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On Higher Order Approximation for Nonlinear Variational Problems in Nonsmooth Mechanics

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Abstract

This paper is concerned with the hp-version of the finite element method (hp-FEM) to treat a variational inequality that models frictional contact in linear elastostatics. Such an approximation of higher order leads to a nonconforming discretization scheme. We employ Gauss-Lobatto quadrature for the approximation of the nonsmooth friction-type functional and take the resulting quadrature error into account in the error analysis. We prove convergence of the hp-FEM Galerkin solution in the energy norm. To this end we investigate Glowinski convergence for the friction-type functional. The key of our norm convergence result for the $hp-$FEM is the used Gauss-Lobatto integration rule with its high exactness order and its positive weights together with a duality argument in the sense of convex analysis. Then we discuss how our convergence analysis can be further extended to other nonlinear variational problems from nonsmooth mechanics. In particular we treat the Bingham fluid problem and propose a mixed hp-FEM discretization scheme with analogous convergence properties.

Keywords: nonsmooth mechanics, contact, friction, hp-FEM, nonconforming approximation, Gauss-Lobatto quadrature.

1 Introduction

This paper is concerned with the hp-version of the finite element method (hp-FEM) to treat a variational inequality in a vectorial Sobolev space that models bilateral frictional contact in linear elastostatics. Thus we extend recent work [1] for the boundary element method to a larger class of nonlinear variational problems that are treatable by the finite element method.

By the pioneering work of Babuška and co-workers, the exponentially fast convergence of the hp-FEM for linear elliptic problems is well-known. Recently M. Maischak and E.P. Stephan [2, 3], respectively P. Dörsek and J.M. Melenk [4, 5] showed superior convergence properties of adaptive $hp-$ boundary element methods ($hp-$BEM) , respectively adaptive hp- finite element methods in numerical experiments also for unilateral, nonsmooth problems compared to the standard $h-$version.

Such an approximation of higher order leads to a nonconforming discretization scheme. In contrast to previous work [6] we employ Gauss-Lobatto quadrature for the approximation of the nonsmooth friction-type functional and take the resulting quadrature error into account of the error analysis.

Here without any regularity assumptions, we prove convergence of the hp-FEM Galerkin solution in the energy norm. To this end we investigate Glowinski convergence for the friction-type functional. The key of our norm convergence result for the $hp-$FEM is the used Gauss-Lobatto integration rule with its high exactness order and its positive weights together with a duality argument in the sense of convex analysis. Thus our convergence analysis complements prior work of M. Maischak and E.P. Stephan [3] on $hp-$BEM for frictionless unilateral contact and more recent work of P. Dörsek and J.M. Melenk [4] on a mixed $hp-$FEM for frictional bilateral contact.

Then we discuss how our convergence analysis can be further extended to other nonlinear variational problems from nonsmooth mechanics. In particular we treat the Bingham fluid problem and propose a mixed hp-FEM discretization scheme with analogous convergence properties.

The paper is organised as follows. In the following section 2, we shortly present two prominent examples in nonsmooth mechanics in the simplified form of a scalar variational inequality; namely a friction problem from contact mechanics and a Bingham flow problem from fluid mechanics. Then in section 3, we focus to a bilateral frictional contact problem in plane linear elastostatics, provide its variational formulation and some preliminaries for our hp -FEM analysis, and describe the Gauss-Lobatto quadrature. Our main result, the convergence result and its proof come in section 4. In the concluding section 5, we discuss how our convergence analysis can be modified for the Bingham flow problem and extended to a mixed hp-FEM formulation.

2 A class of nonlinear variational problems from nonsmooth mechanics

In this section we review a class of problems from nonsmooth mechanics that in the terminology of Glowinski [7] can be modeled as elliptic variational inequalities of the second kind.

All these problems are considered on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with boundary $\Gamma = \partial\Omega$.

2.1 A simplified friction problem

Let $V = H^1(\Omega)$, γ denote the trace map, and define

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega uv \, dx \\ \ell(v) &= \langle f, v\rangle, \ f \in V^* \\ j(v) &= g \int_\Gamma |\gamma v| d\Gamma, \ \text{where} \ \ g > 0. \end{aligned}$$

Then find $u \in V$ such that for all $v \in V$,

$$a(u, v-u) + j(v) - j(u) \geq \ell(v-u).$$

This variational problem is the simplified version of a friction problem in linear elasticity. Here we can refer to Duvaut and Lions [8] and also to the following section of the present paper, where we treat a full vectorial variational problem that describes frictional contact in plane linear elasticity.

