], as:

\[ u_{\text{mat}} = \frac{1}{2} E_y C_{ijkl} F_{kl} \]  

where \( C_{ijkl} \) is the elastic fourth-order tensor and \( E \) is the Green-Lagrange second-order strain expressed respectively by:

\[ C_{ijkl} = \frac{2G \nu}{1-2\nu} \delta_{ij} \delta_{kl} + G \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \]  

\[ E_y = \frac{1}{2} \left( C_{ij} - \delta_{ij} \right) = \frac{1}{2} \left( A_{ij} A_{kj} - \delta_{ij} \right) \]  

The variables \( \mathbf{C} = \mathbf{A}^T \mathbf{A} \) and \( \mathbf{\delta} \) are the right Cauchy-Green stretch tensor and the Kroenecker delta, respectively. In Equation (13), \( G \) is the shear modulus and \( \nu \) is the Poisson’s ratio.

The strain energy accumulated in the finite element is calculated by integrating the specific strain energy over the initial volume, i.e.

\[ U_{\text{mat}} = \int_{V_0} u_{\text{mat}} dV_0 \]  

Considering plates with unitary thickness and writing Equation (14) as a function of dimensionless coordinates \( 1\xi \) and \( 2\xi \) results:

\[ U_{\text{mat}} = \int_{0}^{1} \int_{0}^{1} u_{\text{mat}}(\xi_1, \xi_2) J_0(\xi_1, \xi_2) d\xi_1 d\xi_2 \]  

where \( J_0 \) is the Jacobian of the initial mapping, i.e., \( J_0(\xi_1, \xi_2) = \text{det}(\mathbf{A}^0) \) with \( \mathbf{A}^0 \) given by Equation (10).

If the modelled material has no reinforcement, the strain energy can be derived directly regarding the plate finite element positions finding the conjugate internal forces, as:

\[ F^\beta_{\alpha \text{int}} = \frac{\partial U_{\text{mat}}}{\partial Y_\beta^\alpha} = \int_{V_0} \frac{\partial u_{\text{mat}}}{\partial Y_\beta^\alpha} dV_0 = \int_{0}^{1} \int_{0}^{1} \frac{\partial u_{\text{mat}}}{\partial Y_\beta^\alpha} J_0(\xi_1, \xi_2) d\xi_1 d\xi_2 \]  

The derivative inside the integral term of equation (16) can be developed as:

\[ \frac{\partial u_{\text{mat}}}{\partial Y_\beta^\alpha} = \frac{\partial u_{\text{mat}}}{\partial E} \cdot \frac{\partial E}{\partial C} \cdot \frac{\partial C}{\partial Y_\alpha^\beta} = \frac{1}{2} S : \frac{\partial C}{\partial Y_\alpha^\beta} \]
where \( S = \partial u_{\text{mat}} / \partial E \) is the second Piola-Kirchhoff stress tensor and \( C \) is the right Cauchy-Green stretch tensor.

In the solution process, Section 2, it is necessary to calculate the second derivative of strain energy regarding nodal positions, resulting into the Hessian matrix, that is:

\[
H_{a\beta}^{\text{mat}} = \frac{\partial^2 U_{\text{mat}}}{\partial Y_a^\beta \partial Y_\gamma^\gamma} = \int_{V_0} \frac{\partial^2 u_{\text{mat}}}{\partial Y_a^\beta \partial Y_\gamma^\gamma} dV_0
\]

in which

\[
\frac{\partial^2 u_{\text{mat}}}{\partial Y_a^\beta \partial Y_\gamma^\gamma} = \frac{1}{4} \frac{\partial^2 u_{\text{mat}}}{\partial E^\gamma \partial E^\gamma} \frac{\partial C}{\partial Y_a^\beta} + \frac{1}{2} \frac{\partial u_{\text{mat}}}{\partial Y_a^\beta} \frac{\partial^2 C}{\partial Y_\gamma^\gamma \partial Y_a^\beta}
\]

4 Elastic fibre reinforcement – kinematics and energy considerations

This section presents the necessary expressions to introduce the fibres characteristics in the composite formulation.

4.1 Any order curved fibre finite element

Figure 2 shows the non-deformed initial configuration \( B_0 \), the current configuration \( B \) and a non-dimensional auxiliary configuration \( B_1 \) for the curved fibre finite element of any order.

