Paper 95

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Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Exact Dynamic Stiffness Matrix for a Class of Elastically Supported Structures

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Abstract

Consideration is given to determining the exact natural frequencies and modes of vibration of a class of structures comprising two parallel members with uniform distribution of mass and stiffness, which have independent properties and which are linked to each other, and possibly also to foundations, by uniformly distributed elastic interfaces of unequal stiffness. The formulation is general and applies to any structure in which the motion of the component members is governed by a second order linear differential equation. Closed form solution of the governing differential equations leads either to an exact dynamic stiffness matrix or to a number of exact relationships between the natural frequencies corresponding to coupled and uncoupled motion. An appropriate form of the Wittrick-Williams algorithm is presented for converging on the required natural frequencies to any desired accuracy. Examples are given to confirm the accuracy of the approach and to indicate its range of application.

Keywords: exact dynamic stiffness matrix, Wittrick-Williams algorithm, elastically supported structures.

1 Introduction

The dynamics of a family of simple, but extremely useful structural elements is governed by a linear second order differential equation. This equation allows for the uniform distribution of mass and stiffness and enables the motion of strings and shear beams, together with the axial and torsional motion of bars to be described exactly. As a result, each member type in this family has been treated exhaustively when considered as a single member or when joined contiguously to others, e.g. Rao [1]. However, when such members are linked in parallel by uniformly distributed elastic interfaces, their complexity becomes significantly more intractable and it is this class of structures that has led to renewed interest and which forms the basis of the work that follows.

Spring-mass systems generally have been at the heart of structural dynamics for many years and their synthesis and analysis has been the subject of considerable interest [2-4]. Over time, extremely complex systems have evolved and much research effort has been expended. However, relatively little work has been undertaken on the class of structures considered herein. Perhaps most interest has been directed towards double string systems, which have attracted a number of authors [5-8], as have the problems associated with the longitudinal motion of spring linked bars [9-12], although the torsional vibration problem has seen less activity [13-14].

The theory developed in Section 2 of this paper establishes a unified approach to solving the class of spring-linked structures described above. Initially, differential equations governing the coupled motion of the system are developed from first principles. A common solution procedure then leads either to an exact dynamic stiffness matrix (exact finite element) or to a series of exact relational models that link the uncoupled frequencies to the coupled ones that stem from them. Both forms of solution therefore represent alternative ways of implementing precisely the same theory and can be used interchangeably with identical solution accuracy, depending on the most efficient way to solve the problem in hand. A brief assessment of their relative merits is given below.

The exact dynamic stiffness approach enables all the powerful features of the finite element method to be utilized and hence enables structures with piecewise uniform members and distributed elastic connections to be modelled and analysed with nodal masses, cross-stiffnesses, elastic supports and any combination of classical or non-classical boundary conditions. The corresponding modes of vibration are then easily recovered by any of the well established methods, such as [15]. Furthermore, an appropriate formulation of the Wittrick-Williams algorithm is given, which guarantees that any desired frequency can be calculated to any desired accuracy with the certain knowledge that none have been missed.

The relational approach, on the other hand, enables all the natural frequencies of the coupled system to be determined from a knowledge of any single uncoupled frequency of one of the two component members. Under certain conditions, not considered here, the theory can be extended to cover stepwise uniform members [16]. The approach thus enables 'back of the envelope calculations' to be undertaken quickly and efficiently. In connection with this, it should be noted that the member theory has been developed in the context of two linked shear beams. This leads directly to the possibility of using well established simplification procedures that reduce multi-bay, multi-storey sway frames to equivalent one bay frames and then to simpler global models that can often retain sufficient accuracy for preliminary analysis and design procedures [17-19].

Finally it is shown how the presented theory can be extended to cover systems comprising three or four spring-linked parallel members in which the structure modelled is symmetric about an axis parallel to the members. The paper is concluded with a number of examples that confirm the accuracy of the proposed method and indicate its range of application.

2 Theory

2.1 Dynamic stiffness matrix

The equations of motion for the general two member, spring supported system shown in Figure 1 are initially developed in the context of two shear beams. Transformation to other systems, in which the motion of the members is also governed by a second order, linear differential equation is given in the Appendix.

