# A Variational Multiscale Approach to Recover Perfect Bond in the Finite Element Analysis of Composite Beams 

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#### Abstract

A practical application in the modelling of composite beams, for designers who are usually limited to standard elements, is to connect beam element components by using rigid connections tied to the nodes or use master-slave type kinematic constraints. However, due to numerical issues this type of modelling may lead to the weakening of the intended kinematic constraints; that is, satisfaction of the perfect bond condition between the components in the point-wise sense. Therefore, this type of multiple-point constraint application provides softer behaviour than the intended perfectly bonded composite beam behaviour. The variational multiscale method is adopted herein to recover the perfect bond between the layers in the point-wise sense, based on the idea that the numerical solution space in the multiple-point constraint application can be deemed as the superfluously extended solution space because of the weakening in the kinematic constraints. Therefore, intended perfectly bonded composite beam solution is defined as the coarse-scale solution and thus, the perfect bond between the composite beam layers can be recovered by excluding the identified fine-scale effect from the solution of the multiple point constraint application. The improvements in the accuracy and convergence characteristics based on the proposed variational multiscale approach are illustrated.


Keywords: variational multiscale method; composite beams; multiple-point constraints; perfect bond; interpolation error.

## 1 Introduction

Composite beams that consist of different components juxtaposed with a shear connection find widespread applications. Numerous composite beam theories have been proposed to date to describe the kinematic behaviour and stress states of composite laminates. An early mathematical model for composite beams with
flexible shear connectors was introduced by Newmark et al. [1], in which two beams are connected by assuming that vertical separation does not occur between the components. Subsequently, several displacement-based finite element formulations were developed based on Newmark's model, e.g., [2-4]. However, in many practical cases the interlayer connections are very stiff between the two components such that the interlayer slip is negligible comparison to the deformations of the composite beam. In such cases, displacement-based finite element formulations based on flexible shear connectors suffers due to locking. In order to alleviate locking for the cases of stiff connections the strategies developed in [5-10,] can be used.

On the other hand, in engineering applications with stiff connections the classical theory, which assumes full interaction between the components [11], can also be adopted. A practical application in the modelling of composite beams based on classical theory is to use Multiple-Point Constraints (MPCs) at the nodes to bond the components together. However, this type of modelling does not provide perfect bond between the layers in the point-wise sense, and as a result full-interaction between the layers cannot be always imposed by applying MPC at the nodes. Gupta and Ma [12] pointed out this fact and noted that the source of error in MPC applications of this type is due to the incompatibility in the displacement field. A similar type of error in MPC applications for built-up plates and shells was pointed out by Crisfield [13].

An interpolated displacement field can be conceived as a displacement field of an extended interpolation space under a constraint condition [14]. Since the MPC application imposes a weaker condition than the perfectly bonded case, the interpolation space in the MPC application can be treated as a superfluously extended space, in which unwanted higher order terms or bubbles are contained [15]. Following the ideas introduced in [15-17], this study adopts the variational multiscale approach to recover the perfect bond between the layers in the finite element analysis of composite beams by excluding the identified bubbles from the solution of the MPC application. The improvements in the accuracy and convergence characteristics based on the proposed approach are illustrated.

The paper is organised as follows. The kinematics and the weak form of the equilibrium equations for composite beams are introduced in Section 2. In Section 3, finite element formulations are developed by using MPCs and alternatively by modifying the weak form of the equilibrium equations to enforce perfect bond between the layers in the point-wise sense. In Section 4, it is shown that by using the variational multiscale method the finite element formulation based on the perfect bond can be recovered from the formulation based on MPC application. In Sect. 5, a finite element formulation that provides exact values at the nodes is obtained also using the variational multiscale approach. Numerical examples are presented in Section 6 and conclusions are drawn in Section 7.

## 2 Composite beam kinematics and finite element solution

### 2.1 Displacements and strains

The composite member is made up of a top and a bottom Euler-Bernoulli beam component, which are referred to as layers 1 and 2, respectively. The deformations of the layers can be expressed in terms of the axial displacements $w_{1}$ and $w_{2}$ of their centroids and the vertical displacements $v_{1}$ and $v_{2}$ of layers 1 and 2 , respectively. Thus, the strain expression in each layer can be determined in terms of the axial displacement gradients $\mathscr{D} w_{1}$ and $\mathscr{D} w_{2}$, and the curvatures $\mathscr{D}^{2} v_{1}$ and $\mathscr{D}^{2} v_{2}$ due to bending as

$$
\begin{equation*}
\varepsilon_{1}=\mathscr{D} w_{1}-y_{1} \mathscr{D}^{2} v_{1} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\varepsilon_{2}=\mathscr{D} w_{2}-y_{2} \mathscr{D}^{2} v_{2} \tag{2}
\end{equation*}
$$

in which $\varepsilon_{1}$ and $\varepsilon_{2}$ are the strains in layers 1 and 2, respectively, $\mathscr{D}()=\mathrm{d}() / \mathrm{d} z$, and $y_{1}$ and $y_{2}$ refer to the distance of a point on the cross-section from the centroid of the associated layer.

### 2.2 Weak form of the equilibrium equations

Employing linear elastic materials, a displacement-based finite element formulation can be developed by employing the principle of virtual work, i.e.

$$
\begin{equation*}
\delta \Pi=\iint_{L A_{1}} E_{1} \varepsilon_{1} \delta \varepsilon_{1} \mathrm{~d} A \mathrm{~d} z+\int_{L} \int_{A_{2}} E_{2} \varepsilon_{2} \delta \varepsilon_{2} \mathrm{~d} A \mathrm{~d} z-\delta \Pi_{e x t}=0 \tag{3}
\end{equation*}
$$

where the first and second integrals are the virtual work done due to the deformations of the layers and $\delta \Pi_{e x t}$ is the virtual work done by the external forces. In Eq. (3), $L$ is the span of the analysis domain, $A_{1}$ and $A_{2}$ are the cross-sectional areas, and $E_{1}$ and $E_{2}$ are the moduli of elasticity for layers 1 and 2 respectively. Routinely, by substituting Eqs. (1) and (2) into Eq. (3), the weak form of equilibrium equations can be written as

$$
\begin{equation*}
\int_{L} \delta \partial \mathbf{u}^{\mathrm{T}}(z) \mathbf{D} \partial \mathbf{u}(z) \mathrm{d} z-\int_{L} \delta \mathbf{u}^{\mathrm{T}}(z) \mathbf{q}(z) \mathrm{d} z=0 \tag{4}
\end{equation*}
$$

where $\mathbf{D}$ is the matrix of the cross-sectional properties, i.e.

