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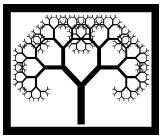

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Bipenalty Methods for Single-Point and Multi-Point Constraints in Computational Dynamics

J. Hetherington, A. Rodriguez-Ferran and H. Askes
Department of Civil and Structural Engineering
University of Sheffield, United Kingdom

Abstract

Penalty methods are used in finite element analysis to model constraints for a wide range of problem types. A major drawback of the method, however, is that the traditional formulation can introduce spurious eigenfrequencies of large magnitude. In explicit dynamics the critical time step of an analysis depends on the maximum eigenfrequency in such a way that it may be drastically decreased by the addition of penalty constraints, increasing total expense and the risk of instability. This problem may be solved using the bipenalty method, which includes mass penalties alongside the standard stiffness penalty formulation. In this paper, we formulate the bipenalty method for an arbitrary set of constraint equations and show that the spurious eigenfrequencies tend to a finite value determined by the ratio of the stiffness and mass penalty parameters. Through numerical examples, we demonstrate that the method may be utilised such that time step stability is ensured, whilst also displaying superiority over the standard mass penalty method in terms of accuracy and versatility.

Keywords: finite element methods, constraints, penalty methods, stability, explicit dynamics, critical time step.

1 Introduction

The imposition of constraints is necessary for many types of numerical analysis. They are expressed as a set of equations that accompany the usual system of simultaneous equations to be solved, often enforcing boundary conditions and any other special relationships between degrees of freedom (DOF). In finite element (FE) analysis, one of the most popular techniques for imposing constraints is the use of penalty functions. Typically, penalty functions are applied by modification of the stiffness matrix of a system. In fact, they can be understood to represent artificial springs of large stiffness

which act to enforce the given constraints. Increasing these virtual stiffness values gives greater accuracy of constraint imposition. Compared to other techniques (e.g. Lagrange multipliers, direct transformation), the penalty method is simple to formulate and implement, and it has the advantage of preserving the size of the system. The main disadvantage is that the constraints are not enforced exactly (although, if care is taken, errors in constraint imposition can usually be kept within acceptable bounds).

However, when standard penalty methods are transferred to the field of explicit dynamic analysis the method has a further disadvantage. The stability of conditionally stable time integrators is reliant on the time step of the analysis being below the critical time step Δt_{crit}, which is related to the maximum eigenfrequency of the system. Adding additional stiffness into the system causes an increase in the maximum eigenfrequency, which decreases the critical time step. When this is accounted for by the analyst the total simulation time may be drastically increased; if it is not accounted for, instability may occur.

A related method which does not exhibit this disadvantage is the mass penalty method [1]. Mass penalties operate on the mass matrix of the system rather than the stiffness matrix (and therefore can only be used in dynamic analyses); they can be understood as virtual inertia values which enforce constraints in a similar way to the virtual springs of the stiffness penalty method. Adding additional mass to the system does not increase the maximum eigenfrequency and so the critical time step is not affected. However, they are ineffectual in circumstances where a violation of the displacement constraint is not accompanied by a violation of the corresponding *acceleration* constraint. Perhaps the most prominent example of this is the field of contact-impact, where penetration (displacement constraint violation) may occur without any relative difference in acceleration.

In this paper, we investigate the bipenalty method: the use of both stiffness and mass penalty functions simultaneously. The goal is to combine the accuracy and versatility of the well-established stiffness penalty method with the superior stability behaviour of mass penalties. In order to investigate this we examine dynamic finite element systems, with time integration carried out using the explicit central difference method (CDM). Some treatments of the bipenalty method already exist in the literature [2–6]; our goal in the present contribution is to formulate the method involving a set of k arbitrary constraints (involving any number of DOF) and to give a precise analysis of the resulting eigenproblem, so that robust guidelines for the selection of penalty parameters can be developed.

2 Formulation of the bipenalty method

Presently, we consider the standard finite element discretisation of the equations of elastodynamics, written in matrix form as

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f} \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively, $\mathbf{u}$ is the displacement vector containing the n solution variables, $\mathbf{f}$ is the external force vector and dot notation is used to indicate time derivatives. We assume this system has n degrees of freedom in total.

In this section we give a bipenalty formulation for a set of k constraint equations written as

$$\mathbf{h} = \mathbf{C}\mathbf{u} - \mathbf{q} \tag{2}$$

where $\mathbf{C}$ is the constraint matrix of size $k \times n$, $\mathbf{q}$ is a vector of prescribed displacements and $\mathbf{h} = \mathbf{0}$ implies exact satisfaction of the constraints. The constraint matrix describes the relationships between DOF for each constraint and we assume here that its rows are linearly independent (i.e., there are no duplicated constraints).

