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On the Convergence of a Refined Nonconforming Thin Plate Bending Finite Element

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Abstract

In this paper an improvement of the displacement-based nonconforming quadrilateral thin plate bending finite element RPQ4 proposed by Wanji and Cheung [1] is presented. It is found that element RPQ4 is only conditionally unisolvent. The improvement is achieved by adding three new interpolation base shape functions, which reduces the nonunisolvence and decreases the stiffness matrix condition number. This convenient property makes the element more robust and thus better suited for computations. The convergence of such an improved element is proved and the rate of convergence estimated. The mathematically proof of convergence is based on Stummel's generalized patch test and the consideration of the element approximability condition, which are both necessary and sufficient for convergence.

Keywords: nonconforming thin plate bending finite element, convergence, generalized patch test, nonconvex quadrilateral.

1 Introduction

Practical modeling of plate-like structures is often performed using displacement-based thin plate bending finite elements. Due to the C1-continuity requirement across the element borders their theoretical formulation is difficult. This problem could be resolved by using a weakly continouos nonconforming plate element, yet its convergence should be theoretically proved.

The element RPQ4 introduced by Wanji and Cheung [1] serves as an example of an innovative nonconforming quadrilateral thin plate bending finite element which fulfills the so called weak continuity conditions across the element borders. After extensive numerical comparisons with a number of different plate elements, the authors [1] concluded the high accuracy of the element. The element passes the Irons' numerical patch test [2, 3, 4, 5, 6]. However it has been shown [7, 8, 9, 10] that this test is neither a necessary nor a sufficient condition for convergence. So its convergence should be theoretically proved.

The present paper is aimed to mathematically prove that necessary and sufficient conditions for convergence of RPQ4 element are satisfied indeed. Our derivation of the proof is based on Stummel's generalized patch test and the approximability condition [7, 11], but is somewhat unusual and unique regarding to [12] in order to incorporate the specific type of the weak continuity conditions [1]. The generalized patch test cannot generally be applied to a broad class of elements and should normally be performed on each particular element as also discussed by Wanji [5]. A rare example of the convergence analysis proof according to Stummel's generalized patch test performed on a whole class of nonconforming simplex elements was presented by Wang [4]. Due to a different element geometry and specific type of the weak continuity conditions in [1], his findings cannot be directly reproduced here. See [13] for the approach in this direction. In addition to the convergence proof, the error estimates are also derived using partially the methodology of Shi [12] and Flajs and Saje [14] and some inequalities derived by Brenner and Scott [15] and Verfürth [16].

These theoretical results show that the RPQ4 finite element is convergent even if its shape is nonconvex, provided that its degrees of freedom are unisolvent [11]. Element RPQ4 is thus advantageous over standard isoparametric thin plate elements due to its capability of describing nonconvexly shaped elements.

A disadvantage of the RPQ4 elements is possible violence of the unisolvence condition, being the fundamental requirement for convergence [11] which may endanger the applicability of element RPQ4 for randomly designed and/or very dense element meshes. In the present paper, the unisolvence problem of element RPQ4 is theoretically and practically resolved by a unique introduction of the three additional interpolation base shape functions, thus turning off this drawback while introducing a minor correction of the element. The interpolation base shape functions construction is presented in detail and the corresponding finite element degrees of freedom unisolvence is proved.

The outline of the paper is as follows. In Section 2.1, element RPQ4 is briefly presented and Ciarlet's mathematical definition [11] of the finite element is set up. In Section 3 the three new base interpolation shape functions are introduced the unisolvence condition is proved. The boundary problem to be solved is defined in Section 4. The error is estimated in Section 5 for both the consistency and the approximability terms. Numerical examples are presented and discussed in Section 6. The paper ends with Conclusions.

2 Thin plate finite element RPQ4

Finite element RPQ4 is a nonconforming thin plate bending quadrilateral element (Figure 1), developed directly in the Cartesian coordinates and characterized by the satisfaction of the so called 'weak continuity of displacements on the interelement boundaries'. The ideas behind the formulation and the technical derivation of the stiffness matrix are fully described in Wanji and Cheung [1] and will, thus, not be repeated here. In what follows we somewhat generalize the geometry of the element and include the elements with nonconvex shape.

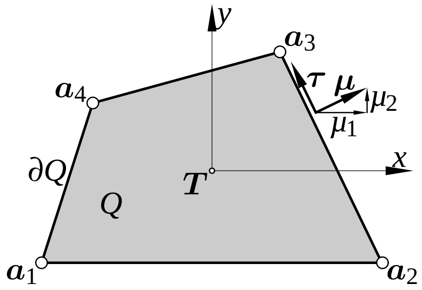

Figure 1: Quadrilateral thin plate bending finite element RPQ4 [1]

In order to prove convergence and estimate the error of finite element RPQ4, we have to recast the original equations of Wanji and Cheung [1] into the form appropriate for our convergence analysis. For this purpose the following notations are introduced:

$$
\begin{aligned}
\partial_1 \bullet &:= \frac{\partial \bullet}{\partial x},\ \partial_2 \bullet := \frac{\partial \bullet}{\partial y},\ \partial_{ij} \bullet := \frac{\partial^2 \bullet}{\partial x_i\, \partial x_j},\ 1 \le i, j \le 2, \\
\partial_\mu \bullet &:= \frac{\partial \bullet}{\partial \mu},\ \partial_\tau \bullet = \frac{\partial \bullet}{\partial \tau},\quad \partial^{\boldsymbol{\alpha}} \bullet \equiv \partial^{(\alpha_1,\alpha_2)} \bullet := \frac{\partial^{(\alpha_1+\alpha_2)} \bullet}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \\
\boldsymbol{X} &:= \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2 y & xy^2 & y^3 & x^3 y & xy^3 \end{bmatrix}^T, \\
\boldsymbol{q} &:= \begin{bmatrix} \ldots w_h(\boldsymbol{a}_i) & \partial_1 w_h(\boldsymbol{a}_i) & \partial_2 w_h(\boldsymbol{a}_i) \ldots \end{bmatrix}^T,\ 1 \le i \le 4
\end{aligned}
$$

$$
A := \begin{bmatrix} A_n(\boldsymbol{a}_1) \\ \vdots \\ A_n(\boldsymbol{a}_4) \end{bmatrix}, \quad A_n((x,y)) := \begin{bmatrix} \boldsymbol{X}^T \\ \partial_1 \boldsymbol{X}^T \\ \partial_2 \boldsymbol{X}^T \end{bmatrix}. \tag{2.1}
$$

The derivatives in the above equations are understood in the generalized sense, see [17]. $\boldsymbol{X}$ is the nonconforming interpolation basis of the element, and $\boldsymbol{q}$ its vector of nodal degrees of freedom $q_i, i = 1 \ldots 12$; $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_4$ are the position vectors of the nodal points. Let the origin of the Cartesian coordinate system (x, y) be the geometric center T of the quadrilateral Q. The components of the outer normal of its border ∂Q in x and y directions are denoted by μ_1 and μ_2, respectively (Figure 1). The area of the quadrilateral is denoted by $|Q|$. Let $w_h|_Q$ denote the nonconforming displacement approximation described by basis $\boldsymbol{X}$, and $v_h|_Q$ the Wanji and Cheung refined nonconforming displacement approximation on Q given by [1]

$$
v_h|_Q := w_h|_Q + \lambda_1 \frac{x^2}{2} + \lambda_2 \frac{y^2}{2} + \lambda_3 \frac{xy}{2} = w_h|_Q + \Lambda_Q = \boldsymbol{X}^T A^{-1} \boldsymbol{q} + \Lambda_Q. \tag{2.2}
$$