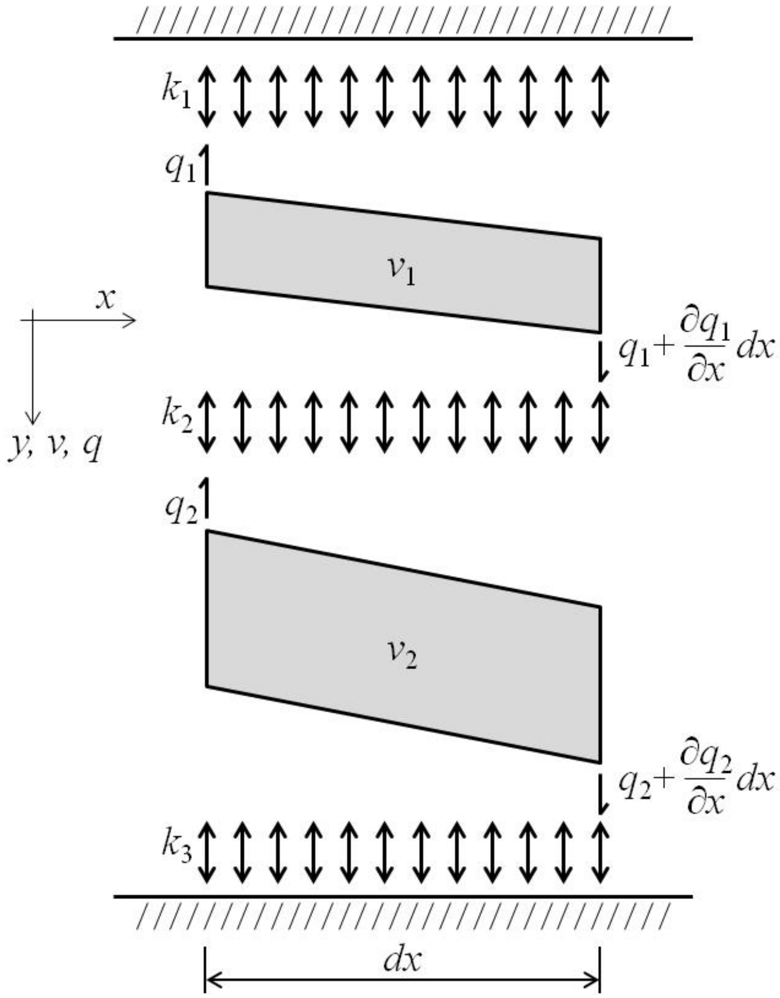


Figure 1. Positive sign convention for the instantaneous forces and displacements associated with a typical elemental length of the spring linked members in local co-ordinates.

The required equations of motion for the system of Figure 1 are then easily shown to be

$$\frac{\partial q_1}{\partial x} - (k_1 + k_2)v_1 + k_2 v_2 = m_1 \frac{\partial^2 v_1}{\partial t^2} \tag{1a}$$

$$\frac{\partial q_2}{\partial x} + k_2 v_1 - (k_2 + k_3)v_2 = m_2 \frac{\partial^2 v_2}{\partial t^2} \tag{1b}$$

where q is the shear force, m is the mass / unit length, k is the stiffness / unit length of an elastic interface and the subscripts refer to the respective components shown in Figure 1. Imposing the assumption of harmonic motion and introducing the non-dimensional parameter, $\xi = x / L$, the constitutive relationships and Eqs.(1) can be written, respectively, as

$$Q_i = r_i L \frac{dV_i}{d\xi} \qquad i = 1,2 \tag{2}$$

and

$$\Gamma \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \mathbf{0} \tag{3}$$

where

$$\Gamma = \begin{bmatrix} \tau + \phi & k_2 / r_1 \\ k_2 / r_2 & \tau + \psi \end{bmatrix}; \quad \tau = D^2 = d^2 / dx^2; \quad \phi = \gamma_1^2 \omega^2 - (k_1 + k_2) / r_1;$$

$\psi = \gamma_2^2 \omega^2 - (k_2 + k_3) / r_2$; and for i = 1,2 $\gamma_i^2 = m_i / r_i$, $r_i = k' AG_i / L^2$, where k' is the shear coefficient, $k' AG_i$ is the effective shear rigidity; and ω is the circular frequency.

Equation (3) can be combined into one equation by eliminating V_1 or V_2 to give the fourth order differential equation

$$|\Gamma| W = 0 \tag{4a}$$

in which $W = V_1$ or V_2.

