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Dynamic Analysis of a Timoshenko Beam on a Semi-Infinite Elastic Subgrade using an Ordinary Differential Equation Method

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Abstract

A novel numerically analytical method is proposed for the analysis of a Timoshenko beam resting on a semi-infinite elastic subgrade in this paper. The deflection of the neutral surface of a Timoshenko beam and the cross- sectional rotation around the neutral axis are chosen as two basic unknown functions. By using a Hamilton principle, the free vibration problem of the Timoshenko beam is then translated into ordinary differential equations composed of the two unknown functions, which can be solved by an ordinary differential equation solver. Consequently, the natural frequencies and modes of the beam are obtained, and the influence of the subgrade stiffness and size as well as the material of the beam on dynamic property of the system can be readily analysed. The numerical results demonstrate that the proposed method is rational and powerful for the dynamic analysis of various Timoshenko beams.

Keywords: Timoshenko beam, semi-infinite elastic subgrade, free vibration analysis, ordinary differential equation solver, subgrade stiffness, mode shape.

1 Introduction

Many practical engineering problems can be modelled as a Timoshenko beam supported on an elastic subgrade, such as the analysis of a suspension bridge, a cable stayed bridge, the structure beneath a highway, an underground tunnel, and etc. Thusly, the mechanical problems pertinent to Timoshenko beams have been attracting the attentions of many scientists and engineers. For instance, the stability problem of a Timoshenko beam resting on a Winkler foundation was studied by Lee S. Y., Kou Y. H. and Lin F. Y. [1]. By using Green's functions, the exact solutions for Timoshenko beams on two-parameter elastic foundations were attained by Wang C. M., Lam K. Y. and He X. Q. [2]. By means of a weak form quadrature element method, the natural frequencies of the Timoshenko beam on nonlinear elastic foundation were studied by Mo Yihua, Ou Li and Zhong Hongzhi [3]. The difference of the free vibration equation of a Timoshenko beam, a Rayleigh beam, a Shear beam and an Euler-Bernoulli beam and how to gain the analytic solutions in some particular conditions were summarized by X. F. Li, Z. W. Yu and H. Zhang [4]. By solving the governing equations of transverse motion of a Timoshenko beam with power series method, the natural frequencies and mode shapes of a uniform simple Timoshenko beam and a non-uniform cantilever Timoshenko beam were studied by Reza Attarnejad, Shabnam Jandaghi Semnani and Ahmad Shahba [5]. A hierarchical finite element was presented for the geometrically nonlinear free and forced vibration of a non-uniform Timoshenko beam resting on a two-parameter foundation and the arc-length iterative method was used to solve the nonlinear eigenvalue equation which was obtained by applying the harmonic balance method by B. Zhu and A.Y.T. Leung [6]. The analytical solutions of the natural frequencies and mode shapes of an Euler beam, a Rayleigh beam, a Shear beam and Timoshenko beam with rigid support were obtained by Seon M. Han, Haym Benaroya and Timothy Wei [7]. The eigenvalue problem of cantilever and simple Bernoulli-Euler beams on elastic foundation with a single edge crack, an axial loading and an excitation force were formulated using the differential quadrature method, and the influence of the crack position were also studied by Ming-Hung Hsu [8].

However, the previous literatures exhibit two aspects of deficiencies. The one is how to quantify the scope of influenced subgrade as well as the compatibility relationship when quantifying the interactions between the beam and subgrade. The two is the exact solutions can only work out more simple engineering problems and will be incapable for more difficult practical problems. These two deficiencies will thusly limit their applications to practical engineering.

For instance, damage always happens in the highway pavement due to the repeated action of moving vehicles, which leads to a great loss of the national economy, ad hoc in China. Thusly, the research of the dynamic characteristics of the structure beneath highway pavement has a great significance for designing the highway or reducing the destruction of highway. A unit width of the highway deck might be selected and simplified as a Timoshenko beam resting on a semi-infinite elastic subgrade as shown in Figure 1. It is very hard for previous studies to apply their results to the engineering problem.

Thusly, the motivation of the paper is to propose a novel method for the analysis of the dynamic problem of a Timoshenko beam supported on a semi-infinite elastic subgrade using Ordinary Differential Equation (ODE) solver, by which an arbitrary desired precision for the solution can be obtained [9,10,11].