The initial configuration \( B_0 \) whose points have coordinates \( x_i \) is mapped from the dimensionless space \( B_1 \) with coordinates \( \xi \) using shape functions of any order, \( \phi_p(\xi) \), and by the coordinates of the nodes \( P \) in the initial configuration, \( X_i^P \), such as:

\[
x_i = f_i^0 = \phi_p(\xi) X_i^P
\]

The current configuration \( B \) is mapped from the dimensionless space \( B_1 \) by the expression

\[
y_i = f_i^1 = \phi_p(\xi) Y_i^P
\]

where \( y_i \) are the coordinates of points in the current configuration \( B \) and \( Y_i^P \) are the current coordinates of fibre nodes in the current configuration. In Equations (20)-
(21) index \( P = 1, \ldots, n \) and \( i = 1, 2 \) represent, respectively, the fibre finite element nodes and the degrees of freedom associated with these nodes.

The tangent vector of the fibre and its modulus are calculated at the initial configuration as:

\[
\mathbf{T}^B = \mathbf{d} \mathbf{X}^P \quad \text{and} \quad \left| \mathbf{T}^B \right|^2 = \left( \frac{d\mathbf{X}^P}{d\xi} \right)^2 + \left( \frac{d\mathbf{X}^P}{d\bar{\xi}} \right)^2 \quad \text{(22)}
\]

It is important to mention that \( \left| \mathbf{T}^B \right|^2 \) is the differential Jacobian of \( f_i^B \). For the current configuration one finds:

\[
\mathbf{T}^B = \mathbf{d} \mathbf{Y}^P \quad \text{and} \quad \left| \mathbf{T}^B \right|^2 = \left( \frac{d\mathbf{Y}^P}{d\xi} \right)^2 + \left( \frac{d\mathbf{Y}^P}{d\bar{\xi}} \right)^2 \quad \text{(23)}
\]

From the tangent vector modulus, Equations (22)-(23), the one-dimensional Green strain is written as:

\[
E = \frac{1}{2} \left( \frac{\left| \mathbf{T}^B \right|^2 - \left| \mathbf{T}^B \right|^2}{\left| \mathbf{T}^B \right|^2} \right) \quad \text{(24)}
\]

which in its expanded form is given by:
Using the Saint-Venant-Kirchhoff constitutive law one writes the specific strain energy at a point of the fibre as:

\[ u_f(\xi) = \frac{1}{2} \mathcal{E} [E(\xi)]^2 \]  

(26)

where \( \mathcal{E} \) is the elastic modulus and \( E(\xi) \) is the Green strain measure defined in Equations (24) or (25).

The strain energy of a fibre is given by:

\[ U_f = \int_{V_0} u_f dV \]  

(27)

in which \( V_0 \) is the initial volume of the fibre element. In order to proceed with the equilibrium analysis it is necessary to know the first derivative of strain energy regarding positions. Based on the energy conjugate concept the natural internal fibre force vector, \( F_{int}^{ij} \) is calculated regarding fibre parameters as:

\[ F_{k}^{ij} = \frac{\partial U_f}{\partial Y_k} = \int_{V_0} \frac{\partial u_f}{\partial Y_k} dV \]  

(28)

From Equations (25) and (26) follows that

\[ F_{k}^{ij} = \int_{V_0} \mathcal{E} \frac{(\frac{d\phi(\xi)}{d\xi}) Y_i^j \frac{d\phi(\xi)}{d\xi}}{|T^0|} d\xi = \int_{V_0} \mathcal{E} \frac{(\frac{d\phi(\xi)}{d\xi}) Y_i^j \frac{d\phi(\xi)}{d\xi}}{|T^0|} J_0(\xi) d\xi \]  

(29)

where \( J_0(\xi) = |\bar{T}^0| = \sqrt{\left(\frac{dx_1}{d\xi}\right)^2 + \left(\frac{dx_2}{d\xi}\right)^2} \).

The Hessian matrix components for the fibre element are obtained by the second derivative of the strain energy, i.e.:

\[ H_{\alpha\beta}^{ij} = \frac{\partial^2 U_f}{\partial Y_i^j \partial Y_a^p} = \int_{V_0} \frac{\partial^2 u_f}{\partial Y_i^j \partial Y_a^p} dV \]  

(30)
Developing the necessary calculations one achieves:

\[
H_{\beta k \alpha}^{ij} = \int_{-1}^{1} \left( \frac{E}{T^0} \left( \frac{d\phi_i(\xi)}{d\xi} Y_a^i \right) \frac{d\phi_p(\xi)}{d\xi} \left( \frac{d\phi_j(\xi)}{d\xi} Y_a^j \right) + \frac{E}{T^0} \frac{d\phi_p(\xi)}{d\xi} \frac{d\phi_j(\xi)}{d\xi} \delta_{ka} \right) A J(\xi) d\xi
\]

(31)

Integrals (29) and (31) are solved using Gauss-Legendre quadrature.