The general solution of Eq.(4a) is found by substituting the trial solution $W = A\mathrm{e}^{s\xi}$ to yield the characteristic equation

$$|\Gamma| = 0 \tag{4b}$$

in which $\tau = s^2$.

It is easy to show that the discriminant of the frequency equation stemming from Eq.(4b) is always positive and hence that the two roots, s_1^2 and s_2^2, can be real or imaginary, but not complex. The three possible combinations of roots are thus shown in Table 1, where α and β are real, positive values to be determined. The values of J_α and J_β are required in Eq.(19) and int( ) is the highest integer less than the bracketed term.

Case	s_1^2	s_2^2	a	b	$C_{\alpha\xi}$	$S_{\alpha\xi}$	$C_{\beta\xi}$	$S_{\beta\xi}$	J_α	J_β
1	$a\alpha^2$	$b\beta^2$	-1	-1	$\cos\alpha\xi$	$\sin\alpha\xi$	$\cos\beta\xi$	$\sin\beta\xi$	$\mathrm{int}(\alpha/\pi)$	$\mathrm{int}(\beta/\pi)$
2	$a\alpha^2$	$b\beta^2$	-1	1	$\cos\alpha\xi$	$\sin\alpha\xi$	$\cosh\beta\xi$	$\sinh\beta\xi$	$\mathrm{int}(\alpha/\pi)$	0
3	$a\alpha^2$	$b\beta^2$	1	1	$\cosh\alpha\xi$	$\sinh\alpha\xi$	$\cosh\beta\xi$	$\sinh\beta\xi$	0	0

Table 1. Possible combinations of the roots of the characteristic equation stemming from Eq.(4b).

Hence the general solution to Eq.(4a) can be written for each case as

$$W = A_1 C_{\alpha\xi} + A_2 S_{\alpha\xi} + A_3 C_{\beta\xi} + A_4 S_{\beta\xi} \tag{5}$$

Since $W = V_1$ or V_2 from Eq.(4a), Eq.(5) yields

$$V_1 = A_1 C_{\alpha\xi} + A_2 S_{\alpha\xi} + A_3 C_{\beta\xi} + A_4 S_{\beta\xi} \tag{6a}$$

and

$$V_2 = B_1 C_{\alpha\xi} + B_2 S_{\alpha\xi} + B_3 C_{\beta\xi} + B_4 S_{\beta\xi} \tag{6b}$$

where A_i and B_i $(i = 1,2,...,4)$ are independent sets of constants. Substituting Eqs.(6) into Eq.(3) yields the following relationships between the constants

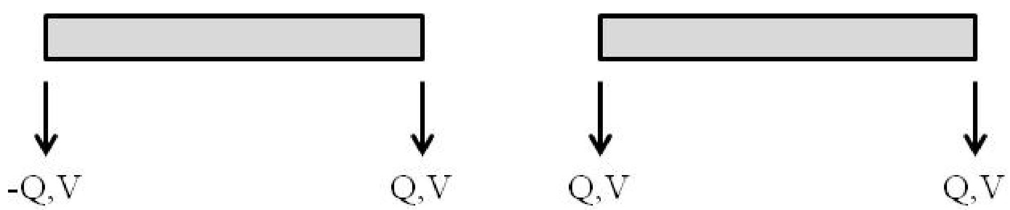
$$B_{1,2} = \eta A_{1,2} \text{ and } B_{3,4} = \mu A_{3,4} \tag{7}$$

where

$$\eta = -r_1(\phi + a\alpha^2)/k_2 = -k_2/r_2(\psi + a\alpha^2) \tag{8a}$$

and

$$\mu = -r_1(\phi + b\beta^2)/k_2 = -k_2/r_2(\psi + b\beta^2) \tag{8b}$$



(a) (b)

Figure 2. Positive sign convention for amplitudes of the nodal forces and displacements in (a) local; and (b) member co-ordinates.