Constants λ_1, λ_2 and λ_3 are determined from the weak continuity conditions [1] resulting in

$$\begin{bmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{bmatrix} = \frac{1}{|Q|}\int_{\partial Q}\left(\widetilde{\partial_\mu w_h}\begin{bmatrix}\mu_1^2\\ \mu_2^2\\ 2\mu_1\mu_2\end{bmatrix} + \widetilde{\partial_\tau w_h}\begin{bmatrix}-\mu_1\mu_2\\ \mu_1\mu_2\\ \mu_1^2-\mu_2^2\end{bmatrix}\right) ds - \frac{1}{|Q|}\int_Q \begin{bmatrix}\partial_{11} w_h\\ \partial_{22} w_h\\ 2\,\partial_{12} w_h\end{bmatrix} d\boldsymbol{x}. \tag{2.3}$$

Consequently the finite element was shown numerically to pass Irons' patch test [1]. Functions $\widetilde{\partial_\tau w_h}$ and $\widetilde{\partial_\mu w_h}$ denote a piecewise linear or parabolic interpolation of the tangential derivative and a piecewise linear interpolation of the normal derivative on the border ∂Q, respectively, interpolated solely by the nodal values $q_i, i = 1, \dots, 12$. With such a choice of constants λ_1, λ_2 and λ_3, the approximation functions fulfill the weak continuity conditions as introduced in [1, Equation (1)]:

$$\begin{aligned}
&\int_Q \partial_{11} v_h\, d\boldsymbol{x} - \int_{\partial Q}\left(\widetilde{\partial_\mu w_h}\,\mu_1^2 - \widetilde{\partial_\tau w_h}\,\mu_1\mu_2\right) ds = 0,\\
&\int_Q \partial_{22} v_h\, d\boldsymbol{x} - \int_{\partial Q}\left(\widetilde{\partial_\mu w_h}\,\mu_2^2 + \widetilde{\partial_\tau w_h}\,\mu_1\mu_2\right) ds = 0,\\
&\int_Q 2\,\partial_{12} v_h\, d\boldsymbol{x} - \int_{\partial Q}\left(2\,\widetilde{\partial_\mu w_h}\,\mu_1\mu_2 + \widetilde{\partial_\tau w_h}\,(\mu_1^2-\mu_2^2)\right) ds = 0.
\end{aligned} \tag{2.4}$$

Remark 2.1. *Note that the refined displacement v_h is nonconforming both across the boundaries of the elements and in the nodal points.*

The Ciarlet's mathematical definition of the finite element [11, p. 78] should be set up in order to prove the convergence.

2.1 Finite element (Q, P_Q, Φ_Q)

Let V_h denote a finite element space, $V_h|_Q := P_Q \subset \mathscr{P}_3(Q) \oplus \mathscr{L}\{x^3\, y, x\, y^3\}$, u_h^* a finite element approximation of the weak solution, u^* the weak solution, v an arbitrary function and $W_2^m(Q) \equiv H^m(Q)$ the Sobolev spaces with norms $\|\cdot\|_{m,2,Q} \equiv \|\cdot\|_{m,Q}$ and subnorms $|\cdot|_{m,2,Q} \equiv |\cdot|_{m,Q}$ for $0 \le m \le 4$.

With the help of Equations (2.2) and (2.3) it is easy to show that

$$\widetilde{\partial_\alpha v_h} - \partial_\alpha v_h = \widetilde{\partial_\alpha w_h} - \partial_\alpha w_h, \quad \alpha \in \{\mu, \tau\}. \tag{2.5}$$

The above relations hold true for both linear and parabolic interpolation of $\partial_\alpha w_h$ on the element border. Consequently, we have $\lambda_i(w_h) = \lambda_i(v_h),\ 1 \le i \le 3$. We introduce two sets of linear functionals, $\Sigma_Q := \{\varphi_i \equiv \varphi_i^Q, i = 1, ..., 12\}$ and $\Phi_Q := \{\phi_i \equiv \phi_i^Q, i = 1, ..., 12\}$, as

$$\begin{aligned}
\varphi_{3\,i-2}(w_h) &:= w_h(\boldsymbol{a}_i) = \boldsymbol{q}_{3\,i-2} = v_h(\boldsymbol{a}_i) - \Lambda_Q(\boldsymbol{a}_i) =: \phi_{3\,i-2}(v_h), \quad 1 \le i \le 4,\\
\varphi_{3\,i-1}(w_h) &:= \partial_1 w_h(\boldsymbol{a}_i) = \boldsymbol{q}_{3\,i-1} = \partial_1 v_h(\boldsymbol{a}_i) - \partial_1\Lambda_Q(\boldsymbol{a}_i) =: \phi_{3\,i-1}(v_h),\ 1 \le i \le 4,\\
\varphi_{3\,i}(w_h) &:= \partial_2 w_h(\boldsymbol{a}_i) = \boldsymbol{q}_{3\,i} = \partial_2 v_h(\boldsymbol{a}_i) - \partial_2\Lambda_Q(\boldsymbol{a}_i) =: \phi_{3\,i}(v_h), \quad 1 \le i \le 4.
\end{aligned}$$

By Ciarlet's definition of a finite element [11], the set Φ_Q must be P_Q*–unisolvent* in the following sense: given any real scalars $\alpha_i, i = 1, \dots, 12$, there exists a unique function $p \in P_Q$ which satisfies the conditions

$$\phi_i(p) = \alpha_i, \quad 1 \le i \le 12 \tag{2.6}$$

[11, p. 78].

The proof of unisolvence becomes straightforward, if the following lemma is proved first. This lemma will convert the P_Q–unisolvence problem of the set Φ_Q into the P_Q–unisolvence problem of the set Σ_Q, which is equivalent to requiring the regularity of the interpolation matrix A.

Lemma 2.2. *Let the space P'_Q denote an algebraic dual of space P_Q. The set Φ_Q is the base for P'_Q, if and only if the set Σ_Q is the base for P'_Q.*

Element RPQ4 of convex shape (rhombus) and nonconvex shape [1] is inherently prone to singularity of the interpolation matrix A.

In order to exclude triangles, we require

Condition 2.3. *Let us denote h_Q the diameter of the quadrilateral Q. Assuming the existence of a constant $c_1 > 0$, such that for every quadrilateral Q the inequality*

$$\min(|\Delta_1|, |\Delta_2|, |\Delta_3|, |\Delta_4|) \ge c_1 h_Q^2. \tag{2.7}$$

holds.