$$
\mathbf{D}=\left[\begin{array}{cccc}
E_{1} A_{1} & 0 & 0 & 0  \tag{5}\\
0 & E_{1} I_{1} & 0 & 0 \\
0 & 0 & E_{2} A_{2} & 0 \\
0 & 0 & 0 & E_{2} I_{2}
\end{array}\right]
$$

in which $I_{1}=\int_{A_{1}} y_{1}^{2} \mathrm{~d} A$ and $I_{2}=\int_{A_{2}} y_{2}^{2} \mathrm{~d} A$ are the second moments of the areas of the cross-sections of layers 1 and 2 with respect to horizontal axes passing through their own centroids respectively. In Eq. (4), the displacement vector $\mathbf{u}$ can be written in terms of the axial and vertical displacement components as

$$
\begin{equation*}
\mathbf{u}(z)=\left\langle w_{1}(z) \quad v_{1}(z) \quad w_{2}(z) \quad v_{2}(z)\right\rangle^{\mathrm{T}}, \tag{6}
\end{equation*}
$$

while the deformation vector $\partial \mathbf{u}$ can be written as

$$
\begin{equation*}
\partial \mathbf{u}(z)=\left\langle\mathscr{D} w_{1}(z) \quad \mathscr{D}^{2} v_{1}(z) \quad \mathscr{D} w_{2}(z) \quad \mathscr{D}^{2} v_{2}(z)\right\rangle^{\mathrm{T}}, \tag{7}
\end{equation*}
$$

In Eq. (4), $\mathbf{q}(z)$ is the vector of external actions, i.e.

$$
\begin{equation*}
\mathbf{q}(z)=\left\langle q_{w 1}(z) \quad q_{v 1}(z) \quad q_{w 2}(z) \quad q_{v 2}(z)\right\rangle^{\mathrm{T}}, \tag{8}
\end{equation*}
$$

The composite action is enforced in the next section by using two alternative approaches; MPC application at the nodes and enforcing perfect bond kinematics in the point-wise sense.

## 3 Finite element solution to enforce composite action

### 3.1 MPC application to enforce composite action

For a finite element formulation, by using Eq. (4) the weak form of the equilibrium equations can be written as

$$
\begin{equation*}
\delta \mathbf{d}^{\mathrm{T}}\left(\int_{L} \mathbf{N}^{\mathrm{T}}(z) \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathbf{N}(z) \mathrm{d} z\right) \mathbf{d}-\delta \mathbf{d}^{\mathrm{T}} \mathbf{f}=0, \tag{9}
\end{equation*}
$$

where

$$
\mathbf{N}(z)=\left[\begin{array}{cccc}
\mathbf{w}_{1}(z) & \mathbf{0} & \mathbf{0} & \mathbf{0}  \tag{10}\\
\mathbf{0} & \mathbf{v}_{1}(z) & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{w}_{2}(z) & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{v}_{2}(z)
\end{array}\right],
$$

and

$$
\mathbf{B}=\left[\begin{array}{cccc}
\mathscr{D} & 0 & 0 & 0  \tag{11}\\
0 & \mathscr{D}^{2} & 0 & 0 \\
0 & 0 & \mathscr{D} & 0 \\
0 & 0 & 0 & \mathscr{D}^{2}
\end{array}\right] .
$$

In Eq. (9), $\mathbf{f}$ is the energy equivalent nodal force vector, i.e.

$$
\begin{equation*}
\mathbf{f}=\int_{L} \mathbf{N}^{\mathrm{T}}(z) \mathbf{q}(z) \mathrm{d} z \tag{12}
\end{equation*}
$$

and $\mathbf{d}$ is the vector of nodal parameters associated with the selected interpolation functions $\mathbf{w}_{1}(z), \mathbf{w}_{2}(z), \mathbf{v}_{1}(z)$ and $\mathbf{v}_{2}(z)$ used in Eq. (10), from which the displacement field used in Eq. (9) can be expressed as

$$
\begin{equation*}
\mathbf{u}(z)=\mathbf{N}(z) \mathbf{d} \tag{13}
\end{equation*}
$$

and deformation field can be expressed as

$$
\begin{equation*}
\partial \mathbf{u}(z)=\mathbf{B N}(z) \mathbf{d} \tag{14}
\end{equation*}
$$

When two layers are bonded together, vertical separation between the two layers is prevented and the composite cross-section remains planar after the deformation. In the finite element modelling of composite beams, two separate elements can be connected conveniently at nodes by using MPCs. Assuming that the nodes of the second element are the master nodes, the nodal displacements of the first element can be obtained by using the 'no vertical separation rule' at the nodes, i.e. $v_{1}(0)=v_{2}(0)$ and $v_{1}(L)=v_{2}(L)$, and the 'plane sections remain plane rule' at the nodes i.e., $w_{1}(0)=w_{2}(0)-h \mathscr{D} v_{2}(0)$ and $w_{1}(L)=w_{2}(L)-h \mathscr{D} v_{2}(L)$, where $h=y_{1}-y_{2}$ is the distance between the centroids of the layers. By using these nodal constraint conditions in Eq. (9), the weak form of the equilibrium equations can be obtained as

$$
\begin{equation*}
\delta \overline{\mathbf{d}}^{\mathrm{T}} \tilde{\mathbf{K}} \overline{\mathbf{d}}-\delta \overline{\mathbf{d}}^{\mathrm{T}} \tilde{\mathbf{f}}=0 \tag{15}
\end{equation*}
$$

in which

$$
\begin{equation*}
\tilde{\mathbf{K}}=\mathbf{T}^{\mathrm{T}} \int_{L} \mathbf{N}^{\mathrm{T}}(z) \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathbf{N}(z) \mathrm{d} z \mathbf{T} \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\mathbf{f}}=\mathbf{T}^{\mathrm{T}} \int_{L} \mathbf{N}^{\mathrm{T}}(z) \mathbf{q}(z) \mathrm{d} z \tag{17}
\end{equation*}
$$

where $\mathbf{T}$ is the matrix of nodal constraint conditions and $\overline{\mathbf{d}}$ is the vector of restrained nodal degrees of freedom, i.e. $\mathbf{d}=\mathbf{T} \overline{\mathbf{d}}$ and $\tilde{\mathbf{f}}=\mathbf{T}^{\mathrm{T}} \mathbf{f}$.