4.2 Kinematical fibre-matrix coupling

The procedure adopted here to place the fibres at any position of the domain without increasing the number of degrees of freedom is based on the work of [4, 11]. These works were concerned with straight linear fibres and triangular plate elements, moreover only linear applications were developed.

The fibre elements are introduced in matrix by means of nodal kinematic relations. In the initial configuration one writes

\[
X_i^f = \phi_i(\xi, \eta^p \xi) X_i^f
\]

(32)

where \( \phi_i \) are the shape functions of the plate element, \( X_i^p \) are the known physical coordinates of fibre nodes and \( X_i^f \) are the plate nodes.

In the current configuration, the kinematic relation is written as

\[
Y_i^f = \phi_i(\xi, \eta^p \xi) Y_i^f
\]

(33)

where \( Y_i^f \) are the current positions of plate nodes. Equation (33) ensures the connection among nodes of fibres to the matrix.

4.3 General internal force

The strain energy stored in a reinforced body is the sum of the strain energy stored in the matrix and fibre, such as:

\[
U = U_{\text{mat}} + U_f
\]

(34)

where \( U_{\text{mat}} \) is the strain energy stored in the plate finite elements used to discretize the matrix and \( U_f \) is the strain energy stored in the fibre finite elements. Therefore, the internal force at a node \( \beta \) in the direction \( \alpha \) of the plate element considering the contribution of the fibre is found by using the conjugate energy concept, such as:
\[
\frac{\partial(U_{\text{mat}} + U_f)}{\partial Y_a^p} = \frac{\partial U_{\text{mat}}}{\partial Y_a^p} + \frac{\partial U_f}{\partial Y_a^p} = \frac{\partial U_{\text{mat}}}{\partial Y_a^p} + \frac{\partial U_f}{\partial Y_a^p} = F_{a\beta}^\text{mat} + \phi_{\beta}(\xi^p, \xi^p)F_{a\beta}^f = F_{a\beta}^\text{int} \tag{35}
\]

where Equations (33) and (29) have been used and there is no summation over \( P \).

### 4.4 Hessian Matrix

Proceeding in the same way as described for the calculation of internal forces, we develop the second derivative of strain energy of the reinforced finite element in relation to the plate nodal parameters, as follows

\[
\frac{\partial^2 U}{\partial Y_i^p \partial Y_j^p} = \frac{\partial^2 (U_{\text{mat}} + U_f)}{\partial Y_i^p \partial Y_j^p} = \int_{V_0} \frac{\partial^2 (u_{\text{mat}} + u_f)}{\partial Y_i^p \partial Y_j^p} dV_0 = \int_{V_0} \frac{\partial^2 u_{\text{mat}}}{\partial Y_i^p \partial Y_j^p} dV_0 + \int_{V_0} \frac{\partial^2 u_f}{\partial Y_i^p \partial Y_j^p} dV_0^f \tag{36}
\]

It is necessary to observe that the kernel of the last integral is the specific strain energy of a fibre derived twice regarding the plate nodal parameters and Equation (31) gives its value when derived regarding fibre parameters. So one has to apply twice the chain rule described by

\[
\frac{\partial Y_i^p}{\partial Y_a^p} = \frac{\partial Y_i^p}{\partial Y_a^p} \phi_i(\xi^p, \xi^p) = \delta_{\alpha i} \delta_{\beta i} \phi_i(\xi^p, \xi^p) = \delta_{\alpha i} \phi_i(\xi^p, \xi^p) \tag{37}
\]

that is \( \partial Y_i^p / \partial Y_a^p = \phi_i(\xi^p, \xi^p) \) if \( \alpha = i \), over Equation (36), resulting