The nodal displacements then follow from Eqs.(6)-(8) in the member co-ordinate system of Figure 2, as follows when $\xi = 0$, $V_{0i} = V_i$ (i=1,2) and when $\xi = 1$, $V_{1i} = V_i$ (i=1,2) (9)

which leads to

$$\mathbf{d} = \mathbf{sc} \tag{10}$$

where

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix},\ \mathbf{s} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C} & \mathbf{S} \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} \mathbf{A}_o \\ \mathbf{A}_e \end{bmatrix};\ \mathbf{d}_1 = \begin{bmatrix} V_{01} \\ V_{02} \end{bmatrix},\ \mathbf{d}_2 = \begin{bmatrix} V_{11} \\ V_{12} \end{bmatrix},\ \mathbf{A}_o = \begin{bmatrix} A_1 \\ A_3 \end{bmatrix},$$

$$\mathbf{A}_e = \begin{bmatrix} A_2 \\ A_4 \end{bmatrix},\quad \mathbf{H} = \begin{bmatrix} 1 & 1 \\ \eta & \mu \end{bmatrix},\quad \mathbf{C} = \begin{bmatrix} C_\alpha & 0 \\ 0 & C_\beta \end{bmatrix},\quad \mathbf{S} = \begin{bmatrix} S_\alpha & 0 \\ 0 & S_\beta \end{bmatrix},$$

$\mathbf{I}$ is the unit matrix, C_α, C_β, S_α and S_β are respectively, $C_{\alpha\xi}$, $C_{\beta\xi}$, $S_{\alpha\xi}$ and $S_{\beta\xi}$ when $\xi = 1$. (11)

In similar fashion, the general expressions for the corresponding force vector are established by substituting Eqs.(6)-(8) into Eq.(2) to give

$$Q_1 = -A_1 a\eta_1 S_{\alpha\xi} - A_2\eta_1 C_{\alpha\xi} - A_3 b\mu_1 S_{\beta\xi} - A_4\mu_1 C_{\beta\xi} \tag{12a}$$

and

$$Q_2 = -A_1 a\eta_2 S_{\alpha\xi} - A_2\eta_2 C_{\alpha\xi} - A_3 b\mu_2 S_{\beta\xi} - A_4\mu_2 C_{\beta\xi} \tag{12b}$$

where

$$\eta_1 = -r_1\alpha L,\quad \eta_2 = -r_2\eta\alpha L,\quad \mu_1 = -r_1\beta L\quad \text{and}\quad \mu_2 = -r_2\mu\beta L \tag{13}$$

The nodal forces then follow from Eqs.(6)-(8) in the member co-ordinate system of Figure 2, as follows

when $\xi = 0$, $Q_{0i} = -Q_i$ (i=1,2) and when $\xi = 1$, $Q_{1i} = Q_i$ (i=1,2) (14)

which leads to

$$\mathbf{p} = \mathbf{s}^*\mathbf{c} \tag{15}$$

where

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix},\qquad \mathbf{s}^* = \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{tS} & -\mathbf{C} \end{bmatrix},$$

$$\mathbf{p}_1 = \begin{bmatrix} \mathbf{Q}_{01} \\ \mathbf{Q}_{02} \end{bmatrix},\quad \mathbf{p}_2 = \begin{bmatrix} \mathbf{Q}_{11} \\ \mathbf{Q}_{12} \end{bmatrix},\quad \mathbf{G} = \begin{bmatrix} \eta_1 & \mu_1 \\ \eta_2 & \mu_2 \end{bmatrix},\quad \mathbf{t} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \tag{16}$$

and the remaining variables are defined in Eq.(11) above. The member dynamic stiffness matrix can then be developed from Eqs.(10) and (15) as

$$\mathbf{k} = \mathbf{s}^{*}\mathbf{s}^{-1} \tag{17}$$

The symmetric elements of **k** can be stated symbolically using the normal row and column subscripts as

$$k_{11} = k_{33} = (\mu_1 \eta S_\alpha C_\beta - \mu \eta_1 C_\alpha S_\beta)/\Delta; \qquad k_{12} = k_{34} = (\eta_1 C_\alpha S_\beta - \mu_1 S_\alpha C_\beta)/\Delta$$
$$k_{13} = (\mu \eta_1 S_\beta - \mu_1 \eta S_\alpha)/\Delta; \qquad k_{14} = k_{23} = (\mu_1 S_\alpha - \eta_1 S_\beta)/\Delta$$
$$k_{22} = k_{44} = (\eta_2 C_\alpha S_\beta - \mu_2 S_\alpha C_\beta)/\Delta; \qquad k_{24} = (\mu_2 S_\alpha - \eta_2 S_\beta)/\Delta$$
$$\Delta = (\mu - \eta) S_\alpha S_\beta \tag{18}$$

Since a component member has only one translational degree of freedom at each node, the boundary conditions are easily modelled using lateral springs that offer a complete spectrum of support from free to fully clamped conditions. The stiffness matrix itself can be used in the normal way to form a series of piecewise uniform spring linked structures, for which exact natural frequencies can be converged upon to any required accuracy using the Wittrick-Williams algorithm described in the following Section. The mode shapes then follow directly using any appropriate method, such as that described in [15].