The stiffness and mass penalty methods may be derived by considering the potential energy $\mathscr{U}$ and kinetic energy $\mathscr{T}$ of the FE system. The traditional penalty method applies a penalty function to the potential energy only, so that [7]

$$\mathscr{U} = \frac{1}{2}\mathbf{u}^T\mathbf{K}\mathbf{u} - \mathbf{u}^T\mathbf{f} + \frac{1}{2}\mathbf{h}^T\mathbf{P}_\mathrm{s}\mathbf{h} \tag{3}$$

Here, the final term is the penalty function and $\mathbf{P}_\mathrm{s}$ is a diagonal matrix of size k containing the penalty parameters $\alpha_{\mathrm{s},j}$ (where $j = 1 \ldots k$). As the penalty parameters are increased, the minimisation of $\mathscr{U}$ results in a smaller constraint violation.

The mass penalty method follows from writing the above constraints in rate form [3] as

$$\dot{\mathbf{h}} = \mathbf{C}\dot{\mathbf{u}} - \dot{\mathbf{q}} \tag{4}$$

where $\mathbf{C}$ is assumed constant. We then proceed by considering the kinetic energy $\mathscr{T}$ of the system with a similar penalty term included, so that

$$\mathscr{T} = \frac{1}{2}\dot{\mathbf{u}}^T\mathbf{M}\dot{\mathbf{u}} + \frac{1}{2}\dot{\mathbf{h}}^T\mathbf{P}_\mathrm{m}\dot{\mathbf{h}} \tag{5}$$

where $\mathbf{P}_\mathrm{m}$ is a diagonal penalty matrix with the same form as $\mathbf{P}_\mathrm{s}$, but containing instead the mass penalty parameters $\alpha_{\mathrm{m},j}$. The penalised equilibrium equations then follow from

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{T}}{\partial \dot{\mathbf{u}}^T} + \frac{\partial \mathscr{U}}{\partial \mathbf{u}^T} = \left[\mathbf{M} + \mathbf{M}^\mathrm{P}\right]\ddot{\mathbf{u}} + \left[\mathbf{K} + \mathbf{K}^\mathrm{P}\right]\mathbf{u} = \mathbf{f} + \mathbf{f}^\mathrm{P} \tag{6}$$

$$\text{where} \quad \mathbf{K}^\mathrm{P} = \mathbf{C}^T\mathbf{P}_\mathrm{s}\mathbf{C} \tag{7}$$

$$\mathbf{M}^\mathrm{P} = \mathbf{C}^T\mathbf{P}_\mathrm{m}\mathbf{C} \tag{8}$$

$$\mathbf{f}^\mathrm{P} = \mathbf{C}^T\mathbf{P}_\mathrm{s}\mathbf{q} + \mathbf{C}^T\mathbf{P}_\mathrm{m}\ddot{\mathbf{q}} \tag{9}$$

In general, the matrix we must add to implement a constraint using inertial penalties has the same form as the standard stiffness penalty matrix. Implementation of the bipenalty method merely requires us to choose a second set of penalty parameters, and include the mass penalty matrix $\mathbf{M}^\mathrm{P}$ alongside $\mathbf{K}^\mathrm{P}$ and $\mathbf{f}^\mathrm{P}$. Additionally, we note that

it is common for the prescribed values q_j to be zero, so that $\mathbf{q} = \ddot{\mathbf{q}} = \mathbf{0}$ and therefore $\mathbf{f}^{\mathrm{P}} = \mathbf{0}$ in many cases.

The penalty parameters $\alpha_{\mathrm{s},j}$ and $\alpha_{\mathrm{m},j}$ have units of N/m and kg, respectively. However, since the accuracy of constraint imposition is dependent on the magnitude of the parameters relative to the existing entries in $\mathbf{K}$ and $\mathbf{M}$, it is useful to define dimensionless 'penalty factors' that quantify the size of the penalty parameters, denoted by $p_{\mathrm{s},j}$ and $p_{\mathrm{m},j}$. For example, for a constraint acting on the ith DOF we use the corresponding entry in $\mathbf{K}$ and $\mathbf{M}$ to give $p_{\mathrm{s},j} = \alpha_{\mathrm{s},j}/K_{ii}$ and $p_{\mathrm{m},j} = \alpha_{\mathrm{m},j}/M_{ii}$. For constraints acting on multiple DOF we use the largest diagonal entry connected with those DOF.