\[
\frac{\partial^2 u_f}{\partial Y_i^p \partial Y_j^p} = \frac{\partial^2 u_f}{\partial Y_a^p \partial Y_i^p} \frac{\partial Y_i^p}{\partial Y_a^p} + \frac{\partial^2 u_f}{\partial Y_a^p \partial Y_j^p} \frac{\partial Y_j^p}{\partial Y_a^p} + \frac{\partial^2 u_f}{\partial Y_a^p \partial Y_i^p} \frac{\partial Y_i^p}{\partial Y_a^p} \frac{\partial Y_j^p}{\partial Y_a^p} + \frac{\partial^2 u_f}{\partial Y_a^p \partial Y_j^p} \frac{\partial Y_j^p}{\partial Y_a^p} \frac{\partial Y_i^p}{\partial Y_a^p} \tag{38}
\]

or

\[
\frac{\partial^2 u_f}{\partial Y_a^p \partial Y_j^p} = h_{a\beta}^f \frac{\partial Y_i^p}{\partial Y_a^p} \frac{\partial Y_i^p}{\partial Y_a^p} + h_{a\beta}^f \frac{\partial Y_i^p}{\partial Y_a^p} \frac{\partial Y_i^p}{\partial Y_a^p} + h_{a\beta}^f \frac{\partial Y_i^p}{\partial Y_a^p} \frac{\partial Y_i^p}{\partial Y_a^p} + h_{a\beta}^f \frac{\partial Y_i^p}{\partial Y_a^p} \frac{\partial Y_i^p}{\partial Y_a^p} \tag{39}
\]

where \( h^f \) is the kernel of the fibre Hessian matrix described by Equation (31). In Equation (39) index notation is not adopted.

Integrating (39) over fibre volume gives:
\[
\frac{\partial^3 U_f}{\partial Y^\text{m}_a \partial Y^\text{m}_b} = H^f \frac{\partial Y^\text{m}_a}{\partial Y^\text{m}_a} + H^f \frac{\partial Y^\text{m}_b}{\partial Y^\text{m}_a} + H^f \frac{\partial Y^\text{m}_b}{\partial Y^\text{m}_b} + H^f \frac{\partial Y^\text{m}_b}{\partial Y^\text{m}_b} + H^f \frac{\partial Y^\text{m}_b}{\partial Y^\text{m}_b}
\]

(40)

The resulting operation is the consistent spreading of fibres contribution over the matrix properties, represented by:

\[
H = H^\text{m} + H_f
\]

(41)

4.5 Spreading operation for any order fibre

This item changes Equation (40) to a matrix form, simplifying the numerical application.

The Hessian matrix of the fibre \([H^f]_{2(GP^f+1)x2(GP^f+1)}\) is expanded into a matrix of order \((4(GP^f+1)N x 4(GP^f+1)N)\), by means of a sparse matrix \([\phi^\beta]_{2(GP^f+1)x4(GP^f+1)N}\), as:

\[
H_f = \hat{H}_f = [\hat{H}_f]_{4(GP^f+1)N x 4(GP^f+1)N} = [\phi^\beta]^T_{4(GP^f+1)N x 4(GP^f+1)} \cdot [H^f]_{2(GP^f+1)x2(GP^f+1)} \cdot [\phi^\beta]_{2(GP^f+1)x4(GP^f+1)N}
\]

(42)

where \(N\) is the number of nodes of the plate element, \(GP^f\) is the order of approximation of the fibre element and \([H^f]_{2(GP^f+1)x2(GP^f+1)}\) is obtained from Equation (31).

5 Numerical examples

In this section, we present the numerical examples simulated to shown the potential of our formulation.

5.1 Cantilever beam

This example compares the displacement result achieved by the proposed formulation and the one given by a general bar software Acadframe freely distributed at http://www.set.eesc.usp.br/portal/pt/softwares [12, 13] of a clamped reinforced beam subjected to a uniform distributed load, see Figure 3.

The adopted geometrical properties are: \(L = 300\,\text{cm}\), \(h = 10\,\text{cm}\), \(b = 1\,\text{cm}\), \(d = 2.5\,\text{cm}\) and \(h' = 7.5\,\text{cm}\). The transverse applied load is \(q = 50\,N/\text{cm}\). The Young modulus and the Poisson’s ratio of the matrix are \(E_c = 21 \times 10^5\,N/\text{cm}^2\) and \(\nu = 0\),
while the Young modulus and the cross-sectional area of the fibre are 
\[ E_f = 210 \times 10^5 \text{ N/cm}^2 \] and \[ A_f = 0.1 \text{cm}^2 \].

