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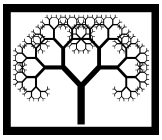

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A New Method for Solving Random Vibration Problems

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Abstract

Current random vibration methods provide efficient solutions for the second moment properties of the states of arbitrary linear systems subjected to random noise. For Gaussian noise, these properties define completely the state probability law. The random vibration methods can also be used to find the distribution of the states of simple linear systems under non-Gaussian noise and simple nonlinear systems under Gaussian/non-Gaussian noise.

Monte Carlo simulation is the only general method for finding the probability laws for the states of arbitrary linear or nonlinear dynamic systems subjected to Gaussian or non-Gaussian noise. Computational effort, that can be significant when dealing with realistic systems, is the essential limitation of Monte Carlo simulation.

A novel method is proposed for analyzing arbitrary linear or nonlinear dynamic systems driven by Gaussian or non-Gaussian noise. The method is based on stochastic reduced order models (SROMs), that is, stochastic processes that have finite number of samples. The proposed method is non-intrusive in the sense that its implementation involves solutions of deterministic versions of random vibration problems corresponding the samples of SROMs describing the inputs to these problems. Numerical examples are presented to illustrate the implementation and performance of the proposed method. It is also shown that for linear systems with non-Gaussian input Itô's calculus can be used to obtain efficient and accurate solutions.

Keywords: Gaussian noise, linear systems, Monte Carlo simulation, non-Gaussian noise, nonlinear systems, random vibration, stochastic reduced order models.

1 Introduction

Uncertainty is an inherent feature of properties of physical systems and inputs to these systems. The states of most dynamic systems are random functions of time and/or space that satisfy equations with random entries, referred to as stochastic equations. The solution of stochastic equations poses notable technical difficulties that are frequently circumvented in applications by heuristic assumptions. The random vibration theory, that is used extensively to analyze dynamic systems subjected to random noise, assume that (1) system properties are deterministic and perfectly known and (2) driving noise is Gaussian, although these assumptions are rarely satisfied in applications. For example, the distribution of the pressure field acting on a spacecraft during atmospheric re-entry differs significantly from the Gaussian distribution and, in this extreme environment, the spacecraft mechanical properties are uncertain. The reliability of this spacecraft cannot be obtained within the framework of the random vibration theory [1].

It is not possible to relax the assumptions of the random vibration theory and retain its conceptually simple framework since some developments in this theory use mathematical objects that are not defined. For example, calculations in random vibration yielding mean and covariance equations for the state of a linear system are formal since they view white noise as a process with constant spectral density, that is, a process with infinite variance that is not defined in the second moment sense [2] (Section 7.2). Alternative methods need to be developed to analyze dynamic systems in a non-Gaussian environment.

The objective of this investigation is the development of accurate, efficient, and rigorous methods for analyzing linear and nonlinear dynamic systems subjected to non-Gaussian noise. The development of these methods requires novel approaches that involve (1) elements of stochastic calculus, for example, Itô integrals, semimartingales, stochastic differential equations, and (2) new representations for stochastic processes, for example, stochastic reduced order models (SROMs) that, in contrast to Karhunen-Loéve expansions that can only describe the first two moments of stochastic processes, can capture the probability law of these processes. The methods provide full probabilistic characterization for the states of dynamic systems in non-Gaussian environments, so that they can be used to solve a broad range of practical problems, for example, optimal design for dynamic systems under reliability constraints.

2 Linear systems

Let $X(t)$ be the state of the dynamic system, that is, a real-valued stochastic process defined by the stochastic differential equation

$$\dot{X}(t) = -\rho\, X(t) + Z(t) \tag{1}$$

driven by the non-Gaussian process

$$Z(t) = Y(t)^2, \tag{2}$$

where $\rho > 0$, $Y(t)$ is a real-valued stationary Gaussian process with $E[Y(t)] = 0$, $E[Y(t+u)\,Y(t)] = \exp(-\alpha\,|u|)$, and $\alpha > 0$. The linear random vibration theory can only delivered the first two moments of $X(t)$ so that, since $X(t)$ is a non-Gaussian process, its probability law is not known. This is a significant limitation since system performance cannot be assessed, for example, the probability $p_f(\tau) = P(\max_{0\le t\le\tau} X(t) > a)$ that the system state exceeds a critical threshold $a > 0$ in a time interval $[0, \tau]$.