2.2 Laminar stationary flow of a Bingham fluid

Let $V = H_0^1(\Omega)$ and define

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \nabla v \, dx \\ \ell(v) &= \langle f, v\rangle, f \in V^* \\ j(v) &= \int_\Omega |\nabla v| dx. \end{aligned}$$

Then with $\mu > 0, g > 0$ fixed, find $u \in V$ such that for all $v \in V$

$$\mu a(u, v-u) + gj(v) - gj(u) \geq \ell(v-u).$$

If $\ell(v) = C \int_\Omega v \, dx$ (with positive C, the linear decay of pressure), then this variational problem models the laminar stationary flow of a Bingham fluid in a cylindrical pipe of cross section Ω, where $u(x)$ describes the velocity at $x \in \Omega$. The parameters μ, g are, respectively, the viscosity and plasticity yield of the fluid. The Bingham fluid behaves like a viscous fluid in

$$\Omega^+ = \{x \in \Omega : |\nabla u(x)| > 0\}$$

and like a rigid medium in

$$\Omega^0 = \{x \in \Omega : \nabla u(x) = 0\}.$$

For a full vectorial numerical analysis of viscoplastic fluid flow problems we refer to [9].

3 The bilateral frictional contact problem and its hp-FEM approximation

Let us consider an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary Γ that splits into three disjoint parts $\Gamma_0, \Gamma_T, \Gamma_c$ such that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_T} \cup \overline{\Gamma_c}$. Zero displacements are prescribed on Γ_0, surface tractions $\underline{T} \in (L^2(\Gamma_T))^2$ act on Γ_T, and on the part Γ_c Tresca friction conditions between the body and a perfectly rigid foundation hold. In the model of Tresca friction (given friction) one assumes a known slip bound $g \in L^\infty(\Gamma_c)$, $g \geq 0$. Moreover, the body is subject to body forces $\underline{F} \in (L^2(\Omega))^2$. To simplify matters, we assume meas $(\Gamma_0) > 0$; otherwise we can resort to a recession analysis to treat the so called semi-coercive case; see [10] for the standard h-FEM.

We denote by $H^s(\,)$ the usual Sobolev spaces on Ω or on parts of Γ with norms defined using the Slobodeckij semi-norms. We also use the short $\underline{H}^s = (H^s)^2$ for the vectorial Sobolev spaces. In particular, we have the space of *virtual displacements*

$$\mathcal{V} = \{\underline{v} \in \underline{H}^1(\Omega) \,|\, \gamma_0\, \underline{v} = 0\}\,,$$

where $\gamma_o = \gamma_{\Gamma 0} : \underline{H}^1(\Omega) \to \underline{H}^{\frac{1}{2}}(\Gamma_0)$ is the trace map onto Γ_0. Here, likewise $\gamma_c = \gamma_{\Gamma c} : \underline{H}^1(\Omega) \to \underline{H}^{\frac{1}{2}}(\Gamma_c) \subset (L^2(\Gamma_c))^2$, and with the unit outer normal $\underline{n} \in (L^\infty(\Gamma))^2$ to the boundary, a vector field $\underline{w}$ at the boundary has its normal component $w_n = \underline{w} \cdot \underline{n}$ and its tangential component $\underline{w}_t = \underline{w} - w_n\, \underline{n}$.