The outline of the analysis can be impressed through following formulation. The deflection of the neutral surface of a Timoshenko beam and the cross-sectional rotation around the neutral axis are chosen as the basic unknown functions, by using a Hamiltonian principle, the dynamic problem of a Timoshenko beam on semi-infinite elastic subgrade is then translated into ordinary differential equations composed of the two unknown functions which can be solved by the ODE solver. And the influence of the subgrade stiffness and size as well as the material of the beam on dynamic property of the system can be readily analyzed. The numerical results demonstrate that the new proposed method is rational and powerful for the dynamic analysis of various Timoshenko beams.

2 The subgrade model and governing equations

2.1 The subgrade model

The subgrade is considered as a sort of elastic medium in a semi-infinite space in this paper, that is, the semi-infinite elastic body (the compression modulus of the semi-infinite elastic body can be obtained by a site measurement). In order to quantify the interaction between the beam and its supporting soil with different elastic properties effectively, Yaoqing Gong and Zhengwei Zhang [12] have already formulated the equivalent stiffness equations of a semi-infinite elastic subgrade corresponding to different kinds of deformations by employing the displacement equations of J. Boussinesq [13]. With these formulas, the interaction between the beam and its subgrade can be quantified effectively and readily.

2.2 Establishment of governing equations

Considering the Timoshenko beam resting on a semi-infinite elastic subgrade as shown in Figure 1, if the deflection of the neutral surface, $w_0(x)$, and the rotatory angle around the neutral axis of the cross section of the beam, $\theta_0(x)$, are chosen as the basic unknown functions , defined on the centroid axis of the beam. The vertical translation of the neutral surface, $w(x,t)$, and the rotatory motion around the neutral axis of the cross section of the beam, $\theta(x,t)$, during the free vibration can be expressed as

$$\left.\begin{aligned} w(x,t) &= w_0(x)\sin(\omega t+\alpha) \\ \theta(x,t) &= \theta_0(x)\sin(\omega t+\alpha) \end{aligned}\right\}. \tag{1}$$

Consequently, by utilizing the normal "plane sections remain plane" assumption, the axial displacement of the beam, $u(z,x,t)$, will be written as

$$u(z,x,t) = -z\theta(x,t), \tag{2}$$

as shown in Fig. 1.

By employing the Hamilton principle,

$$\int_{t_1}^{t_2} \delta(T-U)dt = 0, \tag{3}$$

in which,

$$U = U_b + U_s\,, \tag{4}$$

$$U_b = \frac{1}{2}\int_0^l dx \int_A \left[E\left(\frac{\partial u}{\partial x}\right)^2 + G\left(\frac{\partial w}{\partial x} - \theta\right)^2 \right] dA\,, \tag{5}$$

$$U_s = \frac{1}{2}\int_0^L \left[k_x b u^2\left(0,x,t\right) + k_z b w^2(x,t) \right] dx\,, \tag{6}$$

$$T = \frac{1}{2}\int_0^l dx \int_A \left[\rho\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2 \right] dA\,. \tag{7}$$

are elastic strain energy of the beam, elastic strain energy stored in the subgrade and the kinetic energy of the beam, respectively. The governing equation of the free vibration of the beam can be then obtained as follow

$$\left.\begin{aligned} &\overline{m}\omega^2 w_0 - k_z b w_0 + k'GA(w_0'' - \theta_0') = 0 \\ &J_y \theta_0 \omega^2 - \frac{1}{4}k_x b h^2 \theta_0 + k'GA(w_0' - \theta_0) + EI_y \theta_0'' = 0 \end{aligned}\right\}. \tag{8}$$

The corresponding boundary conditions are

$$x = 0,\ \left.\begin{aligned} &k'GA\left[w_0'\left(0\right) - \theta_0\left(0\right)\right] = 0 \\ &EI_y \theta_0'(0) = 0 \end{aligned}\right\},\quad x = L,\ \left.\begin{aligned} &k'GA\left[w_0'\left(L\right) - \theta_0\left(L\right)\right] = 0 \\ &EI_y \theta_0'(L) = 0 \end{aligned}\right\}, \tag{9}$$

in which, $J_y = \int_A \rho z^2 dA$, ρ is the mass density, and b is the width of the cross section; k_x and k_z represent the equivalent stiffness of subgrade in the horizontal and vertical directions respectively, k' is the shape factor reflecting the nonlinear distribution of the shearing stress along the height of the cross section of the beam, and $5/6$ can be selected for the rectangle cross section, for instance.

Mathematically, equations (8) and (9) consist of an eigenvalue problem of a set of ordinary differential equations which can be solved by applying the ODE solver COLSYS [9,10,11].