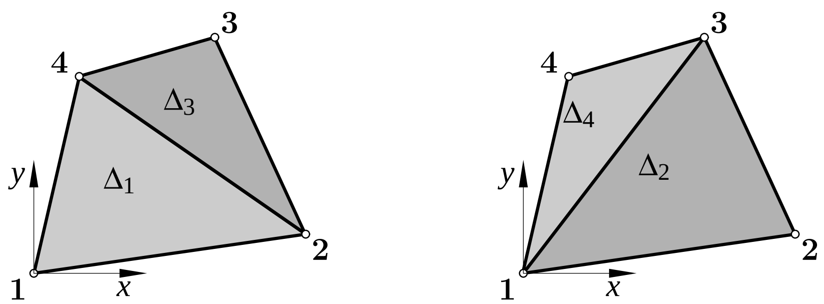

Figure 2: Divisons of quadrilateral into triangles

In the convergence proof we will need a constant γ introduced by

Condition 2.4. *Let h_Q and ℓ_Q denote the diameter of quadrilateral Q and the length of the shortest side on the border ∂Q, respectively. Suppose that Q is star–shaped with respect to the ball B_Q with radius $\rho_Q := \sup\{\mathrm{diam}(\textit{ball } S), S \subset Q, \forall \boldsymbol{x} \in Q \,\forall \boldsymbol{y} \in S \,\forall \lambda \in [0,1] \Rightarrow (1-\lambda)\,\boldsymbol{x} + \lambda\,\boldsymbol{y} \in Q\}$. Then we can define the chunkiness parameter $c_\rho := \frac{h_Q}{\rho_Q}$ and parameter $c_\ell := \frac{h_Q}{\ell_Q}$. We assume that some constant γ exists for which the following inequality holds*

$$\max\left(\cup_{Q \in \mathscr{Q}_h} \max(c_\rho, c_\ell)\right) \le \gamma.$$

Let $\mathcal{N}_h$, $\mathcal{Q}_h$, $\mathcal{Q}_h(\boldsymbol{a})$ and $Q_1(\boldsymbol{a})$ denote the set of all vertices, the set of all quadrilaterals, the set of quadrilaterals with common vertex $\boldsymbol{a}$ and the first quadrilateral from the set $\mathcal{Q}_h(\boldsymbol{a})$, respectively. For a quadrilateral Q with nodes $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$, we rewrite the set of linear functionals as $\Phi_Q =: \{\phi^Q_{\boldsymbol{a}_j,k}, j = 1,\ldots,4,\ \ k = 1,\ldots,3\}$. We can now define the finite element space

$$X_h := \left\{ v_h \in \prod_{Q\in\mathcal{Q}_h} P_Q, \forall \boldsymbol{a} \in \mathcal{N}_h, \forall Q_i, Q_j \in \mathcal{Q}_h(\boldsymbol{a}), \forall k, \phi^{Q_i}_{\boldsymbol{a},k}(v_h|_{Q_i}) = \phi^{Q_j}_{\boldsymbol{a},k}(v_h|_{Q_j}) \right\}$$

and the related set of linear functionals

$$\Phi_h := \{\phi_{\boldsymbol{a},k} = \phi^{Q_1(\boldsymbol{a})}_{\boldsymbol{a},k}, \boldsymbol{a} \in \mathcal{N}_h, 1 \le k \le 3\}.$$

Next we employ the dual functions $p_{\boldsymbol{a},k}$ from V_h for functionals $\phi_{\boldsymbol{a},k}$, on the open set $\Omega_h = \Omega - \cup_{Q\in\mathcal{Q}_h}\partial Q$, and define the interpolation operator

$$I_h : v \mapsto \sum_{\boldsymbol{a}\in\mathcal{N}_h, 1\le k\le 3} \phi_{\boldsymbol{a},k}(v)\, p_{\boldsymbol{a},k}.$$

3 Finite element improvements

The finite element degrees of freedom are V_h unisolvent [11, p. 100] when the interpolation matrix A in Equation (2.2) is nonsingular. The unisolvence can not be achieved with one interpolation shape function. In order to assure that the unisolvence for all quadrilaterals fulfil Condition 2.3, three different sets of interpolation base functions, Equations (3.1a), (3.1b) and (3.1c), are proposed. It is easy to determine the appropriate set of interpolation base functions, usually one with the greatest value of the determinant of the interpolation matrix, see Equations (3.7).

3.1 Interpolation base functions

Employing the interpolation base function coefficient vector $\boldsymbol{\alpha}$ and the interpolation function bases

$$\boldsymbol{X}_1(x,y) := \begin{bmatrix}1 & x & y & x^2 & x\,y & y^2 & x^3 & x^2\,y & x\,y^2 & y^3 & x^3\,y & x^4\end{bmatrix}^T, \tag{3.1a}$$

$$\boldsymbol{X}_2(x,y) := \begin{bmatrix}1 & x & y & x^2 & x\,y & y^2 & x^3 & x^2\,y & x\,y^2 & y^3 & y^3\,x & y^4\end{bmatrix}^T, \tag{3.1b}$$

$$\boldsymbol{X}_3(x,y) := \begin{bmatrix}1 & x & y & x^2 & x\,y & y^2 & x^3 & x^2\,y & x\,y^2 & y^3 & x\,(y+x)^3 & y\,(y+x)^3\end{bmatrix}^T \tag{3.1c}$$

the thin plate displacement interpolation functions could be defined by equations

$$w_{hi}(x,y)|_Q := \boldsymbol{X}_i^T(x,y) \cdot \boldsymbol{\alpha}, \quad 1 \le i \le 3. \tag{3.2}$$

3.2 Interpolation matrices

According to Equation (2.1) the interpolation matrices $A_i,\ (1 \le i \le 3)$ are expressed by

$$A_i = \begin{bmatrix} \dots & \boldsymbol{X}_i(x_j, y_j) & \partial_x \boldsymbol{X}_i(x_j, y_j) & \partial_y \boldsymbol{X}_i(x_j, y_j) & \dots \end{bmatrix}^T, \quad 1 \le j \le 4. \tag{3.3}$$

Since the base of the space $\mathbb{P}_3$ is determined with the first ten functions from (3.1) and the remaining two functions from (3.1) are the members of space $\mathbb{P}_4$ and, consequently, the determinants of interpolation matrices $A_i, (1 \le i \le 3)$ are translation invariant, the origin of the coordinate system can be moved into (x_1, y_1) (see Figure 2). Employing the temporary abbreviations $z_i := x_i + y_i$, $s_i := y_i + 4x_i$, $t_i := 4y_i + x_i$ for $2 \le i \le 4$ and using Equations (3.1a), (3.1b), (3.1c) and (2.1) we get

$$A_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_2 & y_2 & x_2^2 & x_2\,y_2 & y_2^2 & x_2^3 & x_2^2\,y_2 & x_2\,y_2^2 & y_2^3 & x_2^3\,y_2 & x_2^4 \\
0 & 1 & 0 & 2\,x_2 & y_2 & 0 & 3\,x_2^2 & 2\,x_2\,y_2 & y_2^2 & 0 & 3\,x_2^2\,y_2 & 4\,x_2^3 \\
0 & 0 & 1 & 0 & x_2 & 2\,y_2 & 0 & x_2^2 & 2\,x_2\,y_2 & 3\,y_2^2 & x_2^3 & 0 \\
1 & x_3 & y_3 & x_3^2 & x_3\,y_3 & y_3^2 & x_3^3 & x_3^2\,y_3 & x_3\,y_3^2 & y_3^3 & x_3^3\,y_3 & x_3^4 \\
0 & 1 & 0 & 2\,x_3 & y_3 & 0 & 3\,x_3^2 & 2\,x_3\,y_3 & y_3^2 & 0 & 3\,x_3^2\,y_3 & 4\,x_3^3 \\
0 & 0 & 1 & 0 & x_3 & 2\,y_3 & 0 & x_3^2 & 2\,x_3\,y_3 & 3\,y_3^2 & x_3^3 & 0 \\
1 & x_4 & y_4 & x_4^2 & x_4\,y_4 & y_4^2 & x_4^3 & x_4^2\,y_4 & x_4\,y_4^2 & y_4^3 & x_4^3\,y_4 & x_4^4 \\
0 & 1 & 0 & 2\,x_4 & y_4 & 0 & 3\,x_4^2 & 2\,x_4\,y_4 & y_4^2 & 0 & 3\,x_4^2\,y_4 & 4\,x_4^3 \\
0 & 0 & 1 & 0 & x_4 & 2\,y_4 & 0 & x_4^2 & 2\,x_4\,y_4 & 3\,y_4^2 & x_4^3 & 0
\end{bmatrix}, \tag{3.4a}$$