Adopting standard notations from linear elasticity, $\varepsilon(\underline{v}) = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$ denotes the small strain tensor to the displacement field $\underline{v}$ and $\sigma(\underline{v}) = C : \varepsilon(\underline{v})$ the stress tensor. Here, C is the Hooke tensor, assumed to be uniformly positive definite with L^∞ coefficients. This leads to the bilinear form, linear functional, sublinear functional, and to the *total potential energy* of the body, respectively,

$$\begin{aligned}
a(\underline{u}, \underline{v}) &= \int_\Omega \varepsilon(\underline{u}) : C : \varepsilon(\underline{v})\, dx\,, \\
l(\underline{v}) &= \int_\Omega \underline{F} \cdot \underline{v}\, dx + \int_{\Gamma_T} \underline{T} \cdot \underline{v}\, ds\,, \\
j(\underline{v}) &= \int_{\Gamma_c} g\, |\underline{v}_t|\, ds\,, \\
J(\underline{v}) &= \frac{1}{2} a(\underline{v}, \underline{v}) - l(\underline{v}) + j(\underline{v})\,.
\end{aligned}$$

In these terms, the *variational formulation* of the unilateral contact problem with Tresca friction reads as follows: Find a minimizer $\underline{u} \in \mathcal{V}$ of the functional $J(\underline{v}), \underline{v} \in \mathcal{V}$!

Another equivalent formulation is the *variational inequality* problem (π) *of second kind*: Find $\underline{u} \in \mathcal{V}$ such that for all $\underline{v} \in \mathcal{V}$,

$$a(\underline{u}, \underline{v} - \underline{u}) + j(\underline{v}) - j(\underline{u}) \geq l(\underline{v} - \underline{u}). \tag{1}$$

There exists a unique solution $\underline{u}$ (see e.g. [11, 12]), since Γ_0 has positive measure and hence the bilinear form is coercive by Korn's inequality, see e.g. [8] . Moreover there is the a priori estimate

$$a(\underline{u}, \underline{u}) \leq c_0 + c_1 \|\underline{u}\|_{\underline{H}^1(\Omega)} \tag{2}$$

for some constants $c_0, c_1 \geq 0$.

For simplicity let Ω be a polygonal, planar domain and let g be a piecewise constant function on Γ_C. These are no restrictions of generality. In fact, the $hp-$ finite element approximation on curvilinear domains is well-understood, see [13]. The analysis to follow can be extended to higher dimensional domains by tensor product approximation.

To conclude the preliminaries for our finite element analysis we state an essential hypothesis, namely the density relation

$$\overline{\mathcal{V} \cap [C^\infty(\overline{\Omega})]^2} = \mathcal{V}\,. \tag{3}$$

We note that (see [12]) (3) holds true in the polygonal domain Ω, if there is only a finite number of "end–points" $\overline{\Gamma}_c \cap \overline{\Gamma}_T$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_T$, $\overline{\Gamma}_c \cap \overline{\Gamma}_0$.

Let $\mathcal{T}_N$ $(N \in \mathbb{N})$ be a shape regular [14] sequence of meshes consisting of affine quadrilaterals $Q \in \mathcal{T}_N$ with diameter $h_{N,Q}$ such that all corners of Γ and all "end points" $\overline{\Gamma}_c \cap \overline{\Gamma}_T, \overline{\Gamma}_T \cap \overline{\Gamma}_0, \overline{\Gamma}_0 \cap \overline{\Gamma}_c$ are nodes of $\mathcal{T}_N$. Moreover, we introduce the set of edges on the contact boundary,

$$\mathcal{E}_{c,N} = \{E : \ E \subset \Gamma_c \text{ is an edge of } \mathcal{T}_N\}$$

and assume that g is constant on each edge $E \in \mathcal{E}_{c,N}$. Obviously, for every $E \in \mathcal{E}_{c,N}$ there exists a unique $Q_E \in \mathcal{T}_N$ such that E is an edge of Q_E.

Further we denote by $p_{N,Q} \in \mathbb{N}$ a polynomial degree for each $Q \in \mathcal{T}_N$. We assume that neighboring elements have comparable polynomial degrees, i.e. there exists a constant $c > 0$ such that for elements $Q, Q' \in \mathcal{T}_N$ with $\overline{Q} \cap \overline{Q'} \neq \emptyset$ there holds

$$c^{-1}\, p_{N,Q} \leq p_{N,Q'} \leq c\, p_{N,Q}\,.$$

Let $\Pi^p(Q)$ be the tensor product space of polynomials of degree p in each variable. This gives the FE subspace

$$\mathcal{V}_N = \{\underline{v}_N \in \mathcal{V} \,:\, \underline{v}_N \mid Q \in (\Pi^{p_{N,Q}}(Q))^2\,, \forall Q \in \mathcal{T}_N\}\,.$$