Alternative methods are needed to find the probability law of $X(t)$. Samples of $X(t)$ obtained by Monte Carlo simulation can be used to infer the probability law of this process. However, this method can be computationally inefficient when dealing with realistic systems. Two classes of methods are proposed for analyzing dynamic systems, methods based on Itô's calculus and methods based on stochastic reduced order models (SROMs). The first class of methods is useful for linear systems while SROM-based methods work for both linear and nonlinear systems. We use stochastic calculus to find state properties for linear systems in non-Gaussian environment. The implementation of SROM-based is demonstrated for nonlinear systems.

2.1 Stochastic calculus

Concepts of stochastic differential equations and approximate representations for stochastic processes are used to construct approximations for the probability law of $X(t)$. Since $Y(t)$ can be viewed as the output of the linear filter

$$dY(t) = -\alpha\,Y(t)\,dt + (2\,\alpha)^{1/2}\,dB(t) \tag{3}$$

driven by a Brownian motion $B(t)$, the augmented state vector (X, Y) is a bivariate diffusion process satisfying the stochastic differential equation

$$\left\{\begin{array}{ll} dX(t) & = -\rho\,X(t)\,dt + Y(t)^2\,dt \\ dY(t) & = -\alpha\,Y(t)\,dt + (2\,\alpha)^{1/2}\,dB(t). \end{array}\right. \tag{4}$$

Let $p, q \ge 0$ be arbitrary integers. Itô's formula for continuous semimartingales [2] (Section 4.6.2) applied to mapping $(X(t), Y(t)) \mapsto X(t)^p\,Y(t)^q$ gives

$$\dot{\mu}(p,q;t) = -p\,\rho\,\mu(p,q;t) + p\,\mu(p-1,q+2;t) - q\,\alpha\,\mu(p,q;t) + q\,(q-1)\,\mu(p,q-2;t) \tag{5}$$

by expectation, where $\mu(p,q;t) = E[X(t)^p\,Y(t)^q]$ and $\mu(u,v;t) = 0$ if $u < 0$ and/or $v < 0$. This sequence of moment equations is closed, so that moments of any order of (X, Y) can be calculated exactly.

To achieve the objective of finding the probability $p_f(\tau)$, state properties provided by Itô's calculus can be used to construct an approximation for the probability law of $X(t)$. The construction may involve two steps. First, an approximation $\tilde{F}$ needs to be developed for the marginal distribution F of $X(t)$. Second, a translation model $X_T(t)$ needs to be constructed for $X(t)$ based on its covariance function and $\tilde{F}$.

Various techniques can be employed to construct $\tilde{F}$. For example, $\tilde{F}$ can be (1) approximated by the piece-wise constant distribution function

$$\tilde{F}(x) = \sum_{k=1}^{m} p_k \, 1(x_k \leq x) \tag{6}$$

with probability mass $\{p_k\}$ at $\{x_k\}$, $k = 1, \ldots, m$, such that $\sum_{k=1}^{m} x_k^r \, p_i = \mu(r)$, $r = 1, \ldots, n = 2\,m$, [3] (Section II.8), (2) required to maximize the Shannon entropy

$$-\int \tilde{f}(x) \, \ln\big(\tilde{f}(x)\big) \, dx \tag{7}$$

under the constraints $\int \tilde{f}(x) \, dx = 1$ and $\int x^r \, \tilde{f}(x) \, dx = \mu(r)$, $r = 1, \ldots, n$, where $\tilde{f}(x) = d\tilde{F}(x)/dx$ [4] (Section 2.5.3), or (3) postulated to be a member of the linear space spanned by a finite collection of specified distributions $\{F_k(x)\}$, that is,

$$\tilde{F}(x) = \sum_{k=1}^{m} p_k \, F_k(x), \tag{8}$$

where $p_k \geq 0$ are such that $\sum_{k=1}^{m} p_k = 1$. The optimal probabilities $\{p_k\}$ minimize the discrepancy $e(p_1, \ldots, p_m) = \sum_{r=1}^{\bar{r}} w(r) \, \big(\mu(r) - \sum_{k=1}^{m} p_k \, \mu_k(r)\big)^2$ between moments $\mu(r) = E[X(t)^r]$ of $X(t)$ and their approximations $\tilde{\mu}(r) = \sum_{k=1}^{m} p_k \, \mu_k(r)$, $\mu_k(r) = \int x^k \, dF_k(x)$, under the constraints $p_k \geq 0$ and $\sum_{k=1}^{m} p_k = 1$, where $\bar{r} \geq 1$ is an integer and $w(r) \geq 0$ denotes a weighting function.