$$A_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & x_2 & y_2 & x_2^2 & x_2\,y_2 & y_2^2 & x_2^3 & x_2^2\,y_2 & x_2\,y_2^2 & y_2^3 & x_2\,y_2^3 & y_2^4 \\
0 & 1 & 0 & 2\,x_2 & y_2 & 0 & 3\,x_2^2 & 2\,x_2\,y_2 & y_2^2 & 0 & y_2^3 & 0 \\
0 & 0 & 1 & 0 & x_2 & 2\,y_2 & 0 & x_2^2 & 2\,x_2\,y_2 & 3\,y_2^2 & 3\,x_2\,y_2^2 & 4\,y_2^3 \\
1 & x_3 & y_3 & x_3^2 & x_3\,y_3 & y_3^2 & x_3^3 & x_3^2\,y_3 & x_3\,y_3^2 & y_3^3 & x_3\,y_3^3 & y_3^4 \\
0 & 1 & 0 & 2\,x_3 & y_3 & 0 & 3\,x_3^2 & 2\,x_3\,y_3 & y_3^2 & 0 & y_3^3 & 0 \\
0 & 0 & 1 & 0 & x_3 & 2\,y_3 & 0 & x_3^2 & 2\,x_3\,y_3 & 3\,y_3^2 & 3\,x_3\,y_3^2 & 4\,y_3^3 \\
1 & x_4 & y_4 & x_4^2 & x_4\,y_4 & y_4^2 & x_4^3 & x_4^2\,y_4 & x_4\,y_4^2 & y_4^3 & x_4\,y_4^3 & y_4^4 \\
0 & 1 & 0 & 2\,x_4 & y_4 & 0 & 3\,x_4^2 & 2\,x_4\,y_4 & y_4^2 & 0 & y_4^3 & 0 \\
0 & 0 & 1 & 0 & x_4 & 2\,y_4 & 0 & x_4^2 & 2\,x_4\,y_4 & 3\,y_4^2 & 3\,x_4\,y_4^2 & 4\,y_4^3
\end{bmatrix}, \tag{3.4b}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2\,y_2 & y_2^2 & x_2^3 & x_2^2\,y_2 & x_2\,y_2^2 & y_2^3 & x_2\,z_2^3 & y_2\,z_2^3 \\ 0 & 1 & 0 & 2\,x_2 & y_2 & 0 & 3\,x_2^2 & 2\,x_2\,y_2 & y_2^2 & 0 & s_2\,z_2^2 & 3\,y_2\,z_2^2 \\ 0 & 0 & 1 & 0 & x_2 & 2\,y_2 & 0 & x_2^2 & 2\,x_2\,y_2 & 3\,y_2^2 & 3\,x_2\,z_2^2 & t_2\,z_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3\,y_3 & y_3^2 & x_3^3 & x_3^2\,y_3 & x_3\,y_3^2 & y_3^3 & x_3\,z_3^3 & y_3\,z_3^3 \\ 0 & 1 & 0 & 2\,x_3 & y_3 & 0 & 3\,x_3^2 & 2\,x_3\,y_3 & y_3^2 & 0 & s_3\,z_3^2 & 3\,y_3\,z_3^2 \\ 0 & 0 & 1 & 0 & x_3 & 2\,y_3 & 0 & x_3^2 & 2\,x_3\,y_3 & 3\,y_3^2 & 3\,x_3\,z_3^2 & t_3\,z_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4\,y_4 & y_4^2 & x_4^3 & x_4^2\,y_4 & x_4\,y_4^2 & y_4^3 & x_4\,z_4^3 & y_4\,z_4^3 \\ 0 & 1 & 0 & 2\,x_4 & y_4 & 0 & 3\,x_4^2 & 2\,x_4\,y_4 & y_4^2 & 0 & s_4\,z_4^2 & 3\,y_4\,z_4^2 \\ 0 & 0 & 1 & 0 & x_4 & 2\,y_4 & 0 & x_4^2 & 2\,x_4\,y_4 & 3\,y_4^2 & 3\,x_4\,z_4^2 & t_4\,z_4^2 \end{bmatrix}. \tag{3.4c}$$

3.3 Determinants of interpolation matrices

From Figure 2 it is easy to check the following indentities:

$$(x_2\,y_3 - x_3\,y_2) = 2\,\Delta_2, \tag{3.5a}$$

$$(x_4\,y_2 - x_2\,y_4) = -2\,\Delta_1, \tag{3.5b}$$

$$(x_3\,y_4 - x_4\,y_3) = 2\,\Delta_4, \tag{3.5c}$$

$$(x_3\,y_4 - x_2\,y_4 - x_4\,y_3 + x_2\,y_3 + x_4\,y_2 - x_3\,y_2) = 2\,(\Delta_4 - \Delta_1 + \Delta_2) = 2\,\Delta_3. \tag{3.5d}$$

Using wxMaxima and the abbreviations

$$F_A := -16\,\Delta_1\,\Delta_2\,\Delta_3\,\Delta_4, \tag{3.6a}$$

$$F_{XY} := 2\,(x_2\,y_2\,\Delta_4 - x_3\,y_3\,\Delta_1 + x_4\,y_4\,\Delta_2)\,, \tag{3.6b}$$

$$F_X := 2\,\left(x_2^2\,\Delta_4 - x_3^2\,\Delta_1 + x_4^2\,\Delta_2\right), \tag{3.6c}$$

$$F_Y := 2\,\left(y_2^2\,\Delta_4 - y_3^2\,\Delta_1 + y_4^2\,\Delta_2\right) \tag{3.6d}$$

the determinants of the interpolation matrices A_i read

$$D_1 := |A_1| = F_A\,F_X^3, \tag{3.7a}$$

$$D_2 := |A_2| = -F_A\,F_Y^3, \tag{3.7b}$$

$$D_3 := |A_3| = -F_A\,(F_X + F_Y + 2\,F_{XY})^3\,. \tag{3.7c}$$

3.4 Elimination of singularity of the interpolation matrix

According to Condition 2.3 the coefficient F_A does not vanish, so all the determinants in Equation (3.7) vanish only when all factors F_X, F_Y and F_{XY} vanish. However, according to Condition 2.3 this is impossible.

Proof. Equations (3.6b), (3.6c) in (3.6d) can be expressed in a matrix form

$$\begin{bmatrix} x_2\,y_2 & -x_3\,y_3 & x_4\,y_4 \\ x_2^2 & -x_3^2 & x_4^2 \\ y_2^2 & -y_3^2 & y_4^2 \end{bmatrix} \cdot \begin{bmatrix} \Delta_4 \\ \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} F_{XY} \\ F_X \\ F_Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.8}$$

Since the determinant of the matrix, $-\Delta_1\,\Delta_2\,\Delta_4$, does not vanish, the only solution of the system of linear equations is trivial one $\Delta_1 = 0$, $\Delta_2 = 0$, $\Delta_4 = 0$, which cannot fulfil Condition 2.3. □

3.5 Singular cases

- $D_1 = D_2 = 0$: Square with nodes $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.
- $D_1 = D_3 = 0$: Quadrilateral with nodes $(0,0)$, $(-1,0)$, $(-3,-3)$ and $(6,-21)$.
- $D_2 = D_3 = 0$: Quadrilateral with nodes $(0,0)$, $(-21,6)$, $(-3,-3)$ and $(0,-1)$.