![Figure 3: Cantilever beam.](image)

When using the proposed reinforcement procedure the matrix is discretized by 300 quadratic plate elements totalizing 671 nodes and 1342 degrees of freedom and the reinforcement is discretized by 120 linear and 60 quadratic fibre elements. The analysis made by Acadframe applies 10 bar elements of 5th order following the Thimoshenko-Reissner kinematics. The analysis is performed in 10 load steps. The achieved result for the proposed formulation is depicted in Figure 4.

![Figure 4: Deformed configuration in (cm) of the cantilever beam.](image)

Comparing the maximum deflection in the central node of the free end of the cantilever beam obtained with the proposed formulation, \( \delta \approx 187,263 \text{cm} \), with the corresponding result of the AcadFrame, \( \delta \approx 183,168 \text{cm} \), there is a relative difference of 2.235\%, showing the consistency of the proposed method. Obviously the solid solution is larger than the general bar solution, as less kinematical restrictions are made.

Moreover, it is possible to observe that the mechanical coupling between fibre and matrix is properly working.
5.2 Conform coupling between fibre and matrix

This item uses the analysis of some hypothetical plates to demonstrate the total adhesion between fibre and matrix when the approximation order of fibres is higher or equal to the plate element order.

Three analyses of a triangular plate are carried out. The first analysis adopts a single linear plate element reinforced by a vertical fibre. The fibre is modelled by three different approximation orders, from linear to third order. The other two analysis are similar, however the element order is increased from the linear to the third order plate element.

The Young modulus and the Poisson’s ratio of the matrix are $E_c = 1 \text{N/cm}^2$ and $\nu = 0$, while the Young modulus and the cross-sectional area of the fibre are $E_f = 0 \text{N/cm}^2$ and $A_f = 0.1 \text{cm}^2$ for all cases. The lower points are totally constrained and a horizontal force of $F = 5 \text{N}$ is applied on the top node. The triangle side is $L_c = 200 \text{cm}$ and the fibre length is $L_f = 170 \text{cm}$ distant $10 \text{cm}$ from the base.

The objective of the analysis is to verify whether points along the fibre (different from nodes) present relative movement regarding the continuum points after a change of configuration. It is done by choosing the fibre dimensionless coordinates of the analyzed points, in this case 12 equally spaced points. By means of fibre shape functions we determine the physical coordinates of these points at initial configuration. With these physical coordinates we determine the corresponding plate dimensionless coordinates, Equation 50. After applying the external force and finding the equilibrium nodal position, using the fibre shape functions we determine the current positions of that 12 fibre points. With the plate shape functions we determine the corresponding continuum current positions of the same points. In proper tables we compare these values for which when they are different occurs relative movement, when not, total adhesion is guaranteed.

The plate and fibres discretizations for the first analysis (linear plate element) and for the second analysis (second order plate approximation) are shows in the first line of the Figure 5 and Figure 6, respectively. The second line illustrates the horizontal displacements and the third line illustrates the vertical displacements. One observes that the plate approximation, in the first case, is always linear, and in the second case is always quadratic. The fibre approximation grows from left to right.

It is possible to infer from Figure 5 that there is no relative displacement between fibre and continuum points for any adopted fibre order.

Similarly, it is possible to infer from Figure 6 that there is relative displacement between the linear fibre and continuum points, however for quadratic (or over) fibre approximation there is no relative movement.

The first line of Figure 7 shows the plate and fibres discretizations for the third analysis (cubic plate element), the second line illustrates the horizontal displacements and the third line illustrates the vertical displacements. One observes that the plate approximation is always cubic, while the fibre approximation grows from left to right.
Figure 5: Linear two-dimensional plate element.

Figure 6: Quadratic two-dimensional plate element.
It is possible to infer from Figure 7 that there is relative displacement between the linear and quadratic approximations of fibre and continuum points, however for cubic fibre approximation there is no relative movement. This behavior is proved numerically in Table 1 in which the current positions of fibre and continuum points are compared to each other.

![Figure 7: Cubic two-dimensional plate element.](image)

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<th>POSITIONS</th>
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Table 1: Nodal positions of the cubic two-dimensional plate element.
As one can observe the conform coupling between fibre-matrix is guaranteed if the fibre approximation order at least the same as the plate element order.

6 Conclusions

A curved fibre finite element with high order approximations based on a tailoring description is developed in this paper. The use of fibre elements with equal or higher order of approximation to the continuum approximation ensures the conform coupling between the fibre and the matrix. The numerical examples show the generality and accuracy of the proposed formulation. Future works on the subject should comprise the interface stress components calculations.

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References