Let $\tilde{X}_T(t) = \tilde{F}^{-1} \circ \Phi\big(G(t)\big)$ be a translation model for $X(t)$ with marginal distribution $\tilde{F}$, where $G(t)$ is a stationary Gaussian process with mean 0, variance 1, and covariance function $\rho(\tau) = E[G(t+\tau) \, G(t)]$ such that the covariance function of $\tilde{X}_T(t)$ is close to that of $X(t)$ ([5], Section 3.1). Algorithms are available for finding $\rho(\tau)$ with this property [6].

The marginal distribution $\tilde{F}$ and the correlation function ρ specify fully the probability law of $\tilde{X}_T(t)$, so that this process can be used to calculate $p_f(\tau)$ approximately. For a relatively large threshold a, we have

$$p_f(\tau) \simeq 1 - \exp(\nu(a) \, \tau) \simeq 1 - \exp(\nu_T(a) \, \tau), \tag{9}$$

where $\nu(a)$ and $\nu_T(a)$ denote the mean a-upcrossing rate of level a of $X(t)$ and $X_T(a)$, respectively, and

$$\nu_T(a) = \sqrt{\rho^2 + 2\,\rho\,\alpha} \, \exp\big[- \big(\Phi^{-1} \circ \tilde{F}(a)/2\big)\big]/(2\,\pi). \tag{10}$$

Figure 1 shows with solid and dotted lines Monte Carlo estimates for the mean a-upcrossing rate $\nu(a)$ and the mean a-upcrossing rate $\nu_T(a)$ for $\rho = 1$ and $\alpha = 5$. Since $\nu(a) \simeq \nu_T(a)$, the probability $p_f(\tau)$ in Eq. 9 can be approximated by $1 - \exp(-\nu_T(a) \, \tau)$.

In summary, classical methods of linear random vibration can be used to calculate second moment properties of $X(t)$, that is, the mean and correlation functions of this

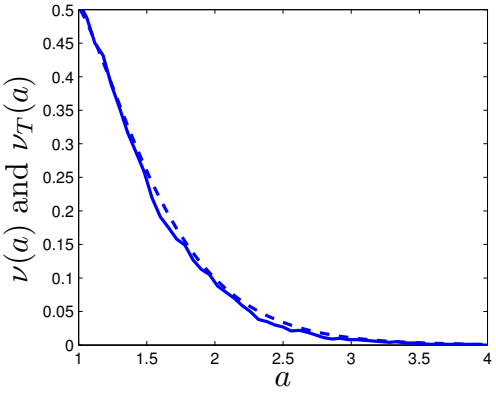

Figure 1: Monte Carlo estimates of the mean a-upcrossing rates of $X(t)$ (solid line) and mean a-upcrossing rates of $\tilde{X}_T(t)$ given by Eq. 10 (dotted line) for $\rho = 1$ and $\alpha = 5$

process. The marginal distribution of $X(t)$ can be obtained from parametric representations of this distribution and moments of $X(t)$, that can be obtained exactly since the moment equations are closed. The probability law of $X(t)$ can be approximated by that of translation models $X_T(t)$ matching the second moment properties and the marginal distribution of $X(t)$.

3 Nonlinear systems

Monte Carlo simulation is the only general method for finding properties of the state $X(t)$ of an arbitrary nonlinear dynamic system with additive/multiplicative Gaussian or non-Gaussian input. The main limitation of the method can be the computation time that is prohibitive for realistic applications. This limitation has promoted the development of approximate methods for finding properties of $X(t)$. Current approximate methods are based on heuristic arguments, for example, equivalent linearization [7, 8] and classical moment closures [9, 10], or rigorous but rather restrictive conditions, for example, stochastic averaging [2, 10] and perturbation [2].