3.6 Preconditioning of the iterpolation matrix

Employing the different interpolations could improve and optimize the element stiffness condition number. We know that all determinants in Equation (3.7) cannot vanish simultaneously. Let us suppose that all of them can be arbitrary small. We prove that this is also impossible, showing that the greatest one is bounded bellow by the vector norm $\|\Delta\| := \sqrt{\Delta_1^2 + \Delta_1^2 + \Delta_4^2}$.

Proof. Changing the right side of Equation (3.8) we get

$$\begin{bmatrix} x_2\,y_2 & -x_3\,y_3 & x_4\,y_4 \\ x_2^2 & -x_3^2 & x_4^2 \\ y_2^2 & -y_3^2 & y_4^2 \end{bmatrix} \cdot \begin{bmatrix} \Delta_4 \\ \Delta_1 \\ \Delta_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} F_{XY} \\ F_X \\ F_Y \end{bmatrix} = \begin{bmatrix} \delta_{xy} \\ \delta_x \\ \delta_y \end{bmatrix}. \tag{3.9}$$

or shortly

$$B\,\Delta = \delta. \tag{3.9'}$$

Since matrix B is nonsigular we can use the matrix inverse

$$B^{-1}\,\delta = \Delta, \tag{3.10}$$

estimate

$$\|\Delta\| \le \|B^{-1}\|\,\|\delta\| \tag{3.11}$$

and derive inequality

$$\|\delta\| \ge \frac{\|\Delta\|}{\|B^{-1}\|}, \tag{3.12}$$

as a simplied form of the general inequality (2.3) [19, p. 50]. In estimate (3.12) the Frobenius matrix norm $\| \, \| : B \equiv [b_{ij}] \mapsto \sqrt{\sum |b_{ij}|^2}$ is used. The inverse of matrix B is

$$B^{-1} = \frac{1}{4} \begin{bmatrix} -\dfrac{x_3\, y_4 + x_4\, y_3}{\Delta_1 \Delta_2} & \dfrac{y_3\, y_4}{\Delta_1 \Delta_2} & \dfrac{x_3\, x_4}{\Delta_1 \Delta_2} \\ -\dfrac{x_2\, y_4 + x_4\, y_2}{\Delta_2 \Delta_4} & \dfrac{y_2\, y_4}{\Delta_2 \Delta_4} & \dfrac{x_2\, x_4}{\Delta_2 \Delta_4} \\ -\dfrac{x_2\, y_3 + x_3\, y_2}{\Delta_1 \Delta_4} & \dfrac{y_2\, y_3}{\Delta_1 \Delta_4} & \dfrac{x_2\, x_3}{\Delta_1 \Delta_4} \end{bmatrix}. \tag{3.13}$$

Employing the abbreviations L_3 and L_4 marking the lengths of the border lines 13 and 14, using the estimate

$$\begin{aligned} (x_3\, y_4 + x_4\, y_3)^2 + (y_3\, y_4)^2 + (x_3\, x_4)^2 &= 2\, x_3\, y_3\, x_4\, y_4 + \left(x_3^2 + y_3^2\right) \left(x_4^2 + y_4^2\right) \\ &\le \frac{3}{2} \left(x_3^2 + y_3^2\right) \left(x_4^2 + y_4^2\right) = \frac{3}{2} L_3^2\, L_4^2, \end{aligned} \tag{3.14}$$

and Condition 2.3 one estimates the matrix inverse as (3.13)

$$\begin{aligned} \|B^{-1}\| &\le \sqrt{\frac{3}{32}} \sqrt{\left(\frac{L_3^2\, L_4^2}{\Delta_1\, \Delta_2}\right)^2 + \left(\frac{L_2^2\, L_4^2}{\Delta_2\, \Delta_4}\right)^2 + \left(\frac{L_2^2\, L_3^2}{\Delta_1\, \Delta_4}\right)^2} \\ &\le \sqrt{\frac{3}{32}}\, h_Q^4 \sqrt{\left(\frac{1}{\Delta_1\, \Delta_2}\right)^2 + \left(\frac{1}{\Delta_2\, \Delta_4}\right)^2 + \left(\frac{1}{\Delta_1\, \Delta_4}\right)^2} \\ &\le \sqrt{\frac{3}{32}}\, h_Q^4 \sqrt{3\, \frac{1}{\left(c_1^2\, h_Q^4\right)^2}} \le c_1^{-2}. \end{aligned} \tag{3.15}$$

Combining Equations (3.12) and (3.15) yields the final estimate

$$\|\delta\| \ge \frac{\|\Delta\|}{\|B^{-1}\|} \ge c_1^2\, \|\Delta\|. \tag{3.16}$$

□

4 Boundary value problem

We seek the weak solution u^* for the deflection of the thin clamped plate subjected to a given surface load f

$$a(u^*, v) = (f, v), \quad u^*, v \in V := H_0^2(\Omega),$$

where

$$\begin{aligned} a(u, v) &= \int_\Omega \left(\nu \Delta u \Delta v + (1 - \nu) \sum_{i=1, j=1}^{2} \partial_{ij} u\, \partial_{ij} v\right) d\boldsymbol{x}, \\ (f, v) &= \int_\Omega f\, v\, d\boldsymbol{x}, \quad f \in L^2(\Omega). \end{aligned}$$

Its nonconforming approximation

$$u_h^* \in V_h := X_{00h} := \{v_h \in X_h, \forall \boldsymbol{a} \in \partial\Omega, \phi_{\boldsymbol{a},k}(v_h) = 0, 1 \le k \le 3\}$$

is obtained by the solution of the variational equation

$$a_h(u_h^*, v_h) = (f, v_h), \quad u_h^*, v_h \in V_h,$$

where

$$a_h(u_h, v_h) = \sum_{Q \in \mathscr{Q}_h} \int_Q \Big(\nu \Delta u_h \Delta v_h + (1-\nu) \sum_{i=1,j=1}^{2} \partial_{ij} u_h \partial_{ij} v_h\Big)\, d\boldsymbol{x},$$
$$(f, v_h) = \int_\Omega f\, v_h\, d\boldsymbol{x}, \quad f \in L^2(\Omega).$$

Here, $\nu \in [0, \frac{1}{2}]$ is Poisson's coefficient of material.

5 Convergence and error estimate

According to the second Strang Lemma [11]

$$\|u^* - u_h^*\|_{2,h} \le c \left(\inf_{v_h \in V_h} \|u^* - v_h\|_{2,h} + + \sup_{v_h \in V_h} \frac{|(f, v_h) - a_h(u^*, v_h)|}{\|v_h\|_{2,h}} \right),$$

the error of the weak nonconforming solution consists of two parts, i.e. the error of approximation and the error of the consistency term. In what follows each part will be estimated separately. The error of the consistency term will be estimated with the help of Stummel's generalized patch test [7]. Since the main steps of the proof are independent of the particular choice of the interpolation base functions and the proof thus coincides with the one already presented in [18], only some definitions and main steps will be outlined.

Let h denote the largest diameter of all quadrilaterals in quasi-uniform quadrangulation $\mathscr{Q}_h$ of polygonal domain Ω. Let c denote a generic constant independent on h, which may have different values at different places. For each quadrilateral Q we introduce the quadrilateral $\hat{Q}$ with the same shape, yet with the diameter $h_{\hat{Q}}$ being equal to 1.