3 Examples and computational results

We give two examples. For verifying the accuracy of the ODE solver, the first example is a simply supported Timoshenko beam, whose analytical solution for the natural frequencies and modes can be obtained by the method provided in literature [7], with the governing equations and corresponding boundary conditions below

$$
\left.\begin{array}{l}
\bar{m}\omega^2 w_0 + k'GA(w_0'' - \theta_0') = 0 \\
J_y \theta_0 \omega^2 + k'GA(w_0' - \theta_0) + EI_y \theta_0'' = 0
\end{array}\right\},
\tag{10}
$$

$$
x = 0, \quad \left.\begin{array}{l} w_0(0) = 0 \\ EI_y \theta_0'(0) = 0 \end{array}\right\}, \quad x = L, \quad \left.\begin{array}{l} w_0(L) = 0 \\ EI_y \theta_0'(L) = 0 \end{array}\right\}.
\tag{11}
$$

3.1 Example 1

A simply supported Timoshenko beam is shown in Fig. 2, the material and geometrical properties of the beam are: mass density $\rho = 7850\text{kg/m}^3$, elastic modulus $E = 2.1\times10^{11}\text{N/m}^2$, shear modulus $G = 3/8E$, the length $L = 0.4\text{m}$, the width and height of the cross section are 0.02m and 0.08m, which is constant along the length, and the shape factor $k' = 5/6$. The first five orders of natural frequencies are computed using the ODE solver are shown in table 1, which agree with the analytical solutions very well.

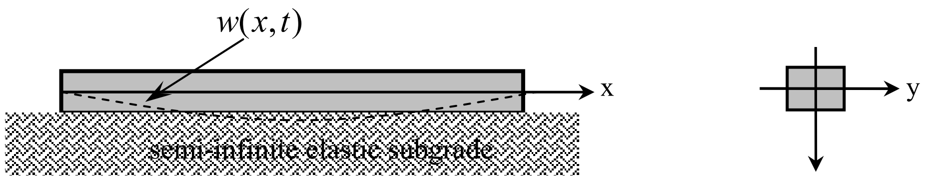

Fig. 1 a Timoshenko beam resting on a semi-infinite elastic subgrade — The cross section of the beam

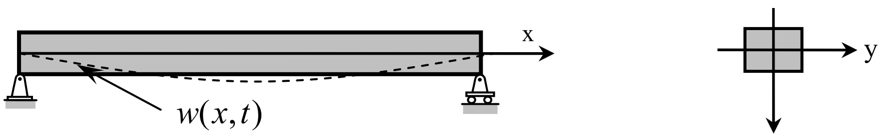

Fig. 2 a simple Timoshenko beam — The cross section of the beam

Natural frequency	ω_1	ω_2	ω_3	ω_4	ω_5
Analytical solution	6956.61	24373.99	47002.37	71849.62	97551.68
ODE solver result	6975.49	24478.21	47285.94	72400.15	98457.49
error	0.27%	0.43%	0.60%	0.77%	0.92%

Table 1 The first five natural frequencies of the simple Timoshenko beam

3.2 Example 2

A Timoshenko beam resting on semi-infinite elastic subgrade is shown in Fig. 1, the material and geometrical properties of the beam are: mass density $\rho = 2500\text{kg}/\text{m}^3$, elastic modulus $E = 32.5\text{GPa}$, shear modulus $G = 13.0\text{GPa}$, the Poisson's ratio is $\nu = 0.4$, the length $L = 4.0\text{m}$, the size of the cross section is $1.0\times0.5\text{m}^2$, the shape factor $k' = 5/6$, and the equivalent stiffness of the subgrade in the horizontal and vertical direction are $k_x = 4.8 \times 10^5 \text{kN/m}^3$ and $k_z = 5.6 \times 10^5 \text{kN/m}^3$ respectively.

By using the method mentioned in the previous section, the first five natural frequencies and the corresponding mode shapes can be obtained and shown in table 2-6, in which w represents the deflection mode and γ stands for the shearing deformation mode.

x	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
w	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000

Table 2 The first mode shape ($\omega_1 = 669.33$)

x	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	-1.4142	-1.4241	-1.4397	-1.4566	-1.4699	-1.4747	-1.4699	-1.4566	-1.4397	-1.4241	-1.4142
w	0.9635	0.7745	0.5838	0.3909	0.1961	0.0000	-0.1961	-0.3909	-0.5838	-0.7745	-0.9635

Table 3 The second mode shape ($\omega_2 = 679.89$)

x	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	1.9964	2.0264	1.8198	1.4052	0.7979	0.0000	-0.7979	-1.4052	-1.8198	-2.0264	-1.9964
w	-0.5821	-0.3129	-0.5720	0.1570	0.3033	0.3573	0.3033	0.1570	-0.5720	-0.3129	-0.5821