We employ Gauss-Lobatto quadrature in the discretization procedure. To this end we introduce for $q \geq 1$ on the reference interval $[-1, 1]$ the $q + 1$ Gauss-Lobatto points, i.e., the zeros ξ_j^{q+1} $(0 \leq j \leq q)$ of $(1 - \xi^2)L_q'(\xi)$, where L_q denotes the Legendre polynomial of degree q. Note that $\xi_0^{q+1} = -1$ and $\xi_q^{q+1} = 1$ are the end points of the reference interval. It is known (see [15], chapter I, section 4) that there exist positive weights

$$\omega_j^{q+1} := \frac{1}{q(q+1)L_q^2(\xi_j^{q+1})}$$

such that the quadrature formula

$$\int_{-1}^{1} \phi(\xi)\, d\xi = \sum_{j=0}^{q} \omega_j^{q+1} \phi(\xi_j^{q+1})$$

is exact for all polynomials ϕ up to degree $2q-1$.

For any $E \in \mathcal{E}_{c,N}$ we introduce the quadrature order $q_{N,E}$ such that $q_{N,E} = p_{N,Q_E}$. By affine transformation $F_E : [-1,1] \to \overline{E}$ we define the set $G_{E,N}$ of $q_{N,E}+1$ Gauss-Lobatto points for each element E of $\mathcal{E}_{c,N}$ and set $G_{c,N} := \bigcup\{G_{E,N} : E \in \mathcal{E}_{c,N}\}$.

We approximate the nonlinear nonsmooth functional j using the above quadrature rule by

$$j_N(\underline{v}) = j_{c,N}(\gamma_c\, \underline{v})_t,\ j_{c,N}(\psi) = \sum_{E \in \mathcal{E}_{c,N}} g_E \sum_{j=0}^{q_{N,E}} \omega_j^{q_{N,E}+1} \left|\psi \circ F_E(\xi_j^{q_{N,E}+1})\right|,$$

where g_E denotes the constant value of the function g on E. Then $j_N, j_{c,N}$ are sublinear functionals, with $j_{c,N}$ uniformly bounded on $C(\overline{\Gamma_c})$. Note that for the piecewise polygonal boundary Γ, $\underline{v}_t \circ F_E$ is piecewise polynomial of the same degree as $\underline{v}$.

Thus we arrive at the following discrete variational problem (π_N) as approximation to our variational problem (π): Find $\underline{u}_N \in \mathcal{V}_N$ such that for all $\underline{v}_N \in \mathcal{V}_N$

$$a(\underline{u}_N, \underline{v}_N - \underline{u}_N) + j_N(\underline{v}_N) - j_N(\underline{u}_N) \geq l(\underline{v}_N - \underline{u}_N)\,. \tag{4}$$

Similarly to the above bound (2) we obtain the a priori bound

$$a(\underline{u}_N, \underline{u}_N) \leq c_0 + c_1 \|\underline{u}_N\|_{\underline{H}^1(\Omega)} \tag{5}$$

for some constants $c_0, c_1 \geq 0$ independent of N.

Note that we only replaced the nonlinear functional j by its approximate j_N. In most computations, however, also a and l have to be replaced by some approximations that take into account e.g. numerical integration or approximation of a curved boundary. Since such approximations are well documented in the literature of $h-$ and $hp-$ finite element analysis of elliptic boundary value problems (see [13, 14]), we omit this aspect here.

Associated to the Gauss-Lobatto points $G_{E,N}$ we have the local interpolation operator $i_{E,q} = i_{E,N} : C^0(\overline{E}) \to \mathcal{P}_q(E)$ with $q = q_{N,E}$ given by

$$(i_{E,N}\eta)(x) = \eta(x),\ \forall\, x \in G_{E,N},\ \eta \in C^0(\overline{E})$$

and the global interpolation operator $i_{c,N}$ on $C^0(\overline{\Gamma_c})$ defined by

$$i_{c,N}\eta = \sum_{E \in \mathcal{E}_{c,N}} (i_{E,N}\eta)|\overline{E},\ \forall\, \eta \in C^0(\overline{\Gamma})\,.$$