Our objective is the development of a practical method for calculating statistics of the state $X(t)$ of nonlinear dynamic systems in Gaussian and non-Gaussian environments that are accurate and efficient. The method is based on stochastic reduced order models (SROMs).

Consider a nonlinear dynamic system subjected to a non-Gaussian input $Y(t)$ that may or may not have stationary independent increments as, for example, the Brownian motion, the compound Poisson, and α-stable processes considered in the classical theory of stochastic differential equations. We propose to represent $Y(t)$ by a simple stochastic process $\tilde{Y}(t)$, that is, a process with a finite number of samples $\{\tilde{y}_k(t)\}$ of probabilities $\{p_k\}$, $k = 1, \ldots, m$, that may not be equal. The samples $\{\tilde{y}_k(t)\}$ and their probabilities $\{p_k\}$ define completely the probability law of $\tilde{Y}(t)$. Optimization

algorithms have been developed to construct SROMs [11]. Two distinct methods, referred to as SROM and extended SROM methods, are discused. We present the essentials of these methods and illustrate their implementation and performance by a simple example.

3.1 SROM-based method

Denote by $\{\tilde{x}_k(t)\}$ the deterministic solutions $X(t)$ for $Y(t)$ set equal to samples $\{\tilde{y}_k(t)\}$ of a SROM $\tilde{y}(t)$ for $Y(t)$. The solutions $\{\tilde{x}_k(t)\}$ and the probabilities $\{p_k\}$ define a SROM $\tilde{X}(t)$ for $X(t)$, that can be used to calculate properties of $X(t)$ approximately. Properties of $\tilde{X}(t)$ can be obtained by elementary calculations. For example, the moment of order $q \geq 1$ and the marginal distribution of $\tilde{X}(t)$ are given by

$$E[\tilde{X}(t)^q] = \sum_{k=1}^{m} p_k \, \tilde{x}_k(t)^q \text{ and}$$

$$\tilde{F}(x) = P(\tilde{X}(t) \leq x) = \sum_{k=1}^{m} p_k \, 1(\tilde{x}_k(t) \leq x). \tag{11}$$

Preliminary studies on the feasibility of the SROM method are in [12]. Simplicity is the main feature of the approximation of $X(t)$ by SROMs. The method involves three steps. First, optimization algorithms are employed to construct a SROM $\tilde{Y}(t)$ for input $Y(t)$. Second, the SROM $\tilde{X}(t)$ of $X(t)$ corresponding to $\tilde{Y}(t)$ is obtained from m deterministic solutions of the defining equation for $X(t)$ with $\tilde{y}_k(t)$, $k = 1, \ldots, m$ in place of $Y(t)$. Third, elementary calculations as in Eq. 11 yield approximations for properties of $X(t)$.

3.2 ESROM-based method

The accuracy of a SROM $\tilde{Y}(t)$ of $Y(t)$ for a selected model size m is guaranteed by construction, that is, the optimization algorithm used to select the samples of this process and their probabilities. The accuracy of $\tilde{X}(t)$ depends on that of $\tilde{Y}(t)$ and the approximation used for the mapping $Y \mapsto X$. The SROM $\tilde{X}(t)$ is based on a rather crude approximation of mapping $Y \mapsto X$. The extended SROM method is based on a more accurate representation of this mapping. The implementation of the extended SROM method involves the following four steps that, for simplicity, are stated for the case in which $X(t)$ is a real-valued stochastic process.

– *Step 1: Input parametric model.* Construct a parametric model $Y_P(t; Z)$ for $Y(t)$, that is, a deterministic function of time that depends on an d-dimensional random vector Z defined on a probability space $(\Omega, \mathcal{F}, P)$. The construction of parametric models for $Y(t)$ involves considerations in [13]. Classical parametric models obtained by truncating Karhunen-Loève series for $Y(t)$ are inadequate since the deterministic functions of time in these representations, which

are eigenfunctions of a linear operator related to the correlation function of $Y(t)$, may have properties that are inconsistent with those of samples of $Y(t)$. For example, the Brownian motion and properly scaled compound Poisson processes have the same second moment properties, that is, the same Karhunen-Loève series. However, the samples of Brownian motion and compound Poisson processes are continuous and piecewise constant, respectively. Moreover, only the first two moments of the random coefficients of Karhunen-Loève series are defined. Unless $Y(t)$ is Gaussian, the distributions of these coefficients are not known. We conclude this step with the observation that the representation of $Y(t)$ by parametric models is essential for computation since the stochastic dimension of $Y_P(t;Z)$ is finite, while the stochastic dimension of $Y(t)$ is, generally, infinite.