3 Eigenvalue analysis of the bipenalised system

Having described the general formulation of the system, we now turn our attention to the associated eigenproblem. As previously described, the critical time step associated with conditionally stable integrators relies on the maximum eigenfrequency of the system $\omega_{\max}$; for the CDM it is given by [8]

$$\Delta t_{\mathrm{crit}} = \frac{2}{\omega_{\max}} \tag{10}$$

The unpenalised eigenvalue problem (UP) can be stated as

$$(\mathbf{K} - \lambda_i \mathbf{M})\,\mathbf{u}_i = \mathbf{0} \tag{11}$$

where $\mathbf{K}$ and $\mathbf{M}$ are the (symmetric) stiffness and mass matrices, respectively, and the eigenvectors $\mathbf{u}_i$ and corresponding eigenvalues λ_i form the n solutions (ordered so that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \equiv \lambda_{\max}$). The corresponding eigenfrequencies are then given by $\omega_i = \sqrt{\lambda_i}$.

The bipenalised problem (BP) is given by

$$\left([\mathbf{K} + \mathbf{K}^{\mathrm{p}}] - \tilde{\lambda}_i\,[\mathbf{M} + \mathbf{M}^{\mathrm{p}}]\right)\tilde{\mathbf{u}}_i = \mathbf{0} \tag{12}$$

The n solutions of the BP are given by $\tilde{\mathbf{u}}_i$ and $\tilde{\lambda}_i$ (once again ordered so that $\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \ldots \leq \tilde{\lambda}_n \equiv \tilde{\lambda}_{\max}$).

In the following proofs, we assume that the ratio between stiffness and mass penalty parameters is constant for all constraint equations. We define this penalty ratio as

$$R = \frac{\alpha_{\mathrm{s},j}}{\alpha_{\mathrm{m},j}} \quad \text{for } j = 1 \ldots k \tag{13}$$

where R is a scalar constant with units s^{-2}. In this case, the magnitudes of the various penalties may vary as long as the *ratio* of parameters is the same for all constraint equations, and, from (7) and (8),

$$\mathbf{K}^{\mathrm{P}} = R\,\mathbf{M}^{\mathrm{P}} \tag{14}$$

We shall also utilise the Rayleigh quotient, given by [8]

$$\rho(\mathbf{v}) = \frac{\mathbf{v}^T\mathbf{K}\mathbf{v}}{\mathbf{v}^T\mathbf{M}\mathbf{v}} \tag{15}$$

in the case of the unpenalised problem, where $\mathbf{v}$ is any non-zero vector. A useful property of this quotient is that

$$\rho(\beta\mathbf{u}_i) = \lambda_i \tag{16}$$

where $\beta \neq 0$. That is, the Rayleigh quotient of any eigendirection $\beta\mathbf{u}_i$ is equal to its associated eigenvalue.

3.1 Physical and non-physical eigenmodes

Our goal is to determine how bipenalisation affects the eigensolutions of an FE system so that we may assess its influence on the maximum eigenfrequency, and therefore the critical time step, of the system. We note also that using direct transformation in order to impose the set of constraints given in (2) results in a reduced system of size $n-k$ [7]. Thus, compared to the equivalent reduced (fully constrained) system, the penalised system has k additional eigensolutions, which are of special interest in this case. In fact, our first observation is that it is not possible for all n eigenmodes to satisfy the given constraints.

Proposition 1. *When applying k linearly independent constraint equations to a system of size n using the bipenalty method, at most $n-k$ of the eigenmodes associated with the system satisfy those constraints.*

Proof. The constraint equations may be written in terms of the bipenalised eigenmodes $\tilde{\mathbf{u}}_i$ as

$$\mathbf{h}_i = \mathbf{C}\tilde{\mathbf{u}}_i \tag{17}$$

where $\mathbf{h}_i = \mathbf{0}$ implies the satisfaction of all constraints for the ith eigenmode. The rank-nullity theorem states that

$$\text{rank}(\mathbf{C}) + \text{nullity}(\mathbf{C}) = n \tag{18}$$

Since it is assumed that the rows of $\mathbf{C}$ are linearly independent, $\text{rank}(\mathbf{C}) = k$. Therefore,

$$\text{nullity}(\mathbf{C}) = n-k \tag{19}$$

meaning that $\mathbf{h}_i = \mathbf{0}$ is possible for $n-k$ eigenmodes at most. □

Using this information, we may go on to that in fact there are exactly $n-k$ eigenmodes that (approximately) satisfy the imposed constraints.