Likewise associated to the Gauss-Lobatto points $G_{Q,N} = F_Q\{(\xi_i^{p+1}, \xi_j^{p+1}) \,|\, 0 \le i,j \le p\}$ with $p = p_{N,Q}$ and the affine transformation $F_Q : [-1,1]^2 \to \overline{Q}$ we have the local interpolation operator $i_{Q,p} = i_{Q,N} : C^0(\overline{Q}) \to \mathcal{P}_p(Q)$ with $p = p_{N,E}$ given by

$$(i_{Q,N}\psi)(x) = \psi(x),\ \forall\, x \in G_{Q,N},\ \psi \in C^0(\overline{Q})$$

and the global interpolation operator i_N on $C^0(\overline{\Omega})$ defined by

$$i_N\psi = \sum_{Q\subset\Omega} (i_{Q,N}\psi)|\overline{Q},\ \forall\, \psi \in C^0(\overline{\Omega})\,.$$

For later use we recall from [15, Theorem 13.4, Theorem 14.2] the following results on the polynomial interpolation error in the reference interval $\hat{E} = (-1,1)$, respectively in the reference square $\hat{Q} = (-1,1)^2$.

Theorem 1 *(i) For any real numbers r and s satisfying $s > (1+r)/2$ and $0 \le r \le 1$, there exists a positive constant c depending only on s such that for any function $\eta \in H^s(\hat{E})$ the following estimate holds*

$$\|\eta - i_{\hat{E},q}\eta\|_{H^r(\hat{E})} \le c\, q^{r-s}\, \|\eta\|_{H^s(\hat{E})}\,. \tag{6}$$

(ii) For any real numbers r and s satisfying $s > 1 + r/2$ and $0 \le r \le 1$, there exists a positive constant c depending only on s such that for any function $\psi \in H^s(\hat{Q})$ the following estimate holds

$$\|\eta - i_{\hat{Q},p}\psi\|_{H^r(\hat{Q})} \le c\, p^{r-s}\, \|\psi\|_{H^s(\hat{Q})}\,. \tag{7}$$

4 The $hp-$ approximation result

Without any regularity assumption for the solution $\underline{u}$ of (π) we can show the following convergence result for the $hp-$ FEM solutions $\underline{u}_N$ of (π_N) in the energy norm.

Theorem 2 *Suppose that for the polygonal domain Ω, there are only a finite number of "end points" $\overline{\Gamma}_c \cap \overline{\Gamma}_0, \overline{\Gamma}_c \cap \overline{\Gamma}_T, \overline{\Gamma}_T \cap \overline{\Gamma}_c$. Then for $N \to \infty$ with $\min_{Q\in\mathcal{T}_N} h_{N,Q}^{-1}\, p_{N,Q} \to \infty$ there holds $\underline{u}_N \to \underline{u}$ with respect to the $\underline{H}^1(\Omega)$ norm.*

Proof.

Here we use the discretization theory of Glowinski [7]. Thus we have to show the following hypotheses:

H1 If $\underline{v}_N \rightharpoonup \underline{v}$ (weak convergence) in $\mathcal{V}$ for $N \to \infty$, then

$$\liminf_{N\to\infty} j_N(\underline{v_N}) \ge j(\underline{v}).$$

H2 There exist a subset $M \subset \mathcal{V}$ dense in $\mathcal{V}$ and mappings $\varrho_N : M \to \mathcal{V}_N$ such that, for each $\underline{w} \in M$, $\varrho_N(\underline{w}) \to \underline{w}$ for $N \to \infty$,

$$\lim_{N\to\infty} j_N\big(\varrho_N(\underline{w})\big) = j(\underline{w})\,.$$

Classical $h-$ FEM convergence for the variational problem under study is already treated in [6], where Newton-Cotes formulas in numerical quadrature are used instead of Gauss-Lobatto quadrature. Inspecting the proof of [6, Theorem 4.1] shows that the norm convergence for a fixed quadrature order hinges on the positiveness of the quadrature weights, what is satisfied for all quadrature orders with Gauss-Lobatto quadrature. Therefore in the following we can focus to the case where $h_{N,Q}$ is fixed for all $Q \in \mathcal{T}_N$ and $\min_{Q\in\mathcal{T}_N} p_{N,Q} \to \infty$.