### 3.2 Stiffness matrix due to MPC application based on the selected interpolation functions

The simplest Euler-Bernoulli beam finite element can be developed by using linear interpolations for the axial displacement and cubic interpolation for the vertical displacements, i.e.

$$
\begin{equation*}
\mathbf{w}_{1}(z)=\mathbf{w}_{2}(z)=\langle(1-z / L) \quad z / L\rangle, \tag{18}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{v}_{1}(z)=\mathbf{v}_{2}(z)=\left\langle\left(1-\frac{3 z^{2}}{L^{2}}+\frac{2 z^{3}}{L^{3}}\right)\left(z-\frac{2 z^{2}}{L}+\frac{z^{3}}{L^{2}}\right)\left(\frac{3 z^{2}}{L^{2}}-\frac{2 z^{3}}{L^{3}}\right)\left(-\frac{z^{2}}{L}+\frac{z^{3}}{L^{2}}\right)\right\rangle . \tag{19}
\end{equation*}
$$

The nodal displacement vector in this case can be written as

$$
\mathbf{d}^{\mathrm{T}}=\left\langle\begin{array}{llll}
\mathbf{w}_{N 1}^{\mathrm{T}} & \mathbf{v}_{N 1}^{\mathrm{T}} & \mathbf{w}_{N 2}^{\mathrm{T}} & \mathbf{v}_{N 2}^{\mathrm{T}} \tag{20}
\end{array}\right\rangle .
$$

where $\mathbf{w}_{N 1}^{\mathrm{T}}=\left\langle w_{1}(0) \quad w_{1}(L)\right\rangle, \mathbf{v}_{N 1}^{\mathrm{T}}=\left\langle v_{1}(0) \quad v_{1}^{\prime}(0) \quad v_{1}(L) \quad v_{1}^{\prime}(L)\right\rangle, \mathbf{w}_{N 2}^{\mathrm{T}}=\left\langle w_{2}(0) \quad w_{2}(L)\right\rangle$ and $\mathbf{v}_{N 1}^{\mathrm{T}}=\left\langle v_{1}(0) \quad v_{1}^{\prime}(0) \quad v_{1}(L) \quad v_{1}^{\prime}(L)\right\rangle$. When the nodes of the second element are the master nodes, the matrix of nodal constraint conditions can be written as

$$
\mathbf{T}=\left[\begin{array}{cccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0  \tag{21}\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-h & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & -h & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]^{\mathrm{T}}
$$

and the vector of retained nodal degrees of freedom can be written as

$$
\overline{\mathbf{d}}^{\mathrm{T}}=\left\langle\begin{array}{ll}
\mathbf{w}_{N 2}^{\mathrm{T}} & \mathbf{v}_{N 2}^{\mathrm{T}} \tag{22}
\end{array}\right\rangle,
$$

By substituting Eqs.(5), (7), (10), (18) and (19) into Eq.(16), the stiffness matrix can be obtained as

$$
\tilde{\mathbf{K}}=\left[\begin{array}{c:c}
\tilde{\mathbf{K}}_{11} & \tilde{\mathbf{R}}_{12}  \tag{23}\\
\tilde{\mathbf{K}}_{12} & \tilde{\mathbf{K}}_{22}
\end{array}\right],
$$

in which

$$
\begin{gather*}
\tilde{\mathbf{K}}_{11}=\left[\begin{array}{cc}
\frac{E_{1} A_{1}+E_{2} A_{2}}{L} & -\frac{E_{1} A_{1}+E_{2} A_{2}}{L} \\
-\frac{E_{1} A_{1}+E_{2} A_{2}}{L} & \frac{E_{1} A_{1}+E_{2} A_{2}}{L}
\end{array}\right],  \tag{24}\\
\tilde{\mathbf{K}}_{12}=\left[\begin{array}{cccc}
0 & -\frac{E_{1} A_{1} h}{L} & 0 & \frac{E_{1} A_{1} h}{L} \\
0 & \frac{E_{1} A_{1} h}{L} & 0 & -\frac{E_{1} A_{1} h}{L}
\end{array}\right], \tag{25}
\end{gather*}
$$

and

$$
\begin{equation*}
\tilde{\mathbf{K}}_{22}=\left(E_{1} I_{1}+E_{2} I_{2}\right) \mathbf{A}+E_{1} A_{1} h^{2} \mathbf{S}, \tag{26}
\end{equation*}
$$

where

$$
\mathbf{A}=\frac{1}{L^{3}}\left[\begin{array}{cccc}
12 & 6 L & -12 & 6 L  \tag{27}\\
6 L & 4 L^{2} & -6 L & 2 L^{2} \\
-12 & -6 L & 12 & -6 L \\
6 L & 2 L^{2} & -6 L & 4 L^{2}
\end{array}\right],
$$

and

$$
\mathbf{S}=\frac{1}{L}\left[\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{28}\\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{array}\right],
$$

This solution will be referred herein as Multiple-Point Constraint Solution (MPCS). It will be shown, however, that the imposition of the MPCs at the nodes is deficient in that it does not inherit the kinematic properties of the continuous problem introduced in the next section. However, the deficiency in MPCS can be remedied by using the variational multiscale method as will be shown in section 4 .

### 3.3 Finite element solution satisfying kinematic constraints in the point-wise sense

When two layers are juxtaposed, in order to enforce perfect bond between the layers (i.e. full interaction in the point-wise sense), the kinematic conditions can be modified prior to imposing finite element interpolation functions. By using the kinematic constraint conditions of no vertical separation (i.e. $v_{1}=v_{2}$ ) and the crosssection remains planar after the deformation (i.e. $w_{1}=w_{2}-h \mathscr{D} v_{2}$ ) in Eq. (3), the weak form of the equilibrium equations becomes

$$
\int_{L}\left(\left\langle\delta \mathscr{D} w_{2} \quad \delta \mathscr{D}^{2} v_{2}\right\rangle\left[\begin{array}{cc}
E_{1} A_{1}+E_{2} A_{2} & -E_{1} A_{1} h  \tag{29}\\
-E_{1} A_{1} h & E_{1} A_{1} h^{2}+E_{1} I_{1}+E_{2} I_{2}
\end{array}\right]\left\{\begin{array}{l}
\mathscr{D} w_{2} \\
\mathscr{D}^{2} v_{2}
\end{array}\right\}\right) \mathrm{d} z-\delta \Pi_{e x t}=0
$$

Similarly to the MPCS in section 3.1, a finite element solution can be developed by using linear interpolation for the axial displacement field $w_{2}$ and cubic interpolation for the vertical displacement field $v_{2}$, i.e.