Proposition 2. *When applying k linearly independent constraint equations to a system of size n using the bipenalty method, exactly $n-k$ of the eigenmodes associated with the system satisfy those constraints, while exactly k of the eigenmodes associated with the system do not, for large α_m.*

Proof. Firstly, we note the orthonormal property of eigenvectors, so that for the BP,

$$\tilde{\mathbf{u}}_i^T(\mathbf{M}+\mathbf{M}^{\mathrm{p}})\tilde{\mathbf{u}}_j = \delta_{ij} \tag{20}$$

where δ_{ij} is the Kronecker delta. For the case of $i \neq j$ and taking into account (8) we have

$$\tilde{\mathbf{u}}_i^T\mathbf{M}\tilde{\mathbf{u}}_j + (\mathbf{C}\tilde{\mathbf{u}}_i)^T\mathbf{P}_{\mathrm{m}}(\mathbf{C}\tilde{\mathbf{u}}_j) = 0 \tag{21}$$

Next, we rewrite the matrix of penalty parameters so that $\mathbf{P}_{\mathrm{m}} = \alpha_{\mathrm{m}}\mathbf{D}_{\mathrm{m}}$, where $\mathbf{D}_{\mathrm{m}}$ is a square diagonal matrix of size k and all its entries have the same sign (i.e., all positive or all negative). In this case, negative penalties can be used (a useful technique in certain situations [1]) as long as they are not used alongside positive penalties in the same analysis. This substitution leads to

$$\tilde{\mathbf{u}}_i^T\mathbf{M}\tilde{\mathbf{u}}_j + \alpha_{\mathrm{m}}(\mathbf{C}\tilde{\mathbf{u}}_i)^T\mathbf{D}_{\mathrm{m}}(\mathbf{C}\tilde{\mathbf{u}}_j) = 0 \tag{22}$$

from which,

$$\lim_{\alpha_{\mathrm{m}}\to\infty}\left[(\mathbf{C}\tilde{\mathbf{u}}_i)^T\mathbf{D}_{\mathrm{m}}(\mathbf{C}\tilde{\mathbf{u}}_j)\right] = 0 \tag{23}$$

With the assumption that all entries in $\mathbf{D}_{\mathrm{m}}$ have the same sign (which rules out compensation during matrix multiplication) we are left with two possibilities for the vector $\mathbf{C}\tilde{\mathbf{u}}_i$, assuming large α_{m}:

1. $\mathbf{C}\tilde{\mathbf{u}}_i = \mathbf{0}$,
2. $\mathbf{C}\tilde{\mathbf{u}}_i$ and $\mathbf{C}\tilde{\mathbf{u}}_j$ are non-zero and orthogonal.

From Proposition 1, the first case is possible for at most $n-k$ of the n eigenmodes. The second case is possible for at most k of the n modes, since the vector $\mathbf{C}\tilde{\mathbf{u}}_i$ is of dimension k. Therefore, we can say that

1. $\mathbf{C}\tilde{\mathbf{u}}_i = \mathbf{0}$ for $n-k$ of the n eigenvectors,
2. $\mathbf{C}\tilde{\mathbf{u}}_i \neq \mathbf{0}$ for k of the n eigenvectors. □

This implies that the bipenalty method results in k non-physical (spurious) eigenmodes: the corresponding eigenvalues are as yet unknown. The $n-k$ constrained modes, on the other hand, tend to those of the fully constrained system for large penalty parameters, and therefore, from Rayleigh's theorem of separation, the eigenvalues are bounded by those of the unconstrained system. Hence, it is clearly one or more of the k non-physical eigensolutions which are responsible for introducing problematic eigenfrequencies.

Proposition 3. *For any system subject to k bipenalty constraints with large α_m, k of the associated eigenvalues tend to the penalty ratio, R.*

Proof. Substituting the n bipenalised eigenvectors into the Rayleigh quotient of the BP, we find

$$\rho_{\mathrm{BP}}(\tilde{\mathbf{u}}_i) = \frac{\tilde{\mathbf{u}}_i^T \mathbf{K}\tilde{\mathbf{u}}_i + \tilde{\mathbf{u}}_i^T \mathbf{K}^{\mathrm{p}}\tilde{\mathbf{u}}_i}{\tilde{\mathbf{u}}_i^T \mathbf{M}\tilde{\mathbf{u}}_i + \tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i} = \frac{\tilde{\mathbf{u}}_i^T \mathbf{K}\tilde{\mathbf{u}}_i + R \cdot \tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i}{\tilde{\mathbf{u}}_i^T \mathbf{M}\tilde{\mathbf{u}}_i + \tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i} \tag{24}$$

For eigenmodes with $\tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i \neq 0$ the penalty terms dominate, so that in the limit

$$\lim_{\alpha_{\mathrm{m}} \to \infty} \rho_{\mathrm{BP}}(\tilde{\mathbf{u}}_i) = \frac{R \cdot \tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i}{\tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i} = R \tag{25}$$

Hence, any eigenvalue whose eigenvector gives a non-zero $\tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i$ tends to R for large α_{m}. From Lemma 2, in the limit exactly k eigenmodes have $\tilde{\mathbf{u}}_i^T \mathbf{M}^{\mathrm{p}}\tilde{\mathbf{u}}_i \neq 0$. Therefore, the k eigenvalues associated with these modes tend to R for large α_{m}. □

The unpenalised system has a set of n eigensolutions: the eigenvectors and eigenvalues associated with the unconstrained problem. As the constraints are introduced (by increasing the magnitudes of the penalty parameters) $n-k$ of the eigensolutions tend to those of the fully constrained system. About the remaining eigensolutions we may say this: the eigenmodes do not satisfy the constraints, and the eigenvalues tend to R, the ratio of penalty parameters. This means that the spurious eigenvalues can controlled by variation of the penalty parameters α_{s} and α_{m}.