– *Step 2: SROM for Z.* Let $\{\tilde{z}_k,\ k=1,\ldots,m\}$ be samples of Z, and denote by

$$\Gamma_k = \{z \in \Gamma : \|z-\tilde{z}_k\| \le \|z-\tilde{z}_l\|,\ l \ne k\}, \quad k=1,\ldots,m, \tag{12}$$

the Voronoi tessellation with centers $\{\tilde{z}_k\}$ in the range $\Gamma = Z(\Omega)$ of Z. Any set of samples $\{\tilde{z}_k\}$ and probabilities $\{p_k = P(Z^{-1}(\Gamma_k))\}$ defines a SROM $\tilde{Z}$ for Z. We are interested in that set of pairs $\{\tilde{z}_k, p_k\}$, $k=1,\ldots,m$, that minimizes the discrepancy between probability laws of $\tilde{Z}$ and Z. An optimization algorithm is used to select the optimal pair $\{\tilde{z}_k, p_k\}$ defining $\tilde{Z}$.

– *Step 3: Approximate solutions.* Calculate the deterministic solutions $\{\tilde{x}_k(t)\}$ corresponding to $\{Z=\tilde{z}_k\}$ and the gradients

$$\{\nabla\tilde{x}_k(t) = (\partial\tilde{x}_k(t)/\partial z_1,\ldots,\partial\tilde{x}_k(t)/\partial z_d)\}$$

of these solutions with respect to the coordinates of Z. The gradients $\{\nabla\tilde{x}_k(t)\}$ can be interpreted as sensitivity factors with respect to the coordinates of Z. The deterministic solutions $\{\tilde{x}_k(t)\}$ and $\{\nabla\tilde{x}_k(t)\}$ can be used to construct piecewise linear approximation

$$X_L(t) = \sum_{k=1}^{m} \left[\tilde{x}_k(t) + \nabla\tilde{x}_k(t)\cdot(Z-\tilde{z}_k)\right] 1(Z\in\Gamma_k) \tag{13}$$

for the mapping $Z \mapsto X(t)$, under the assumption that the mapping is sufficiently smooth in Z. The representation in Eq. 13 approximates $X(t)$ in each cell Γ_k of the tessellation by a hyperplane tangent to the mapping $Z \mapsto X(t)$ at $(\tilde{z}_k, \tilde{x}_k(t))$, $k=1,\ldots,m$. Piecewise quadratic approximations can be constructed in a similar manner, but require derivatives of order two of $\tilde{x}_k(t)$ with respect to the coordinates of Z.

– *Step 4: Solution properties.* The properties of $X_L(t)$ in Eq. 13 depends on the samples of $\tilde{Z}$, the probability law of Z, and the Voronoi tessellation $\{\Gamma_k\}$. The construction of the tessellation $\{\Gamma_k\}$ in high dimension is a rather difficult task. However, it is possible to estimate properties of $X_L(t)$ efficiently by

Monte Carlo simulation based on the definition of $\{\Gamma_k\}$ in Eq. 12 without precalculating these sets. Let $\{z_i,\ i = 1, \ldots, n\}$ be $n \gg m$ independent samples of Z. The members of $\{z_i,\ i = 1, \ldots, n\}$ that are in Γ_k have the property $\|z_i - \tilde{z}_k\| \leq \|z_i - \tilde{z}_l\|$, $l \neq k$. A simple algorithm is used to identify the subsets of $\{z_i,\ i = 1, \ldots, n\}$ that belong to the cells of the Voronoi tessellation in $\Gamma = Z(\Omega)$. Once these subsets have been identified, properties of $X_L(t)$ can be estimated simply. For example, moments of order $q \geq 1$ and marginal distributions of this process can be estimated from

$$E[X_L(t)^q] \simeq \sum_{k=1}^{m} \frac{n_k}{n} \left[\frac{1}{n_k} \sum_{z_i \in \Gamma_k} \left[\tilde{x}_k(t) + \nabla \tilde{x}_k(t) \cdot (z_i - \tilde{z}_k) \right]^q \right] \text{ and}$$