$$
\begin{equation*}
\delta \overline{\mathbf{d}}^{\mathrm{T}} \overline{\overline{\mathbf{K}}} \overline{\mathbf{d}}-\delta \overline{\mathbf{d}}^{\mathrm{T}} \overline{\mathbf{f}}=0 \tag{30}
\end{equation*}
$$

in which

$$
\begin{equation*}
\overline{\overline{\mathbf{K}}}=\int_{L} \overline{\mathbf{N}}^{\mathrm{T}}(z) \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{T}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{T}} \overline{\mathbf{B}} \overline{\mathbf{N}}(z) \mathrm{d} z, \tag{31}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathbf{f}}=\int_{L} \overline{\mathbf{N}}^{\mathrm{T}}(z) \overline{\mathbf{t}}^{\mathrm{T}} \mathbf{q}(z) \mathrm{d} z, \tag{32}
\end{equation*}
$$

where $\overline{\mathbf{f}}$ is the energy equivalent nodal force vector. In Eqs. (31) and (32),

$$
\begin{gather*}
\overline{\mathbf{T}}=\left[\begin{array}{cccc}
1 & 0 & 1 & 0 \\
-h & 1 & 0 & 1
\end{array}\right]^{\mathrm{T}},  \tag{33}\\
\overline{\mathbf{B}}=\left[\begin{array}{cc}
\mathscr{D} & 0 \\
0 & \mathscr{D}^{2}
\end{array}\right],  \tag{34}\\
\overline{\mathbf{N}}(z)=\left[\begin{array}{cc}
\mathbf{w}_{2}(z) & 0 \\
0 & \mathbf{v}_{2}(z)
\end{array}\right], \tag{35}
\end{gather*}
$$

and

$$
\overline{\mathbf{t}}=\left[\begin{array}{cccc}
1 & 0 & 1 & 0  \tag{36}\\
-h \mathscr{D} & 1 & 0 & 1
\end{array}\right]^{\mathrm{T}}
$$

are used. By substituting Eqs. (5), (7), (10), (18) and (19) into Eq. (16), the stiffness matrix can be obtained as

$$
\overline{\overline{\mathbf{K}}}=\left[\begin{array}{c:c}
\overline{\overline{\mathbf{K}}}_{11} & \overline{\overline{\mathbf{K}}}_{12}  \tag{37}\\
\hdashline \overline{\overline{\mathbf{K}}}_{12} & \overline{\mathbf{K}}_{22}
\end{array}\right],
$$

in which $\overline{\overline{\mathbf{K}}}_{11}=\tilde{\mathbf{K}}_{11}, \overline{\overline{\mathbf{K}}}_{12}=\tilde{\mathbf{K}}_{12}$ and $\overline{\overline{\mathbf{K}}}_{22}$ can be written as

$$
\begin{equation*}
\overline{\overline{\mathbf{K}}}_{22}=\left(E_{1} I_{1}+E_{2} I_{2}+E_{1} A_{1} h^{2}\right) \mathbf{A} \tag{38}
\end{equation*}
$$

This solution will be referred herein as Perfect-Bond Kinematics Finite element Solution (PBKFS).

## 4 Enforcement of perfect bond kinematics

### 4.1 Variational multiscale approach

In this section, by using the variational multiscale approach, the finite element formulation of the perfectly bonded case in section 3.3 will be produced by using the finite element formulation based on MPC application introduced in section 3.2. In the variational multiscale approach [16], the displacement field vector $\mathbf{u}$ is decomposed into coarse and fine-scale displacement fields, $\overline{\mathbf{u}}$ and $\mathbf{u}^{\prime}$ respectively, i.e. $\mathbf{u}=\overline{\mathbf{u}}+\mathbf{u}^{\prime}$. The fine-scale displacement field vanishes at the element boundaries, i.e. $\mathbf{u}^{\prime}(0)=\mathbf{u}^{\prime}(L)=\mathbf{0}$ and the spaces of the course- and fine-scale functions are linearly independent. Likewise, the deformation field is decomposed herein into two linearly independent components, i.e. $\partial \mathbf{u}=\partial \overline{\mathbf{u}}+\partial \mathbf{u}^{\prime}$ where the coarse-scale deformation field is $\partial \overline{\mathbf{u}}=\overline{\mathbf{T}} \overline{\mathbf{B}} \overline{\mathbf{N}}(z) \overline{\mathbf{d}}$, in which $\overline{\mathbf{T}}, \overline{\mathbf{B}}$ and $\overline{\mathbf{N}}(z)$ are as shown in Eqs. (33), (34) and (35), respectively. The fine-scale displacement field can in general be expressed as $\mathbf{u}^{\prime}=\mathbf{N}^{\prime}(z) \mathbf{d}^{\prime}$, where $\mathbf{N}^{\prime}$ denotes the vector of fine-scale interpolation functions such as bubble functions or $p$-refinements, and $\mathbf{d}^{\prime}$ denotes the vector of fine-scale nodal parameters. From the fine-scale displacement field, the fine-scale deformation field can be obtained as $\partial \mathbf{u}^{\prime}=\mathbf{B N}^{\prime}(z) \mathbf{d}^{\prime}$. The aim is to recover the finite element formulation based on perfect bond between the composite beam layers in the point-wise sense by excluding the fine-scale effect from the finite element formulation based on MPC application. For this purpose, Eq. (15) is split into two interacting forms by using the variational multiscale approach, i.e.

$$
\begin{equation*}
\delta \overline{\mathbf{d}}^{\mathrm{T}} \overline{\overline{\mathbf{K}}} \overline{\mathbf{d}}+\delta \overline{\mathbf{d}}^{\mathrm{T}} \overline{\mathbf{K}}^{\prime} \mathbf{d}^{\prime}-\delta \overline{\mathbf{d}}^{\mathrm{T}} \overline{\mathbf{f}}=0 \tag{39}
\end{equation*}
$$

and

$$
\begin{equation*}
\delta \mathbf{d}^{\mathrm{T}} \overline{\mathbf{K}}^{\prime \mathrm{T}} \overline{\mathbf{d}}+\delta \mathbf{d}^{\mathrm{T}} \mathbf{K}^{\prime \prime} \mathbf{d}^{\prime}-\delta \mathbf{d}^{\mathrm{T}} \mathbf{f}^{\prime}=0 \tag{40}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{K}^{\prime \prime}=\int_{L} \mathbf{N}^{\prime \mathrm{T}}(z) \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \mathbf{N}^{\prime}(z) \mathrm{d} z, \tag{41}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathbf{K}}^{\prime}=\int_{L} \overline{\mathbf{N}}^{\mathrm{T}}(z) \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{T}}^{\mathrm{T}} \mathbf{D B} \mathbf{N}^{\prime}(z) \mathrm{d} z, \tag{42}
\end{equation*}
$$

It should be noted that in obtaining Eqs. (39) and (40) from Eq. (15), an additive split for the weighting functions was also introduced, i.e. $\delta \mathbf{u}=\delta \overline{\mathbf{u}}+\delta \mathbf{u}^{\prime}$. In between the above two interacting equations, the fine-scale nodal parameters, i.e. $\mathbf{d}^{\prime}$ in Eq. (39) can be eliminated in order to be left with a problem entirely in terms of the coarse-scale solution. The elimination of $\mathbf{d}^{\prime}$ can be achieved by using the usual static condensation, and thus from Eq. (40) the vector of fine scale nodal parameters $\mathbf{d}^{\prime}$ can be calculated as

$$
\begin{equation*}
\mathbf{d}^{\prime}=\mathbf{K}^{\prime \prime-1}\left[\overline{\mathbf{f}}-\overline{\mathbf{K}}^{\prime T} \overline{\mathbf{d}}\right] \tag{43}
\end{equation*}
$$