If our goal is to ensure that the bipenalised eigenfrequencies $\tilde{\omega}_i$ do not exceed those of the unpenalised system (so as not to decrease the critical time step, for instance) then there is a maximum ratio that should not be exceeded,

$$R_{\mathrm{crit}} = \lambda_{\max} \tag{26}$$

which we shall refer to as the *critical penalty ratio* (CPR). If a penalty ratio $R \leq R_{\mathrm{crit}}$ is selected then we can safely use a time step based on the critical time step of the unconstrained system. To find R_{crit} we need only to know the maximum eigenvalue of the unconstrained system.

3.2 Selecting a suitable penalty ratio

Calculating (or finding a good estimate for) the maximum eigenvalue of a system $\lambda_{\max}$ is a well-known problem, since it is required to ensure a safe selection of the critical time step via Equation (10). Methods such as direct iteration provide efficient methods of computing only the maximum eigenvalue (without requiring a full solution of the eigenproblem). However, if a suitable time step has already been selected using an approximate method it may be easier to calculate a penalty ratio directly. Combining (10) and (26) we can say that that the critical time step of the BP is lower than the chosen time step Δt if

$$R \leq \frac{4}{\Delta t^2} \tag{27}$$

A safe time step, calculated based on the unconstrained system, sets an upper limit on the eigenfrequencies which may be present in the constrained system. Using the bipenalty method, the spurious eigenfrequencies can be set using the penalty ratio R to fall within this limit by ensuring it conforms to the above inequality.

4 Numerical examples

In the present work we limit ourselves to one-dimensional FE analysis, specifically to two-noded linear elements (structural bar elements), with elemental stiffness and mass matrices given by

$$\mathbf{K}^{\mathrm{e}} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{M}^{\mathrm{e}} = \frac{\rho A h}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{28}$$

where E is the Young's modulus, ρ the mass density, A the cross-sectional area and h the length of the element.

4.1 Node-to-node tyings

One of the simplest multipoint constraints is the enforcement of zero relative displacement between two nodes (referred to here as a tying or interface element). In this section we will implement tyings in one dimension in order to assess the accuracy and time step stability of the bipenalty method in comparison to the standard stiffness and mass penalty methods.

We consider a 1D bar of length 1 m, discretised into 100 finite elements of equal length, each with a Young's modulus $E = 1$ N/m^2, mass density $\rho = 1$ kg/m^3 and cross-sectional area $A = 0.01$ m^2. The bar is fixed at position $x = 0$ and a force of $F = 1$ mN is applied at $x = 1$ m for a duration of 0.01 s. The displacement profile is computed using the CDM until $t = 0.5$ s. This final solution forms the reference solution, as shown in Figure 1.

To obtain the comparison solutions, the topology of the problem is altered so that the elements are unconnected; tyings of zero length are then added between all elements using a penalty method. As the relevant penalty parameters are increased the penalised solutions are expected to converge towards the unpenalised reference solution. The error $\mathbf{e}$ between the penalised solution and the reference solution is calculated according to

$$\mathbf{e} = \mathbf{u}^{\mathrm{test}} - \mathbf{u}^{\mathrm{ref}} \tag{29}$$

in each case, where $\mathbf{u}^{\mathrm{test}}$ and $\mathbf{u}^{\mathrm{ref}}$ are the computed displacement vectors for the solutions with and without tyings, respectively. The L_2 norm of the error vector is then taken to obtain a scalar measure of the overall error, which is denoted by $\|\mathbf{e}\|$. This value measures only the error introduced by the constraint imposition.