$$F_l(x) = P(X_L(t) \leq x) \simeq \sum_{k=1}^{m} \frac{n_k}{n} \left[\frac{1}{n_k} \sum_{z_i \in \Gamma_k} 1\big(\tilde{x}_k(t) + \nabla \tilde{x}_k(t) \cdot (z_i - \tilde{z}_k) \leq x \big) \right], \tag{14}$$

where n_k denotes the number of samples $\{z_i\}$ in Γ_k. Monte Carlo estimates for properties of $X_L(t)$ of the type in Eq. 14 are efficient since the functional form of the piecewise linear approximation of $X(t)$ is known.

3.3 Numerical example

Let $X(t)$ be the state of the dynamic system

$$\dot{X}(t) = \alpha\, X(t) + \beta\, X(t)^3 + X(t)\, Y(t), \quad t \geq 0, \tag{15}$$

where $Y(t) = \sum_{j=1}^{d} Z_j\, \varphi_j(t)$ is a parametric input process, $Z = (Z_1, \ldots, Z_d)$ denotes an $\mathbf{R}^d$-valued random variable, $\{\varphi_j(t)\}$ are specified deterministic functions, and α, β denote real constants. Note that $X(t)$ is the state of a nonlinear system with non-Gaussian multiplicative noise. The random input $Y(t)$ will be Gaussian if the random variables $\{Z_i\}$ are Gaussian. Let $\{\tilde{z}_k\}$ be the samples of a SROM model $\tilde{Z}$ for Z and $\{\Gamma_k\}$ the cells of a Voronoi tessellation centered on the samples of $\tilde{Z}$. The deterministic solutions $\tilde{x}_k(t)$ and the coordinates $\tilde{w}_{k,j}(t) = \partial \tilde{x}_k(t)/\partial z_j$ of the gradients of these solutions can be calculated from

$$\dot{\tilde{x}}_k(t) = \alpha\, \tilde{x}_k(t) + \beta\, \tilde{x}_k(t)^3 + \tilde{x}_k(t) \sum_{j=1}^{d} \tilde{z}_{k,j}\, \varphi_j(t) \text{ and}$$

$$\dot{\tilde{w}}_{k,j}(t) = \left(\alpha + 3\,\beta\, \tilde{x}_k(t)^2 + \sum_{j=1}^{d} \tilde{z}_{k,j}\, \varphi_j(t) \right) \tilde{w}_{k,j}(t) + \tilde{x}_k(t)\, \varphi_j(t), \quad j = 1, \ldots, d. \tag{16}$$

The piecewise linear approximation $X_L(t)$ of $X(t)$ is given by Eq. 13. Following numerical results are for $d = 2$, (Z_1, Z_2) = translation Beta random variables with range $(1, 4)$, shape parameters $(p, q) = (1, 3)$, and correlation coefficient $\rho = 0.2$ for

their image in the Gaussian space, $\alpha = 1$, $\beta = -1$, $\nu = 1$, initial state $X(0) = 1$, time interval $[0, 10]$, $\varphi_1(t) = \cos(\nu\, t)$, and $\varphi_2(t) = \sin(\nu\, t)$. The samples and probabilities of a SROM $\tilde{Z}$ with $m = 10$ are

$$\tilde{z}_{k,1} = 1.36;\ 1.39;\ 1.37;\ 2.77;\ 1.58;\ 2.30;\ 1.19;\ 1.22;\ 2.05\ \ 1.71$$
$$\tilde{z}_{k,2} = 1.33;\ 1.00;\ 2.81;\ 1.25;\ 1.01;\ 2.55;\ 2.29;\ 1.53;\ 1.37;\ 2.03$$

and

$$p_k = 0.0001;\ 0.0001;\ 0.2075;\ 0.2435;\ 0.0882;\ 0.0598;\ 0.0713;\ 0.0967;\ 0.1403;\ 0.0924.$$

Figure 2 shows the deterministic solutions $\tilde{x}_k(t)$ corresponding to $Z = \tilde{z}_k$, $k =$