By substituting Eq. (43) into Eq. (39), the equilibrium equations that consider the effect of the fine-scale solution can be written as

$$
\begin{equation*}
\tilde{\mathbf{K}} \overline{\mathbf{d}}=\tilde{\mathbf{f}}, \tag{44}
\end{equation*}
$$

in which $\tilde{\mathbf{K}}=\overline{\mathbf{K}}-\overline{\mathbf{K}}^{\prime} \mathbf{K}^{\prime \prime-1} \overline{\mathbf{K}}^{\prime \mathrm{T}}$ and $\tilde{\mathbf{f}}=\overline{\mathbf{f}}-\overline{\mathbf{K}}^{\prime} \mathbf{K}^{\prime \prime-1} \mathbf{f}^{\prime}$. From the results based on Eq. (15), by substituting into Eq. (44) the coarse scale solution can be extracted as

$$
\begin{equation*}
\overline{\overline{\mathbf{K}}}=\tilde{\mathbf{K}}+\overline{\mathbf{K}}^{\prime} \mathbf{K}^{\prime \prime-1} \overline{\mathbf{K}}^{\prime \mathrm{T}} \tag{45}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathbf{f}}=\tilde{\mathbf{f}}+\overline{\mathbf{K}}^{\prime} \mathbf{K}^{\prime-1} \mathbf{f}^{\prime} \tag{46}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{f}^{\prime}=\int_{L} \mathbf{N}^{\prime \mathrm{T}}(z) \mathbf{q}(z) \mathrm{d} z, \tag{47}
\end{equation*}
$$

By using Eqs. (45) and (46), the equilibrium equations for the composite beam with perfect bond in Eq. (30) can be recovered. In the next section, the interpolation space of the fine-scale displacement field is identified.

### 4.2 Fine-scale displacement field and the associated Green's function

The displacement field used in MPCS, i.e. $\mathbf{u}=\mathbf{N T} \overline{\mathbf{d}}$, is based on linear interpolations for both the axial displacement fields $w_{1}$ and $w_{2}$, and cubic interpolations for both the vertical displacements fields $v_{1}$ and $v_{2}$. On the other
hand, the displacement field in PBKFS, i.e. $\overline{\mathbf{u}}=\overline{\mathbf{t}} \overline{\mathbf{N}} \overline{\mathbf{d}}$, is based on a quadratic axial displacement field for $w_{1}$ because the vertical displacement field $v_{2}$ is cubic, i.e. $w_{1}=w_{2}-h \mathscr{D} v_{2}$. Considering the difference in the interpolation spaces of the two solutions, the vector of the fine-scale interpolation functions can be determined as

$$
\mathbf{N}^{\prime}(z)=\left\{\begin{array}{c}
\frac{z}{L}\left(1-\frac{z}{L}\right)  \tag{48}\\
0 \\
0 \\
0
\end{array}\right\}
$$

Despite the fact that it contains higher order terms PBKFS is identified herein as the coarse-scale solution, which may seem initially counter-intuitive. Therefore, in order to clarify why PBKFS is the coarse scale solution, an argument based on the hierarchical enrichment procedure and static condensation of the associated hierarchical degrees-of-freedom [6] is presented in the following. For this purpose, it is crucial to note that in PBKFS the quadratic bubble is enforced without having its own associated hierarchical degree-of-freedom, which prevents PBKFS from attaining the optimal solution considering the space of all interpolations of the axial displacement field $w_{1}$ up to quadratic order. Comparison to PBKFS, MPCS is the optimal solution in the sense that when the axial displacement fields $w_{1}$ are enriched using quadratic bubble functions with associated hierarchical degrees-of-freedom, i.e. $\quad w_{1}(z)=\mathbf{w}_{1}(z) \mathbf{w}_{N_{1}}^{\mathrm{T}}+b \frac{z}{L}\left(1-\frac{z}{L}\right)$, degrees-of-freedom associated with the quadratic bubble, i.e. $b$ will not be activated. This means, MPCS is the optimal solution within the space of all interpolations of the axial displacement field $w_{1}$ up to quadratic order. When the extra degrees-of-freedom, i.e. $b$ is condensed out, there will be no change in the MPCS stiffness matrix $\tilde{\mathbf{K}}$. This is because there is no coupling between the hierarchical degrees of freedom $b$ and those based on the linear interpolation $\mathbf{w}_{N_{1}}^{\mathrm{T}}$, i.e.

$$
\begin{equation*}
\int_{L}\left(\frac{\mathrm{~d} \mathbf{w}_{1}(z)}{\mathrm{d} z} \mathbf{w}_{N_{1}}^{\mathrm{T}} E_{1} A_{1} \frac{\mathrm{~d}\left(\frac{z}{L}-\frac{z^{2}}{L^{2}}\right)}{\mathrm{d} z} b\right) \mathrm{d} z=E_{1} A_{1}\left[\left(\frac{w_{1}(L)-w_{1}(0)}{L}\right) \times\left.\left(\frac{z}{L}-\frac{z^{2}}{L^{2}}\right)\right|_{0} ^{L} b\right]=0 \tag{49}
\end{equation*}
$$

Among those solutions where multiple-point constraints are applied to connect composite beam layers, PBKFS is prevented from attaining the optimal solution due to further enforcement of the perfect bond in the point-wise sense, and therefore it is considered herein as the coarse-scale solution.