The results are shown in Figure 2 for a range of penalty factors. The penalty factor p shown on the plots refers to the mass penalty factor p_{m} for the mass penalty

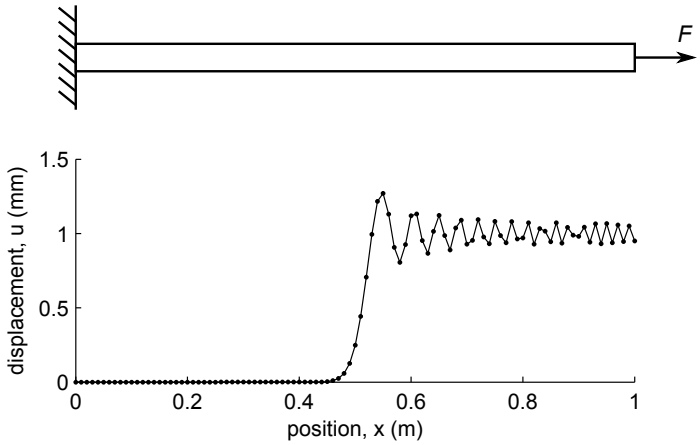

Figure 1: The bar and its reference displacement profile at time $t = 0.5$ s, generated from an analysis without interface elements.

analysis, and the stiffness penalty factor p_s otherwise. For the bipenalty analyses, a penalty ratio of $R = R_{crit} = 4 \times 10^{-4}$ s^{-2} is used.

The plots show that for equal penalty ratios the stiffness penalty method gives the lowest error. However, with standard stiffness penalties, penalty factors of 10^4 and above lead to instability, even when the time step is set two orders of magnitude lower than the original Δt_{crit}. Above this limit the error measure for both the mass and bipenalty methods continues to decrease monotonically, although the bipenalty method consistently produces lower errors for equal penalty factors $p_s = p_m$.

4.2 Contact-impact

The field of contact-impact provides a class of problems for which penalty methods are commonly employed in practice. Here, we consider a one-dimensional problem (shown in Figure 3), which is taken from the work of Huněk [9]. The first bar has length $L = 10$ m and the second has a length of $2L = 20$ m, discretised into 50 and 100 finite elements, respectively. Each bar has the same material properties, with Young's modulus $E = 100$ N/m^2, density $\rho = 0.01$ kg/m^3 and a cross-sectional area $A = 1$ m^2. Bar 2 is fixed at one end, while bar 1 is given an initial velocity of $v_1 = 0.1$ m/s. The analytical solution for the first 0.6 s is shown in Figure 4.

It should be noted that the mass penalty method cannot be used in this case; all nodes have zero initial acceleration and therefore there is no difference in relative acceleration to activate the mass penalty force, even as penetration occurs. Stiffness penalties are therefore required in order to ensure that the displacement constraint is enforced.

In Reference [9], Huněk shows is that contact-impact using the standard stiffness penalty method is very sensitive to the choice of penalty parameter, with low penalties leading to high levels of (non-physical) penetration and high penalties creating spu-

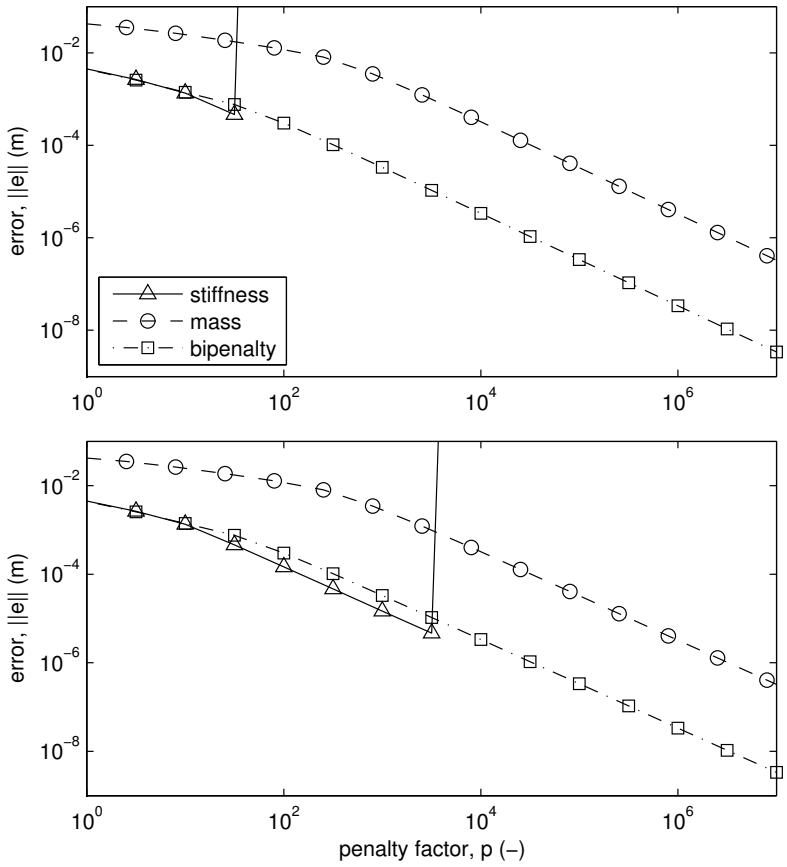

Figure 2: Measures of error at time $t = 0.5$ s for the case of wave propagation through a bar with node-to-node tyings between each element. The time steps used are $0.1\Delta t_{\text{crit}}$ (top) and $0.01\Delta t_{\text{crit}}$ (bottom).