5.1 Error estimate of the consistency term

According to [7, 4] the sequence $\{V_h\} \cup H_0^2(\Omega)$ passes the generalized patch test if and only if

$$\lim_{h \to 0} T_{\alpha,i}(\psi, v_h) := \lim_{h \to 0} \sum_{Q \in \mathscr{Q}_h} T_{\alpha,i}^Q(\psi, v_h) = \lim_{h \to 0} \sum_{Q \in \mathscr{Q}_h} \int_{\partial Q} \psi\, \partial^\alpha v_h\, \mu_i\, ds = 0$$

for all $i = 1, 2$, all $|\alpha| \leq 1$, all bounded sequences $\{V_h\}$ and all $\psi \in C^\infty(\overline{\Omega})$. We employ the operators $P_0 : v \mapsto Q^1 v := \int_{B_Q} v(\boldsymbol{x})\phi(\boldsymbol{x})\, d\boldsymbol{x}$, $R_0 : v \mapsto v - Q^1 v$ and $\hat{R}_0 : \hat{v} \mapsto \hat{v} - \hat{Q}^1 \hat{v}$, where we have used the cut-off function ϕ (supp$\phi \in \overline{B_Q}, \int_{\boldsymbol{R}^2} \phi(\boldsymbol{x})\, d\boldsymbol{x} = 1$) [15, p. 97]. Introducing the affine mapping $F_Q : \hat{\boldsymbol{x}} \mapsto h_Q \hat{\boldsymbol{x}}$ one can write $v := \hat{v} \circ F_Q^{-1}$, $Q^1 v := \hat{Q}^1 \hat{v} \circ F_Q^{-1}$.

Let us define the border interpolation functions

$$\widetilde{\partial_1 w_h} := \mu_1 \widetilde{\partial_\mu w_h} - \mu_2 \widetilde{\partial_\tau w_h}, \quad \widetilde{\partial_2 w_h} := \mu_2 \widetilde{\partial_\mu w_h} + \mu_1 \widetilde{\partial_\tau w_h}.$$

We have to estimate the following terms:

$$\begin{aligned}
T_{(0,0),i}(\psi, v_h) &= T_i(\psi, v_h), \\
T_{(1,0),i}(\psi, v_h) &= T_i(P_0\psi, \partial_1 v_h - \widetilde{\partial_1 w_h}) + T_i(\psi, \widetilde{\partial_1 w_h}) + \\
&\qquad + T_i(R_0\psi, \partial_1 w_h - \widetilde{\partial_1 w_h} + \partial_1 \Lambda), \\
T_{(0,1),i}(\psi, v_h) &= T_i(P_0\psi, \partial_2 v_h - \widetilde{\partial_2 w_h}) + T_i(\psi, \widetilde{\partial_2 w_h}) + \\
&\qquad + T_i(R_0\psi, \partial_2 w_h - \widetilde{\partial_2 w_h} + \partial_2 \Lambda).
\end{aligned} \tag{5.1}$$

Using the relations $dx = -\mu_2\, ds$, $dy = \mu_1\, ds$, the Green formula and Equation (5.2),

$$\begin{bmatrix} \partial_\mu w_h \\ \partial_\tau w_h \end{bmatrix} = \begin{bmatrix} \mu_1 & \mu_2 \\ -\mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} \partial_1 w_h \\ \partial_2 w_h \end{bmatrix}, \quad \begin{bmatrix} \partial_1 w_h \\ \partial_2 w_h \end{bmatrix} = \begin{bmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{bmatrix} \begin{bmatrix} \partial_\mu w_h \\ \partial_\tau w_h \end{bmatrix}, \tag{5.2}$$

we can rewrite Equations (2.4) as

$$\begin{aligned}
&T_1^Q(1, \partial_1 v_h - \widetilde{\partial_1 w_h}) := \int_{\partial Q} \partial_1 v_h \mu_1\, ds - \int_{\partial Q} \widetilde{\partial_1 w_h}\, \mu_1\, ds = 0, \\
&T_2^Q(1, \partial_2 v_h - \widetilde{\partial_2 w_h}) := \int_{\partial Q} \partial_2 v_h \mu_2\, ds - \int_{\partial Q} \widetilde{\partial_2 w_h}\, \mu_2\, ds = 0, \\
&\int_{\partial Q} \left(2\, \partial_1 v_h \mu_2 - \widetilde{\partial_1 w_h}\, \mu_2 - \widetilde{\partial_2 w_h}\, \mu_1\right) ds =: 2\, T_2^Q(1, \partial_1 v_h - \widetilde{\partial_1 w_h}) - \int_{\partial Q} \widetilde{\partial_\tau w_h}\, ds, \\
&\int_{\partial Q} \left(2\, \partial_2 v_h \mu_1 - \widetilde{\partial_1 w_h}\, \mu_2 - \widetilde{\partial_2 w_h}\, \mu_1\right) ds =: 2\, T_1^Q(1, \partial_2 v_h - \widetilde{\partial_2 w_h}) + \int_{\partial Q} \widetilde{\partial_\tau w_h}\, ds.
\end{aligned}$$

In order to estimate the first and the second terms in the second and in the third equation, we have to estimate the term $\int_{\partial Q} \widetilde{\partial_\tau w_h}\, ds$. A short derivation shows that the sum $\sum_{Q \in \mathscr{Q}_h} \int_{\partial Q} \psi\, \widetilde{\partial_\tau w_h}\, ds$ vanishes for both linear and parabolic interpolations of the tangential derivative.

$$\sum_{Q \in \mathscr{Q}_h} \int_{\partial Q} \psi\, \widetilde{\partial_\tau w_h}\, ds = 0.$$

Remark 5.1. *From relations* $T_i(1, \partial_j v_h) = 0,\ 1 \le i, j \le 2,$ *which hold for all* $v_h \in V_{P0} := \{v_h \in V_h, \forall \boldsymbol{a} \in \partial(\cup_{Q\in\mathscr{Q}_P} Q), \phi_{\boldsymbol{a},k}(v_h) = 0, 1 \le k \le 3\}$, *it immediately follows*

$$\sum_{Q\in\mathscr{Q}_P} \int_Q \partial^\alpha v_h \, d\boldsymbol{x} = 0 \quad \forall v_h \in V_{P0}, \quad |\alpha| = 2.$$

So, according to [4, Lemma 4.1], the element passes Irons' patch test.

Now we can estimate the first term in both the second and the third equation of Equations (5.1). From

$$T_2(P_0\psi, \partial_1 v_h - \widetilde{\partial_1 w_h}) = -\frac{1}{2} \sum_{Q\in\mathscr{Q}_h} \int_{\partial Q} R_0\psi \, \widetilde{\partial_\tau w_{hd}} \, ds$$

one can derive the estimate:

$$|T_2(P_0\psi, \partial_1 v_h - \widetilde{\partial_1 w_h})| \le \sum_{Q\in\mathscr{Q}_h} c(\gamma) |\psi|_{1,2,Q} \, |v_h|_{2,2,Q} \, h_Q.$$

With the help of the function $\widetilde{w_h}$ we split the first term into the sum of three terms:

$$T_i(\psi, v_h) = T_i(\psi, \widetilde{w_h}) + T_i(P_0\psi, w_h - \widetilde{w_h} + \Lambda) + T_i(R_0\psi, w_h - \widetilde{w_h} + \Lambda).$$