To verify H1 it is enough to show that for any $\mu \in C^0(\Gamma)$ with $|\mu| \le 1$ on Γ_c there holds

$$\int_{\Gamma_C} g\,\underline{v}_t\,\mu\,ds \le \liminf_{N\to\infty} j_N(\underline{v}_N)\,, \tag{8}$$

since by duality with respect to (L^1, L^∞) and density

$$j(\underline{v}) = \sup\Big\{ \int_{\Gamma_c} g\,\underline{v}_t \mu\,ds\ :\ \mu \in C^0(\Gamma), |\mu| \le 1 \Big\}.$$

Moreover, since the mesh $\mathcal{T}_N$ is independent of N, we can simply consider the above integrals on any fixed edge $E \in \mathcal{E}_{c,N}$. Thus fix $\mu \in C^0[\overline{E}]$ with $|\mu| \le 1$ and also $q := q_{N,E}$. We approximate these functions by a combination of Bernstein polynomials B_q with the local mapping $F_E : [-1,1] \to \overline{E}$ to define μ_q via

$$\mu_q(t) = (B_q\mu \circ F_E)(t) := \sum_{k=0}^{q} \binom{q}{k} (\frac{1+t}{2})^k (\frac{1-t}{2})^{q-k} (\mu \circ F_E)\left(\frac{2k}{q} - 1\right).$$

Since the Bernstein operators are monotone, $|\mu_q| \le 1$. By [16, Chapter 1, Theorem 2.3],

$$\lim_{q\to\infty} \|\mu_q - \mu\|_{L^\infty(E)} = 0\,. \tag{9}$$

Since the embedding $H^{1/2}(\Gamma) \hookrightarrow L^1(E)$ is weakly continuous, $\underline{v}_N \rightharpoonup \underline{v}$ in $L^1(E)$ and $\|\underline{v}_N\|_{(L^1(E))^2}$ is bounded. Therefore from

$$\begin{aligned}
&\Big|\int_E [\underline{v}_{N,t}\,\mu_{q_{N,E}-1} - \underline{v}_t\mu]\,dt\Big| \\
&\le \big\|\underline{v}_{N,t}\big\|_{L^1(E)} \big\|\mu_{q_{N,E}-1} - \mu\big\|_{L^\infty(E)} \\
&\quad + \Big|\int_E [\underline{v}_{N,t} - \underline{v}_t]\mu\,dt\Big|,
\end{aligned}$$

(9) and using $\mu \in L^\infty(e) = \big(L_1(e)\big)^*$, we conclude

$$\lim_{N\to\infty} \int_E \underline{v}_{N,t}\, \mu_{q_{N,E}-1} dt = \int_E \underline{v}_t\, \mu\, dt\,. \tag{10}$$

On the other hand, $\underline{v}_{N,t}|E\, \mu_{q_{N,E}-1}$ are polynomials of degree $2q-1$. Hence the above integrals can be evaluated exactly by the Gauss-Lobatto quadrature formula to obtain

$$\int_E \underline{v}_{N,t}\, \mu_{q_{N,E}-1}\, dt = \sum_{j=0}^{q} \omega_j^{q+1} \left(\underline{v}_{N,t}\, \mu_{q-1}\right) \circ F_E(\xi_j^{q+1})\,.$$

Since the weights $\omega_j^{q+1} > 0$, $|\mu_{q-1}| \le 1, g_e \ge 0$ we arrive at

$$\begin{aligned} &g_E \int_E \underline{v}_{N,t}\, \mu_{q_{N,E}-1}\, dt \le g_E \sum_{j=0}^{q} \omega_j^{q+1} |\underline{v}_{N,t} \circ F_E(\xi_j^{q+1})| =: j_{E,N}(\underline{v}_N)\,, \\ &\sum_{E\in\mathcal{E}_{c,N}} j_{E,N}(\underline{v}_N) = j_N(\underline{v}_N)\,. \end{aligned}$$