### 4.3 Green's function associated with the fine-scale field

From Eq. (43), the fine-scale displacement field, i.e. $\mathbf{u}^{\prime}=\mathbf{N}^{\prime} \mathbf{d}^{\prime}$, can be expressed as

$$
\begin{equation*}
\mathbf{u}^{\prime}(\varsigma)=-\int_{L} \tilde{\mathbf{g}}^{\prime}(\varsigma, z)\left[\overline{\mathbf{K}}^{\prime \mathrm{T}} \overline{\mathbf{d}}-\mathbf{f}^{\prime}\right] \mathrm{d} z, \tag{50}
\end{equation*}
$$

in which $\tilde{\mathbf{g}}^{\prime}(\varsigma, z)$ is the fine-scale Green's function, i.e.

$$
\begin{equation*}
\tilde{\mathbf{g}}^{\prime}(\varsigma, z)=\delta(\varsigma-z) \mathbf{N}^{\prime}(z) \mathbf{K}^{\prime \prime-1}, \tag{51}
\end{equation*}
$$

where $\delta$ denotes the Dirac delta function and matrix the $\mathbf{K}^{\prime \prime}$ can be obtained by substituting Eq. (48) into Eq. (41) as

$$
\begin{equation*}
\mathbf{K}^{\prime \prime}=\frac{E_{1} A_{1}}{3 L} \tag{52}
\end{equation*}
$$

By substituting Eqs. (35) and (48) into Eq. (42), matrix $\overline{\mathbf{K}}^{\prime}$ can be obtained as

$$
\overline{\mathbf{K}}^{\prime}=\left\langle\begin{array}{llllll}
0 & 0 & \frac{2 E_{1} A_{1} h}{L^{2}} & \frac{E_{1} A_{1} h}{L} & -\frac{2 E_{1} A_{1} h}{L^{2}} & \frac{E_{1} A_{1} h}{L} \tag{53}
\end{array}\right\rangle^{\mathbf{T}},
$$

### 4.4 Fine-scale effect on the stiffness matrix and the load vector

By using Eqs. (52) and (53) in Eq. (45), the difference between the stiffness matrices, i.e. $\overline{\overline{\mathbf{K}}}-\tilde{\mathbf{K}}$ can be calculated as

$$
\overline{\mathbf{K}}^{\prime} \mathbf{K}^{\prime \prime-1} \overline{\mathbf{K}}^{\prime \mathrm{T}}=\left[\begin{array}{c:c}
\mathbf{0} & \mathbf{0}  \tag{54}\\
\hdashline \mathbf{0} & E_{1} A_{1} \bar{h}^{2} \mathbf{A}-E_{1} A_{1} \bar{h}^{2} \mathbf{S}
\end{array}\right],
$$

From Eqs. (26) and (38) it can be verified that the difference between the stiffness matrix that provides perfect bond in the point-wise sense (i.e. $\overline{\overline{\mathbf{K}}}$ ) and the stiffness matrix of the multiple-point constraint application (i.e. $\tilde{\mathbf{K}}$ ) can be obtained by using the variational multiscale method. On the other hand, the difference in the energy equivalent load vector ( $\overline{\mathbf{f}}-\tilde{\mathbf{f}}$ in Eq. (46)) for the vector of external actions given in Eq. (8) can be calculated as

$$
\overline{\mathbf{K}}^{\prime} \mathbf{K}^{\prime \prime-1} \mathbf{f}^{\prime}=\int_{L} q_{w 1}(z) \frac{z}{L}\left(1-\frac{z}{L}\right) \mathrm{d} z\left\langle\begin{array}{lllll}
0 & 0 & \frac{6 h}{L} & 3 h & -\frac{6 h}{L} \tag{55}
\end{array} 3 h\right\rangle^{\mathbf{T}},
$$

From Eqs. (17) and (32) it can be verified that the difference is between the two vectors is as given in Eq. (55). It should be noted that for a directly applied nodal load $P$, i.e. $q_{w 1}(z)=\delta(z) P$ and/or $q_{w 1}(z)=\delta(z-L) P$, Eq. (55) vanishes, i.e. $\overline{\mathbf{f}}=\tilde{\mathbf{f}}$.

## 5 Exact solution at the nodes

### 5.1 Differential equations and interpolation functions

From Eq.(29), by using integration by parts, the weak form of the differential equilibrium equations are obtained as

$$
\left.\int_{L}\left(\begin{array}{ll}
\left\langle\delta w_{2}\right. & \left.\delta v_{2}\right)
\end{array}\right)\left[\begin{array}{cc}
\left(E_{1} A_{1}+E_{2} A_{2}\right) \mathscr{D}^{2} & -E_{1} A_{1} h \mathcal{D}^{3}  \tag{56}\\
-E_{1} A_{1} h \mathscr{D}^{3} & \left(E_{1} A_{1} h^{2}+E_{1} I_{1}+E_{2} I_{2}\right) \mathcal{D}^{4}
\end{array}\right]\left\{\begin{array}{l}
w_{2} \\
v_{2}
\end{array}\right\}\right) \mathrm{d} z-\delta \Pi_{\text {ext }}-\left.\delta \Pi_{\text {Boun }}\right|_{0} ^{L}=0,(.
$$

The boundary conditions are satisfied at $z=0$ and $z=L$, thus the last term in Eq. (56), i.e. $\left.\delta \Pi_{B o u n}\right|_{0} ^{L}$ vanishes. The solution of the above differential equations contained in Eq. (56) can be obtained as

$$
\left\{\begin{array}{c}
w_{2}(z)  \tag{57}\\
v_{2}(z)
\end{array}\right\}=\left\{\begin{array}{c}
w_{2 h}(z) \\
v_{2 h}(z)
\end{array}\right\}+\left\{\begin{array}{c}
w_{2 p}(z) \\
v_{2 p}(z)
\end{array}\right\}=\overline{\mathbf{N}}_{e}(z) \overline{\mathbf{d}}+\overline{\mathbf{u}}_{p}(z)
$$

in which the first and second terms are the homogenous and particular solutions, respectively. In Eq. (57), $\overline{\mathbf{N}}_{e}(z)$ can be written as

$$
\overline{\mathbf{N}}_{e}(z)=\left[\begin{array}{cc}
\mathbf{w}_{2}(z) & \frac{E_{1} A_{1} h}{E_{1} A_{1}+E_{2} A_{2}}\left[\mathscr{D} \mathbf{v}_{2}(z)-\mathbf{M}^{*}(z)\right]  \tag{58}\\
0 & \mathbf{v}_{2}(z)
\end{array}\right]
$$

in which

$$
\mathbf{M}^{*}(z)=\left\langle\begin{array}{llll}
0 & (1-z / L) & 0 & z / L \tag{59}
\end{array}\right.
$$

is used. By using the homogenous solution of Eq. (56) as the interpolation, and thus replacing $\overline{\mathbf{N}}_{e}(z)$ with $\overline{\mathbf{N}}(z)$ in Eq.(31), the stiffness matrix of the finite element formulation can be obtained as

$$
\begin{equation*}
\overline{\overline{\mathbf{K}}}_{e}=\int_{L} \overline{\mathbf{N}}_{e}^{\mathrm{T}}(z) \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{T}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{T}} \overline{\mathbf{B}} \overline{\mathbf{N}}_{e}(z) \mathrm{d} z, \tag{60}
\end{equation*}
$$

where

$$
\overline{\overline{\mathrm{K}}}_{e}=\left[\begin{array}{c:c}
\overline{\overline{\mathrm{K}}}_{11} & \overline{\overline{\mathrm{~K}}}_{12}  \tag{61}\\
\hdashline \overline{\mathrm{~K}}_{12} & \overline{\overline{\mathrm{~K}}}_{e 22}
\end{array}\right],
$$