Because of conformity of the function $\widetilde{w_h}$, the first term vanishes; thus the remaining two terms only need be elaborated upon. Performing some calculations one can estimate

$$T_i(P_0\psi, w_h - \widetilde{w_h} + \Lambda) \le c(\gamma) \sum_{Q\in\mathscr{Q}_h} |\psi|_{0,2,Q} \, \|v_h\|_{2,2,Q} \, h_Q, \tag{5.3}$$

$$T_i(R_0\psi, w_h - \widetilde{w_h} + \Lambda) \le c(\gamma) \sum_{Q\in\mathscr{Q}_h} |\psi|_{1,2,Q} \, \|v_h\|_{2,2,Q} \, h_Q^2. \tag{5.4}$$

The last terms of the second and the third equations of Equation (5.1) are estimated in a similar way:

$$T_i(R_0\psi, \partial_j w_h - \widetilde{\partial_j w_h} + \partial_j\Lambda) \le c(\gamma) \sum_{Q\in\mathscr{Q}_h} |\psi|_{1,2,Q} \, \|v_h\|_{2,2,Q} \, h_Q. \tag{5.5}$$

In the second and the third inequalities of Equation (5.5), we have used the Sobolev Imbedding Theorem [17], Friedrichs' inequality [15, 16] and the inverse inequality [15, Lemma 4.5.3]. Employing the integration by parts or the equivalent procedure from [11] we can write the error functional E_h in the form

$$\begin{aligned} E_h(u^*, v_h) &:= (f, v_h) - a_h(u^*, v_h) \\ &= \sum_{Q\in\mathscr{Q}_h} \int_{\partial Q} \partial_\nu \Delta u^* \, v_h + (1-\sigma)\partial_{\nu\tau} u^* \partial_\tau v_h \, ds \\ &- \sum_{Q\in\mathscr{Q}_h} \int_{\partial Q} \big(\Delta u^* - (1-\sigma)\partial_{\tau\tau} u^*\big) \partial_\nu v_h \, ds. \end{aligned}$$

With the help of the inequalities (5.3), (5.4) and (5.5) we finally derive the estimate of the error of the consistency term

$$|E_h(u^*, v_h)| \leq c\, h\, (\|u^*\|_{3,h} + h\, \|u^*\|_{4,h})\, \|v_h\|_{2,h} \tag{5.6}$$

valid for all $u^* \in H^4(\Omega)$.

5.2 Estimate of the approximability term

Let us first adapt Definition 4.4.2 and Theorem 4.4.4 from [15] to obtain the form suitable for the present purposes.

Lemma 5.2. *Let (Q, P, Φ) be a finite element satisfying*

(i) Q is star-shaped with respect to some ball, such that

$$|\hat{A}| \geq c(\gamma), \tag{5.7}$$

(ii) $\mathscr{P}_2 \subset P \subset W^3_\infty(Q)$ and

(iii) $\Phi \subset (C^1(\overline{Q}))'$.

Assume $p = 2$. Then for $0 \leq i \leq 2$ and $v \in W^3_2(Q)$ we have

$$|v - I_h\, v|_{i,2,Q} \leq C_{\gamma,\sigma(\hat{Q})} h_Q^{3-i}\, |v|_{3,2,Q},$$

where $\hat{Q} := \{\frac{\boldsymbol{x}}{h_Q},\ \boldsymbol{x} \in Q\}$, a constant $C_{\gamma,\sigma(\hat{Q})}$ is dependent on parameter γ, introduced in Condition 2.4, and $\sigma(\hat{Q})$ is the operator norm of $\hat{I}_h : W^3_2(\hat{Q}) \to C^1(\hat{Q})$.

Let us point out that the interpolation operator $\hat{I}_h$ is well defined on $W^3_2(\hat{Q})$. This follows from the Sobolev Imbedding Theorems. Our aim is to estimate the norm $\sigma(\hat{Q})$ of operator $\hat{I}_h : W^3_2(\hat{Q}) \to C^1(\hat{Q})$. For the sake of simplification of notation, we skip the hat over symbols. Employing the inequalities

$$\|I_h\, u\|_{3,2,Q} \leq \sum_{i=1}^{12} |\phi_i(u)|\, \|p_i\|_{3,2,Q} \leq \sum_{i=1}^{12} \|\phi_i\|_{W^3_2(Q)'}\, \|p_i\|_{3,2,Q}\, \|u\|_{3,2,Q}$$

gives

$$\sigma(Q) \leq \sum_{i=1}^{12} \|\phi_i\|_{W^3_2(Q)'}\, \|p_i\|_{3,2,Q}\, . \tag{5.8}$$

Next we show that the norm $\sigma(Q)$ is uniformly bounded for all quadrilaterals Q. As has been shown by Ciarlet [11] and Adams [17], the identity from $W^3_2(Q)$ to $C_1(Q)$ is uniformly continuous. First we estimate the norm $\|\phi_i\|_{W^3_2(Q)'}$:

$$|\phi_i(v)| \leq c\, \|v\|_{1,\infty,Q} \leq c\, \|v\|_{3,2,Q}, \quad 1 \leq i \leq 12. \tag{5.9}$$

From Equation (5.9) it immediately follows that the norms $\|\phi_i\|_{W^3_2(Q)'}$ are bounded from above by the constant $c = c(\gamma, c_1)$.

Equation (5.7) assures that the base functions p_i are also bounded. Thus we have

Lemma 5.3. *The base functions p_i are bounded:*

$$\|p_i\|_{3,2,Q} \le c(\gamma, c_1), \quad 1 \le i \le 12. \tag{5.10}$$

In order to achieve the ellipticity one should prove

Lemma 5.4. *The seminorm $v \mapsto \|v\|_{2,h} := \left(\sum_{Q\in\mathscr{Q}_h} |v|^2_{2,2,\Omega}\right)^{\frac{1}{2}}$ is a norm.*

Taking into account the second Strang Lemma [11], the error functional estimate (5.6), the estimate of the approximability term (Lemma 5.2), and Equations (5.8), (5.9) and (5.10), one can finally derive the estimate of the error in the energy norm:

$$\|u^* - u^*_h\|_{2,h} \le c\, h\, (\|u^*\|_{3,h} + h\, \|u^*\|_{4,h}).$$

Thus, the error in the energy norm decreases at least linearly with h for all $u^* \in H^4(\Omega)$.

Remark 5.5. *Combining the steps of the derivations above with the ideas from [12] we can derive the same error estimate for weak solutions of other fourth–order V–elliptic boundary value problems.*

6 Numerical examples

The theoretically derived error estimate is also verified numerically. We study the convergence behaviour of a thin clamped plate, subjected to the variable surface load $f : (x, y) \mapsto 8\,(10 - 18\,y^2 + 3\,(x^4 + y^4 + 6\,x^2\,(-1 + 2\,y^2)))$, defined on $\Omega = [-1, 1] \times [-1, 1]$. Poisson's ratio is taken to be $\nu = \frac{1}{3}$. The related analytical solution is $u^* = (x^2 - 1)^2\,(y^2 - 1)^2$. In Section 6.1 we show that the numerical results confirm the theoretically predicted linear convergence. In Section 6.2 we show that the improved element is more robust and thus more convenient for practical computations.