In view of (10) this proves our claim (8).
In the last step let us prove H2.
By the finiteness assumption we have due to [12] the density relation

$$\overline{\mathcal{V} \cap [C^\infty(\Omega)]^2} = \mathcal{V}.$$

Therefore we can take $M = [C^\infty(\Omega)]^2$ and define $\varrho_N : M \to \mathcal{V}_N$ by $\varrho_N := i_N$. By Theorem 1(ii) , $\varrho_N \underline{w} \to \underline{w}$ in $\underline{H}^1(\Omega)$. Finally by $j_N(\underline{w}) = j(\varrho_N \underline{w})$, we conclude for $N \to \infty$,

$$\begin{aligned} &|j(\underline{w}) - j_N(\varrho_N \underline{w})| \le |j(\underline{w}) - j(\varrho_N \underline{w})| + |j_N(\underline{w}) - j_N(\varrho_N \underline{w})| \\ &\le \|g\|_{L^\infty(\Gamma_c)} \left[\|\underline{w}_t - (\varrho_N \underline{w})_t\|_{L^1(\Gamma_c)} + \|\underline{w}_t - (\varrho_N \underline{w})_t\|_{L^\infty(\Gamma_c)} \right] \to 0\,. \end{aligned}$$

■

5 Concluding remarks on the Bingham fluid problem

The primal variational formulation of the simplified Bingham fluid problem given in section 2 corresponds to the strong formulation

$$f \in g\, \partial j(u) - \Delta\, u\,,$$

where ∂j denotes the subdifferential of the sublinear functional j in the sense of convex analysis. We modify the well-known reformulation of the classic Poisson problem with zero Dirichlet boundary condition as a saddle point problem and write $\Delta\, u = \text{div}\, \sigma, \sigma = \nabla u$. This leads to the following mixed formulation: Find $(\sigma, u) \in L_2^d(\Omega) \times H_0^1(\Omega)$ such that for all $(\tau, v) \in L_2^d(\Omega) \times H_0^1(\Omega)$

$$\left\{ \begin{array}{lrcl} (\sigma, \tau)_0 & -(\tau, \nabla\, u)_0 & = & 0 \\ & (\sigma, \nabla\, v - \nabla\, u)_0 & \geq & \langle f, v \rangle - g[j(v) - j(u)] \,. \end{array} \right.$$

This is a saddle point problem with the bilinear forms $\alpha(\sigma, \tau) = (\sigma, \tau)_0$ on $X \times X$, $\beta(\sigma, u) = -(\sigma, \nabla\, u)_0$ on $X \times M$, where $X = L_2^d(\Omega), M = H_0^1(\Omega)$ and $(\cdot, \cdot)$ denotes the L_2^d, respectively L_2 scalar product.

Let again $\mathcal{T}_N$ $(N \in \mathbb{N})$ denote a shape regular sequence of meshes consisting of affine quadrilaterals $Q \in \mathcal{T}_N$ with diameter $h_{N,Q}$ and let $\Pi^p(Q)$ be the tensor product space of polynomials of degree p in each variable. Then appropriate FE subspaces are

$$\begin{aligned} X_N &= \{\tau_N \in L_2^d(\Omega) \,:\, \tau_N \mid Q \in (\Pi^{p-1_{N,Q}}(Q))^d \,,\, \forall Q \in \mathcal{T}_N\} \,, \\ M_N &= \{v_N \in H_0^1(\Omega) \,:\, v_N \mid Q \in (\Pi^{p_{N,Q}}(Q))^2 \,,\, \forall Q \in \mathcal{T}_N\} \,. \end{aligned}$$

We again use (L^1, L^∞) duality and now the representation

$$j(v) = \sup \Big\{ \int_\Omega \nabla\, v \cdot \mu \, dx \,:\, \mu \in L_\infty^d(\Omega), |\mu| \leq 1 \Big\} \,,$$

where $L_\infty^d(\Omega)$ is endowed with the norm

$$|\mu| = \text{ ess } \sup_{\xi \in \Omega} |\mu(\xi)| = \text{ ess } \sup_{\xi \in \Omega} [\sum_{k=1}^d \mu_k^2(\xi)]^{1/2} \,,$$

that is equivalent to the norm

$$\|\mu\|_\infty = \text{ ess } \sup_{\xi \in \Omega} \max_{k=1,\ldots,d} |\mu_k(\xi)| \,.$$

Thus we can modify the arguments of the proof of Theorem 2 and arrive at an analogous convergence result for the considered mixed hp-fem approximation.

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