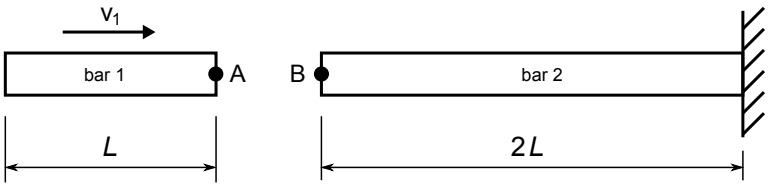

Figure 3: Initial conditions for the bar impact problem.

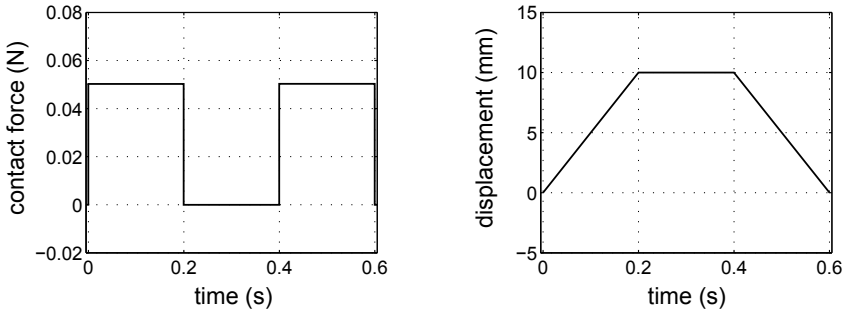

Figure 4: Analytical solution for the bar impact problem.

rious oscillations in the contact force. Figure 5 shows that increasing the penalty parameter, rather than leading to convergence towards the analytical solution, causes the contact force and displacement solutions to become increasingly erratic. By adding mass penalties alongside the standard formulation, we find firstly that for relatively low parameters (the left-hand plot in Figure 6) the spurious contact force oscillations are mitigated by the extra restraint provided by the mass penalties. Secondly, although contact force oscillations are still present for larger penalty magnitudes, the displacement solutions do not deteriorate (see Figure 7). The bipenalty method also ensures time step stability throughout the analysis.

5 Conclusions

The bipenalty formulation given in this paper allows for multiple constraints, with each constraint involving multiple degrees of freedom. It has been shown that the resulting system has as many spurious eigenmodes as there are constraint equations, and that the eigenfrequencies associated with these modes tend to the ratio of stiffness to mass penalty parameters as the parameters are increased. This allows for control over the spurious eigenfrequencies through suitable selection of the relevant parameters, and therefore control over the effect that the penalty constraints have on the critical time step of explicit time integration schemes.

Through simple one-dimensional examples, we have shown that the bipenalty method can therefore be used to obtain results with accuracy approximately equal to, or greater than, the accuracy obtained using traditional stiffness penalty methods, without requiring any adjustment of the time step. It has also been demonstrated that the use of mass penalty methods alone give results of low accuracy (for equal penalty ratios), if they are effective at all. Therefore, the bipenalty method combines the advantages, and avoids the respective disadvantages, of both the stiffness and mass penalty methods.