6.1 Rate of convergence

Two different series of meshes, i.e. those made of only convex quadrilaterals, and meshes made of mostly nonconvex quadrilaterals, have been employed in the convergence analysis. The initial convex mesh and the second convex mesh are shown in Figure 3. The second convex mesh and all the subsequent convex meshes follow the bisection dividing scheme [12]. The construction of the first and the second nonconvex meshes is shown in Figure 4. The first mesh was constructed in two steps. In the first step, the square was divided into parallelograms and trapesoids shown in the left plot of Figure 4. In the second step, the largest diagonal of each quadrilateral was divided into five equal parts with four dividing points, which present some nodes of the final nonconvex quadrilateral division. This is illustrated by shaded quadrilaterals. The second nonconvex mesh and all subsequent nonconvex meshes were obtained in

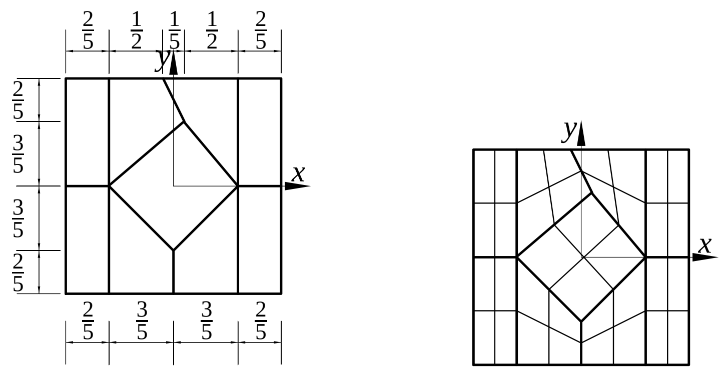

Figure 3: Initial and the first refined convex meshes constructed by the bisection dividing scheme [12]

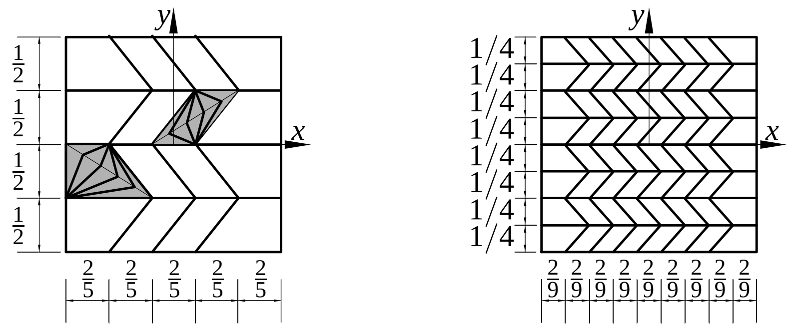

Figure 4: Nonconvex meshes construction

a similar way. The first construction step of the second mesh is shown in the right plot of Figure 4. RPQ4 and its improved version denoted by RPQ4(3) with the linear variation of displacement derivative along the sides of the element have been used in convergence analysis. Nine convex meshes with $h \approx 1.22, \ldots, 0.007$ and seven nonconvex meshes with $h \approx 0.8, \ldots, 0.0125$ have been applied with the related number of linear equations ranging from 12 to 1764867 and from 219 to 981507, respectively. The decrease of the actual error of the solution in the energy norm with h for both elements is depicted in Figure 5. There RPQ4 denotes the original RPQ4 element by Wanji and Cheung [1] and the RPQ4(3) denotes its improved version derived in the present paper. Observe that the actual error in the energy norm decreases linearly with h for element RPQ4 and RPQ4(3) for both meshes, exactly as predicted theoretically. The results of the RPQ4 element for small values of h, which do not fall on the straight line, are discussed in the next section. From the obtained numerical results from Figure 5 it is clearly seen that the results obtained by the dividing schemes using

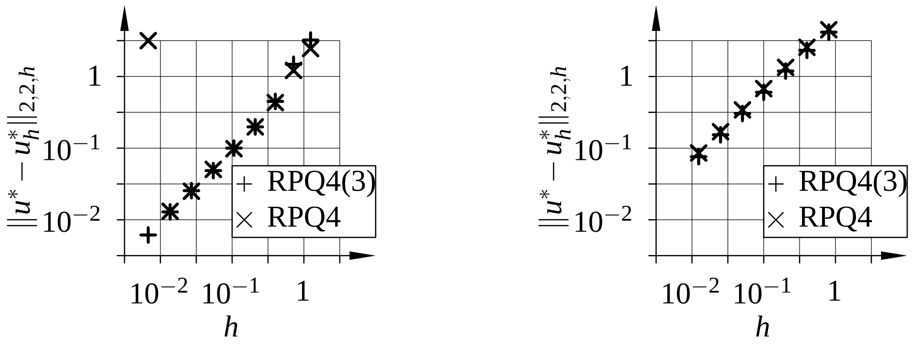

Figure 5: Error in energy norm

only convex quadrilaterals are somewhat more accurate compared to the nonconvex quadrilaterals.

6.2 Condition number of the structure stiffness matrix

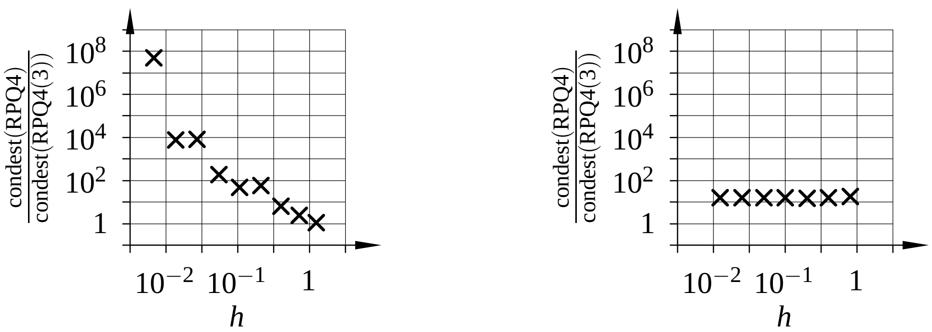

Figure 6: Ratios of stiffness matrix condition numbers

Both theoretical and numerical results for finite elements RPQ4 and RPQ4(3) show that the accuracy of the solution monotonically increases, if the number of finite elements grows.

Clearly, with the increasing number of equations the condition number of the structure stiffness matrix increases, too. The analysis of the present numerical examples has shown that the condition numbers of the stiffness matrices range from, roughly, 10^2 to 10^{12}, if applying improved element RPQ4(3). The condition numbers of the related RPQ4 stiffness matrices obtained by convex meshes increase much faster, see Figure 6, where the ratios of the stiffness matrix condition numbers of elements RPQ4 and

RPQ4(3) are presented for each h. There the condition numbers of elements RPQ4(3) and RPQ4 are denoted by condest(RPQ4(3)) and condest(RPQ4), respectively. The condition numbers were estimated by the MATLAB function `condest`. As clearly seen from Figure 6, the condition number ratios grow from, approx., 1 to 10^8, which indicates that the condition numbers of element RPQ4 grow from, approx. 10^2 to 10^{20}, where a complete lost of the accuracy of solution is observed (see Figures 5). Hence in almost every step of the bisection dividing algorithm, the condition number of RPQ4 structure stiffness matrix increases roughly by factor 7 or more with respect to the condition number of the RPQ4(3) structure stiffness matrix. This indicates an important computational advantage of the improved element version. Since in the case of nonconvex element meshes, the nonconvex element does not change the shape during dividing process, the ratios of condition number of structure matrices remain the same.

7 Conclusion

In the present paper we have proved convergence and estimated the rate of convergence of the improved version of the nonconforming nonconvex quadrilateral thin plate bending finite element derived directly from the finite element RPQ4 proposed by Wanji and Cheung [1].

Our mathematically rigorous proof of convergence is based on Stummel's generalized patch test [7] and the consideration of the element approximability condition [11], which are both necessary and sufficient for convergence.

This improved element has theoretically the same convergence characteristics as its predecessor, RPQ4, only that it is unconditionally unisolvent.

This convenient property of the new element helps to reduce the condition number of the structure stiffness matrices and consequently results in the element being more robust and thus better suited for highly refined finite element meshes.

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