Comparison to the matrix components in Eq. (37), the difference is only due to the sub-matrix $\overline{\overline{\mathbf{K}}}_{e 22}$ which can be written as

$$
\begin{equation*}
\overline{\overline{\mathbf{K}}}_{e 22}=\left[E_{1} I_{1}+E_{2} I_{2}+\frac{E_{1} A_{1} E_{2} A_{2} h^{2}}{E_{1} A_{1}+E_{2} A_{2}}\right] \mathbf{A}+\left[E_{1} A_{1} h^{2}-\frac{E_{1} A_{1} E_{2} A_{2} h^{2}}{E_{1} A_{1}+E_{2} A_{2}}\right] \mathbf{S}, \tag{62}
\end{equation*}
$$

On the other hand, by using Eq. (58), the energy equivalent force vector can be obtained as

$$
\begin{equation*}
\overline{\mathbf{f}}_{e}=\int_{L} \overline{\mathbf{N}}_{e}^{\mathrm{T}}(z) \overline{\mathbf{t}}^{\mathrm{T}} \mathbf{q}(z) \mathrm{d} z, \tag{63}
\end{equation*}
$$

This solution will be referred herein as Perfect-Bond Kinematics Exact element Solution (PBKES). It should be noted that regardless of the load vector $\mathbf{q}(z)$, the nodal displacement values obtained by using $\overline{\overline{\mathbf{K}}}_{e}$ and $\overline{\mathbf{f}}_{e}$ are exact. A proof of exactness can be found in [15].

### 5.2 Variational multiscale approach to derive exact element stiffness matrix and the load vector

By using the variational multi-scale approach it can be shown that PBKFS can be modified to obtain the stiffness matrix and the energy equivalent load vector of PBKES which is an improved solution due to enriched interpolation space however, based on the same perfect-bond kinematics that is satisfied in the point-wise sense. Considering the interpolation spaces of PBKFS which is based on linear interpolations for the axial displacement field $w_{2}$, and cubic interpolations for both the vertical displacements field $v_{2}$ and PBKES which is based on the quadratic interpolation of the axial displacement field $w_{2}$, and cubic interpolations of the vertical displacements field $v_{2}$, the vector of the fine-scale interpolation functions can be determined as

$$
\mathbf{N}_{e}^{\prime}(z)=\left\{\begin{array}{c}
\frac{z}{L}\left(1-\frac{z}{L}\right)  \tag{64}\\
0
\end{array}\right\},
$$

In the variational multiscale approach, by considering PBKFS as the coarse-scale solution and using $\mathbf{N}_{e}^{\prime}(z)$ introduced in Eq. (64) as the vector of fine-scale
interpolation functions, it can be verified that the stiffness matrix $\overline{\overline{\mathbf{K}}}_{e}$ and the energy equivalent load vector of PBKES given in Eqs. (60) and (63), respectively can be obtained as

$$
\begin{equation*}
\overline{\overline{\mathbf{K}}}_{e}=\overline{\overline{\mathbf{K}}}-\overline{\mathbf{K}}_{e}^{\prime} \mathbf{K}^{\prime \prime-1} \overline{\mathbf{K}}_{e}^{\prime \mathrm{T}}, \tag{65}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\mathbf{f}_{e}}=\overline{\mathbf{f}}-\overline{\mathbf{K}}_{e}^{\prime} \mathbf{K}_{e}^{\prime-1} \mathbf{f}_{e}^{\prime}, \tag{66}
\end{equation*}
$$

in which

$$
\begin{align*}
& \mathbf{K}_{e}^{\prime \prime}=\int_{L} \mathbf{N}_{e}^{\prime \mathrm{T}}(z) \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{T}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{B}} \mathbf{N}_{e}^{\prime}(z) \mathrm{d} z,  \tag{67}\\
& \overline{\mathbf{K}}_{e}^{\prime}=\int_{L} \overline{\mathbf{N}}^{\mathrm{T}}(z) \overline{\mathbf{B}}^{\mathrm{T}} \overline{\mathbf{T}}^{\mathrm{T}} \mathbf{D} \overline{\mathbf{T}} \mathbf{N}_{e}^{\prime}(z) \mathrm{d} z, \tag{68}
\end{align*}
$$

and

$$
\begin{equation*}
\mathbf{f}_{e}^{\prime}=\int_{L} \mathbf{N}_{e}^{\prime \mathrm{T}}(z) \overline{\mathbf{t}}^{\mathrm{T}} \mathbf{q}(z) \mathrm{d} z, \tag{69}
\end{equation*}
$$

were used. It is noted that PBKFS is the coarse-scale solution in both kinematic models, i.e. the models in Section 3.1 and Section 3.3. Originality of this study is in the identification of the perfectly-bonded finite element solution PBKFS as the coarse scale solution based on the model in Section 3.1. On the other hand, PBKFS can also be identified as the coarse scale solution as in Section 5.1 when obtaining PBKES. In order to further clarify the differences between MPCS and PBKES, it should be noted that when a finite element formulation is developed according to MPCS formulation axial displacement fields of both layers are interpolated, whereas according to the model PBKES formulation only the axial displacement field $\mathbf{w}_{2}(z)$ is interpolated. The flow of these relationships is presented in Fig.1.


Figure 1: Flow of the relationships

## 6 Applications

### 6.1 Numerical performances of the finite element solutions

In order to illustrate the numerical performance of the solutions discussed herein, a 2 m span composite cantilever beam is analysed. As shown in Fig. 2, the top component of the composite beam has a modulus of elasticity of $E_{1}=200 \times 10^{3} \mathrm{MPa}$, cross-sectional area of $A_{1}=6 \times 10^{3} \mathrm{~mm}^{2}$ and second moment of area of $I_{1}=112.2 \times 10^{3}$ $\mathrm{mm}^{4}$. The bottom component has a modulus of elasticity of $E_{2}=26 \times 10^{3} \mathrm{MPa}$, crosssectional area of $A_{2}=7.1 \times 10^{3} \mathrm{~mm}^{2}$ and second moment of area of $I_{2}=124 \times 10^{6} \mathrm{~mm}^{4}$. For the MPCS the master nodes are selected as the nodes of the bottom component and slave nodes are the nodes of the top component, i.e. $h=163 \mathrm{~mm}$. The beam is analysed under different loading cases in order to illustrate the effect of the loading on the numerical error and deflections are plotted at the centroidal axis of the bottom component.


Figure 2: Cantilever beam with two layers
Under a uniform bending moment, which is imposed by applying a 100 kNm moment at the tip, the vertical deflection of the beam is as shown in Fig. 3.a and the axial deflection of the at the centroid of layer 2 is as shown in Fig. 3.b, from which it can be observed that both the MPCS and PBKFS are in perfect agreement with PBKES which is the exact solution. Thus, it can also be concluded that under uniform bending moment, perfect bond between the layers is satisfied in the pointwise sense in both finite element solutions.