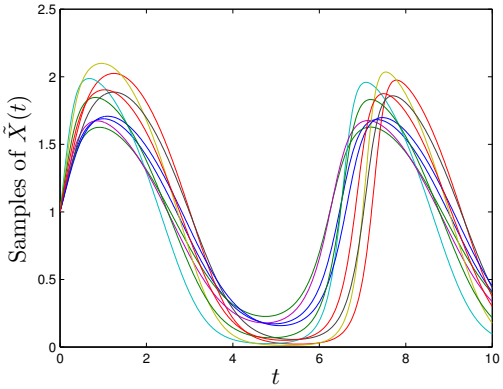

Figure 2: Deterministic solutions $\tilde{x}_k(t)$

$1, \ldots, m$, that is, the samples of $\tilde{X}(t)$. The sensitivity factors $\tilde{w}_{k,j}$, $j = 1, 2$, are shown in Fig. 3. The heavy solid line in Fig. 4 is a Monte Carlo estimate of $E[X(t)^6]$.

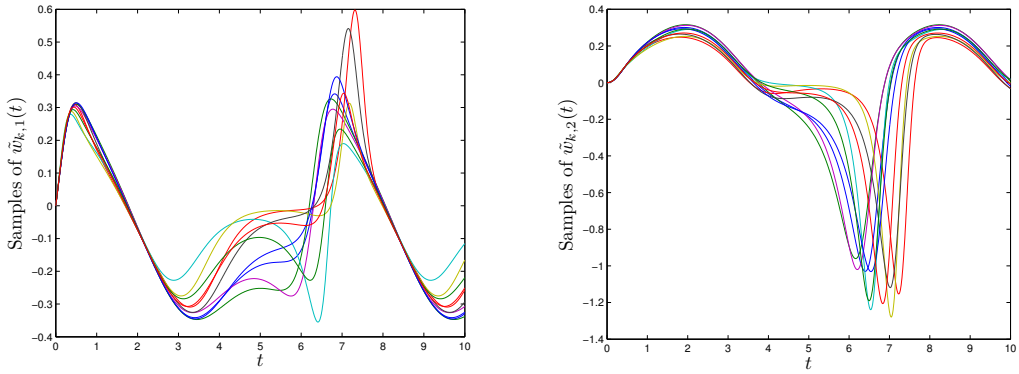

Figure 3: Coordinates of gradient $\nabla \tilde{x}_k(t)$ with respect to Z_1 (left panel) and Z_2 (right panel)

The dashed line is an approximate of this moment delivered by the SROM of $X(t)$

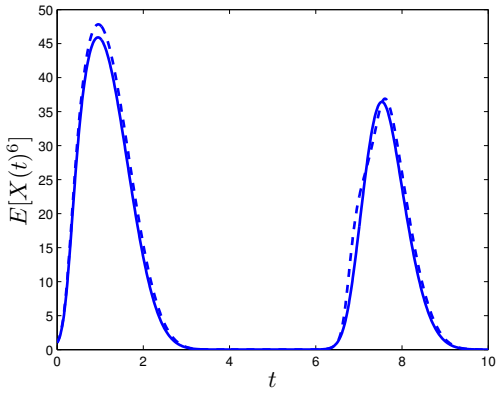

Figure 4: Monte Carlo estimate of $E[X(t)^6]$ (heavy solid line) and SROM-based approximation of this moment (heavy dotted line)

corresponding to $\tilde{Z}$. It has been obtained by using the first formula in Eq. 11. The accuracy of the approximation is remarkable given that it is obtained from only $m = 10$ deterministic solutions. The approximation of $E[X(t)^6]$ given by the extended SROM method in Eq. 14 is indistinguishable from the Monte Carlo estimate at the figure scale. For this system, both SROM-based methods provide accurate approximations for $E[X(t)^6]$. A Monte Carlo estimate and SROM-based approximations for the marginal distribution $F(x;t) = P(X(t) \leq x)$ of $X(t)$ at time $t = 1$ are shown in Fig. 5. The thin solid line and the heavy dashed line are a Monte Carlo estimate