Table 4 The third mode shape ($\omega_3 = 1005.91$)

x	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	-2.4995	-2.6561	-0.5739	1.8636	2.1283	0.0000	-2.1283	-1.8636	0.5739	2.6561	2.4995
w	0.3423	-0.01413	-0.2363	-0.1492	0.1226	0.2677	0.1226	-0.1492	-0.2363	-0.01413	0.3423

Table 5 The fourth mode shape ($\omega_4 = 3384.24$)

x	0.0	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0
γ	2.7017	2.7036	-1.1613	-2.7821	0.4409	2.9458	0.4409	-2.7821	-1.1613	2.7036	2.7017
w	-0.3025	0.09291	0.2063	-0.0755	-0.2431	0.0000	0.2431	-0.0755	-0.2063	-0.09291	0.3025

Table 6 The fifth mode shape ($\omega_5 = 5052.22$)

3 Conclusions

- A unit width of a highway deck might be simplified as a Timoshenko beam resting on a semi-infinite elastic subgrade, the first two fundamental vibration modes of which are a vertical translation oscillation and a vertical shearing vibration as shown in table 2, table 3 and Fig. 3. These imply that designers of a highway should consider the control of the discharge of vehicles moving across the highway as well as both the vertical and horizontal vibrating periods of possible earthquakes happened in the area probably, so as to keep away from the damage induced by the resonance.

- Shearing oscillations play a significant role during the free vibration of a Timoshenko beam. There are also coupling oscillations during high order free vibrations as shown in Table 4, Table 5, Table 6 and Fig. 3. These indicate that we have to pay attention to the shearing vibrations when dealing with the engineering problems pertinent to Timoshenko beams.
- The proposed free-vibration-analysis method can readily provide the designer for insight into the dynamic performance of the structure beneath a highway in the primary design stage. In the primary design stage, a designer might change the arrangement of the structure and the rigidity of the subgrade. This will cause the changes of the stiffness of the dynamic system, so the change of the natural frequencies or periods. Thusly, the designer can adjust the subgrade rigidity and structural system so as to obtain the desired natural frequencies or periods of the structural system beneath a highway.

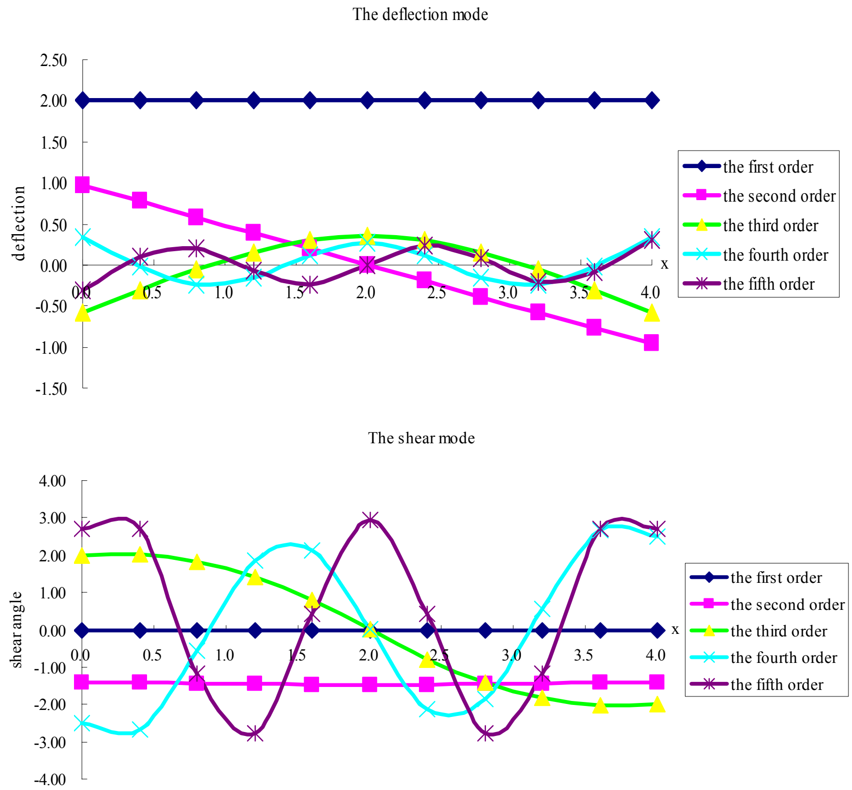

Fig. 3 The first five mode shapes of a Timoshenko beam

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