Figure 3: Deflections under uniform bending moment

Under a vertical tip load of 50 kN which causes a uniform internal shear force distribution along the beam, the vertical and the axial deflection at the centroid of layer 2 are as shown in Fig. 4.a and Fig. 4.b respectively. The differences in the solutions due to numerical errors are clear when beam carries shear load. From Fig. 4.a it can be observed that one-element MPCS is softer than both PBKFS and PBKES; On the other hand, the finite element behaviour based on the perfect bond kinematics, i.e. PBKFS, is stiffer than the exact solution, i.e. PBKES due to interpolation errors. However, for the four-element solutions the interpolation errors reduce significantly and the results converge to the exact solution. As shown in 4.b. linear axial deflection curves in both based on both MPCS and PBKFS converge to the parabolic exact solution when element numbers are increased from one to four.


Figure 4: Deflections under tip load
The accuracies of the numerical solutions are evaluated based on the normalised error according to the strain energy due to exact solution. The convergence rate (i.e. $p$ in $\|e\|_{e n}=C a^{p}$ where $C$ is an arbitrary constant and $a$ is the element size) can be obtained from the slope of the log-log error curve shown in Fig. 5, from which it can be verified that both the accuracy and convergence rate of PBKFS are better than those of MPCS. The slope for MPCS is 9.4, whereas the slope for PBKFS is 10.5 .


Figure 5: Convergence rate of the elements under tip load
A similar pattern can be observed under a uniformly distributed load of $100 \mathrm{kN} / \mathrm{m}$ applied to the composite cantilever beam. Compared to the previous case, the accuracies of the solutions based on both MPCS and PBKFS reduce as can be observed from Fig. 6. From the slope of Fig. 6, the rate of the convergence for

MPCS and PBKFS can be determined as 7.7 and 8.8 respectively. Thus, it can be verified that the convergence rates reduce in comparison to the previous case of tip load.


Figure 6: Convergence rate of the elements under uniformly distributed load
Vertical deflections in this case are shown in Fig. 7.a from which it can be verified that due to numerical errors one-element MPCS is softer and one-element PBKFS is stiffer than the exact solution. Fig. 7.b shows the axial displacement at the centroid of layer 2, from which it can be verified that the nodal values are in perfect agreement however, the curves converge when the number of elements is increased.


Figure 7: Deflections under uniformly distributed load

## 7 Conclusions

In finite element analysis, when composite beams are formed by using multiplepoint constraints at the nodes, perfect bond between the layers cannot be satisfied in the point-wise sense in some cases, and thus the behaviour of the beam can be overly flexible comparison to the behaviour of the perfectly bonded composite beam. In this study, the weakening in the kinematic constraints was considered as being due to the superfluous extension of the interpolation space. By considering this extension in the interpolation space, the numerical solution for the perfectly bonded composite beam could be obtained by using the results of the multiple-point constraint application within the frame work of the variational multiscale approach. By excluding the fine-scale effects, the correction terms in the stiffness matrix and the energy equivalent load vector could be obtained. A finite element formulation
that provides exact values at the nodes was also developed using the variational multiscale approach. Effects of external load types on the numerical error in multiple-point constraints applications were illustrated.

## References

[1] N.M. Newmark, C.P. Siess, I.M. Viest, "Tests and analysis of composite beams with incomplete interaction", Proceedings of the Society of Experimental Stress Analysis, 9(1), 75-92, 1951.
[2] Y. Arizumi, S. Yamada, T.Kajita, "Elastic-plastic analysis of composite beams with incomplete interaction by finite element method", Computers \& Structures, 14, 543-462, 1981.
[3] B. Daniels, M. Crisinel, "Composite slab behaviour and strength analysis. Part I: Calculation procedure", Journal of Structural Engineering (ASCE), 119(1), 16-35, 1993.
[4] G. Ranzi, M.A. Bradford, B. Uy, "A direct stiffness analysis of a composite beam with partial interaction", International Journal for Numerical Methods in Engineering, 61(5), 657-672, 2004.
[5] A. Dall'Asta, A. Zona, "Non-linear analysis of composite beams by a displacement approach", Computers \& Structures, 80, 2217-2228, 2002.
[6] A. Dall'Asta, A. Zona, "Slip locking in finite elements for composite beams with deformable shear connection", Finite Elements in Analysis and Design, 40, 1907-1930, 2004.
[7] R.E. Erkmen, M.A. Bradford, "Elimination of slip-locking in composite beamcolumn analysis by using the element-free Galerkin method", Computational Mechanics, 46, 911-924, 2011.
[8] R.E. Erkmen, M.A. Bradford, "Treatment of slip-locking for displacement based finite element analysis of composite beam-columns International Journal for Numerical Methods in Engineering, 85(11), 805-826, 2011.
[9] R.E. Erkmen, M.A. Bradford, "Locking-free finite element formulation for steel concrete composite members", $9^{\text {th }}$ World Congress on Computational Mechanics, Sydney, Australia, 2010.
[10] R.E. Erkmen, M.M. Attard, "Displacement-based finite element formulations for material-nonlinear analysis of composite beams and treatment of locking behaviour", Finite Elements in Analysis and Design, 47, 1293-1305, 2011.
[11] J.N. Reddy, "An evaluation of equivalent-single-layer and layerwise theories of composite laminates", Composite Structures, 25, 21-35, 1993
[12] A.K. Gupta, S.M. Paul, "Error in eccentric beam formulation", International Journal for Numerical Methods in Engineering, 11, 1473-1483, 1977.
[13] M.A. Crisfield, "The eccentricity issue in the design of plate and shell elements", Communications in Applied Numerical Methods, 7, 47-56,1991.
[14] R. Courant, D. Hilbert, "Methods of mathematical physics", Wiley, New York 1962.
[15] R.E. Erkmen, M.A. Bradford, K. Crews, "Variational multiscale approach to enforce perfect bond in multiple-point constraint applications when forming composite beams", Computational Mechanics (accepted $17^{\text {th }}$ November 2011).
[16] T.J.R. Hughes, "Multiscale phenomena: Green's functions, the Dirichlet-toNeumann formulation, subgrid scale models, bubbles and the origins of stabilized methods", Computer Methods in Applied Mechanics and Engineering, 127, 387-401, 1995.
[17] T.J.R Hughes, G.R. Feijoo, L. Mazzei, J-B. Quincy, "The variational multiscale method-A paradigm for computational mechanics", Computer Methods in Applied Mechanics and Engineering, 166, 3-24, 1998.