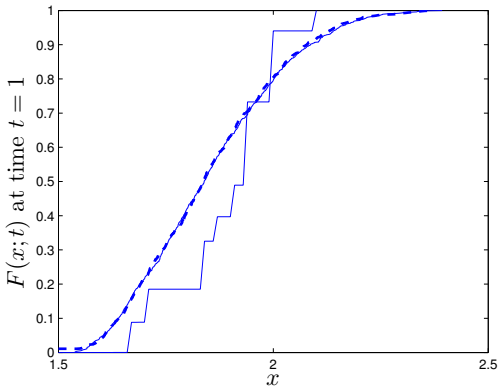

Figure 5: Marginal distribution $F(x;t) = P(X(t) \leq x)$: Monte Carlo estimate (thin solid line), SROM-based approximation (thin solid step function), and extended SROM-based approximation (heavy dashed line)

and a SROM-based approximation for $F(x;t)$ corresponding to the piecewise linear representation $X_L(t)$ in Eq. 13. The staircase thin solid line in the figure is the approximation of $F(x;t)$ based on a SROM $\tilde{X}(t)$ of $X(t)$ of the type in Eq. 11. The approximation of $F(x;t)$ obtained from $X_L(t)$ is significantly better than that delivered by $\tilde{X}(t)$. The notable improvement of the approximation of $F(x;t)$ given by

the extended SROM method is solely due to the superior representation of mapping $Z \mapsto X(t)$ used in this method. The SROM-based method represents the mapping $Z \mapsto X(t)$ by a piecewise constant function, while the extended SROM method approximation represents this mapping by a piecewise linear function.

4 Conclusion

Stochastic reduced order models (SROMs), that is, random processes with a finite number of samples, have been used to represent inputs to linear and nonlinear dynamic systems and construct approximations for the states of these systems. The approximations are non-intrusive in the sense that their implementation involves solutions of small numbers of deterministic versions of random vibration problems. Two types of SROM-based methods have been presented and illustrated numerically. Results show that for some statistics the extended SROM-based method is superior. It was also shown that Itô's calculus and properties of translation processes provide efficient and accurate solutions for the case of linear systems driven by a class of non-Gaussian noise processes.

References

[1] R. V. Jr. Field, M. Grigoriu, "Optimal stochastic models for spacecraft atmospheric re-entry", Journal of Sound and Vibration, 290:991–1014, 2006.

[2] M. Grigoriu, "Stochastic Calculus. Applications in Science and Engineering", Birkhäuser, Boston, 2002.

[3] J. A. Shohat, J. D. Tamarkin, "The Problem of Moments", Mathematical Surveys, Number 1, The American Mathematical Society, New York, 1943.

[4] J. N. Kapur, H. K. Kesavan, "Entropy Optimization Principles with Applications", Academic Press, Inc., Boston, 1992.

[5] M. Grigoriu, "Applied Non-Gaussian Processes: Examples, Theory, Simulation, Linear Random Vibration, and MATLAB Solutions", Prentice Hall, Englewoods Cliffs, NJ, 1995.

[6] B. Puig, F. Poirion, C. Soize, "Non-Gaussian simulation using Hermite polynomials expansion: convergence and algorithms", Probabilistic Engineering Mechanics, 17(3):253–264, 2002.

[7] J. B. Roberts, P. D. Spanos, "Random Vibration and Statistical Linearization", John Wiley & Sons, New York, 1990.

[8] T. T. Soong, M. Grigoriu, "Random Vibration of Mechanical and Structural Systems", Prentice Hall, Englewood Cliffs, N.J., 1993.

[9] V. V. Bolotin, "Random Vibration of Elastic Systems", Martinus Nijhoff Publishers, Boston, 1984.

[10] Y. K. Lin, G. Q. Cai. "Probabilistic Structural Dynamics. Advanced Theory and Applications", McGraw–Hill, New York, 1995.

[11] M. Grigoriu, “Reduced order models for random functions. Applications to stochastic problems“, Applied Mathematical Modelling, 33(1):161–175, 2009.

[12] M. Grigoriu, “Level-cut inhomogeneous filtered poisson field for two-phase microstructures“, International Journal for Numerical Methods in Engineering, 78:215–228, 2009, DOI: 10.1002/nme.2492.

[13] M. Grigoriu, “Probabilisitc models for stochastic elliptic partial differential equations“, Journal of Computational Physics, 229:8406–8429, 2010, DOI: 10.1016/j.jcp.2010.07.023.