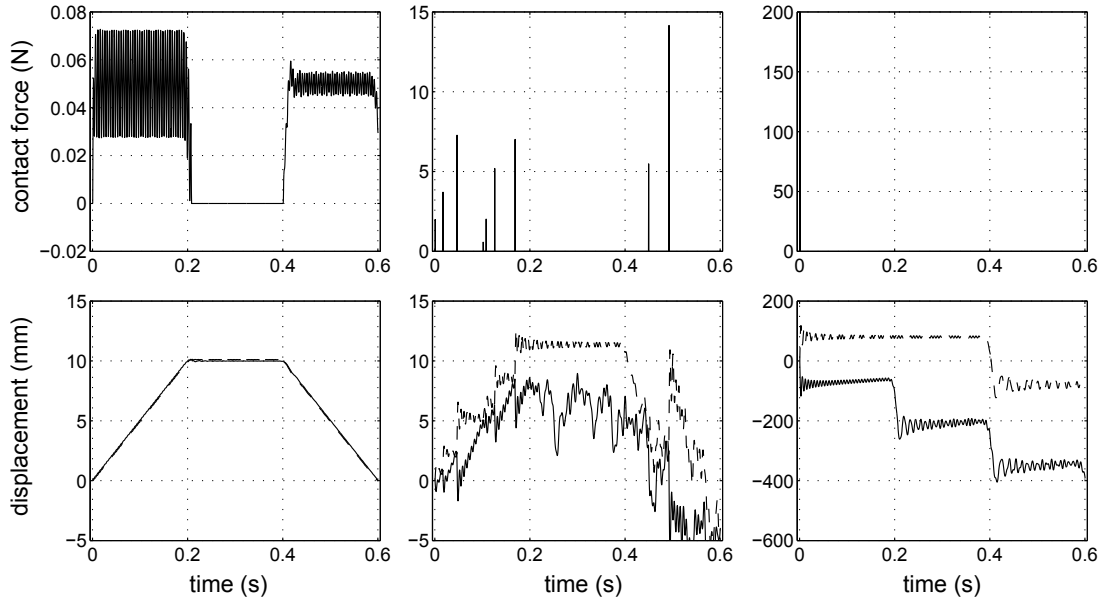

Figure 5: Contact force (on node B) and displacement (at nodes A and B) for the stiffness penalty analysis with varying penalty parameters: from left to right, $\alpha_s = 5 \times 10^2$, 5×10^4 and 5×10^6 N/m.

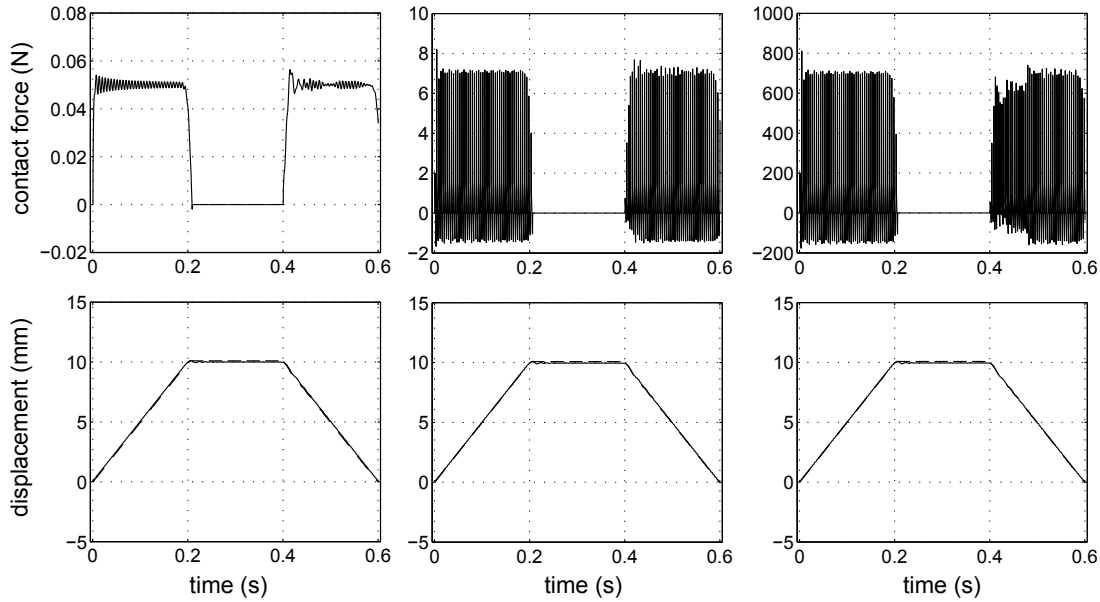

Figure 6: Contact force (on node B) and displacement (at nodes A and B) for the bipenalty analysis with varying penalty parameters: from left to right, $\alpha_s = 5 \times 10^2$, 5×10^4 and 5×10^6 N/m with a constant penalty ratio of $R = R_{\text{crit}} = 10^6$ s^{-2}.

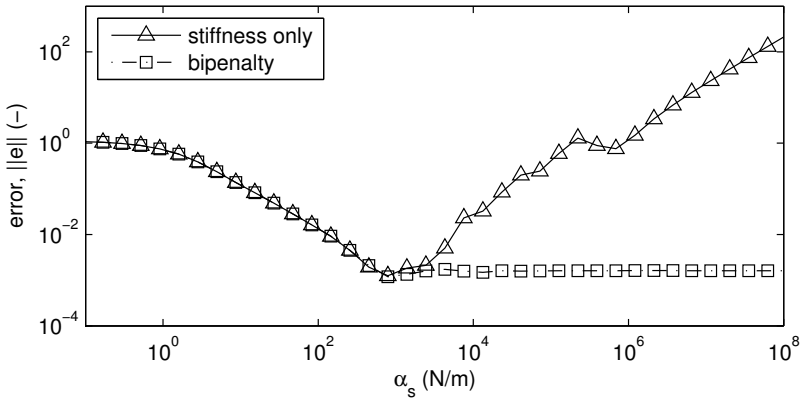

Figure 7: Error in displacement (compared with analytical solution) for increasing stiffness penalty parameter.

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