2.2 Formula for the Wittrick-Williams root counting algorithm

The Wittrick-Williams root counting algorithm has been available for well over thirty years and the following formula can be established easily from many sources, such as reference [20]. In the current notation, it states that the number of coupled natural frequencies passed by a trial frequency, ω^*, is given by

$$J(\omega^*) = \sum_{members} (J_\alpha + J_\beta) + s\{\mathbf{K}\} \tag{19}$$

where J_α and J_β are given in Table 1 and $s\{\mathbf{K}\}$ is the sign count of the dynamic structure stiffness matrix, which is equal to the number of negative elements on the leading diagonal of the upper triangular matrix obtained from **K,** when $\omega = \omega^*$, by the standard form of Gauss elimination without row interchanges.

3 Relational formulation

3.1 $k_1 = k_2 = k_3 = 0$

It is clear from Eq.(3) that when $k_1 = k_2 = k_3 = 0$, the two shear beams in Figure 1 are uncoupled and subjected only to the boundary conditions at their respective nodes. Assuming that the boundary conditions are the same for each member, Eq.(4a) yields their uncoupled natural frequencies when

$$D^2 + \lambda_{i,1}^2 = D^2 + \lambda_{i,2}^2 = 0 \qquad i = 1,2,3... \tag{20}$$

where

$$\lambda_{i,j}^2 = \omega_{i,j}^2 \gamma_j^2 \qquad j = 1,2 \tag{21}$$

and $\omega_{i,1}$ and $\omega_{i,2}$ are the i-th uncoupled natural frequencies of members 1 and 2, respectively. Eqs.(20) and (21) then lead to the following relationship between the frequencies of the two members

$$\omega_{i,2}^2 = \omega_{i,1}^2 \gamma_1^2 / \gamma_2^2 \tag{22}$$

It is important to note here that for each of the two members taken in turn, it is easy to show [16] that the relationship between its i-th and $(i + p)$-th natural frequency is given by

$$\omega_{i,j}^2 / \omega_{i+p,j}^2 = i^2 / (i + p)^2 \tag{23a}$$

when the member has fixed / fixed or free / free boundary conditions and

$$\omega_{i,j}^2 / \omega_{i+p,j}^2 = (2i - 1)^2 / (2(i + p) - 1)^2 \tag{23b}$$

when the member has fixed / free boundary conditions. In turn these equations lead to the following relationship

$$\omega_{i,1}^2 / \omega_{i+p,1}^2 = \omega_{i,2}^2 / \omega_{i+p,2}^2 \tag{24}$$

3.2 $k_2 = 0$ and either k_1 and / or $k_3 \geq 0$

For this case, Eq.(19) remains valid, but with

$$\lambda_{i,j}^2 = \omega_{i,jk}^2 \gamma_j^2 - k_{2j-1} / r_j \qquad j = 1,2 \tag{25}$$

where $\omega_{i,jk}$ $(j = 1,2)$ is the i-th natural frequency of the uncoupled, spring supported members, and leads to the following relationship

$$\omega_{i,2k}^2 = (\omega_{i,1k}^2 \gamma_1^2 - k_1 / r_1 + k_3 / r_2) / \gamma_2^2 \tag{26}$$

Furthermore, using Eqs.(21) and (25) in Eq.(20) yields

$$\omega_{i,1k}^2 = \omega_{i,1}^2 + k_1 / \gamma_1^2 r_1 \quad \text{and} \quad \omega_{i,2k}^2 = \omega_{i,2}^2 + k_3 / \gamma_2^2 r_2 \tag{27a,b}$$

3.3 k_2>0 and either k_1 and / or $k_3 \geq 0$

For this case, Eq.(4a) is satisfied when

$$\left(D^2+\omega^2\gamma_1^2-(k_1+k_2)/r_1\right)\left(D^2+\omega^2\gamma_2^2-(k_2+k_3)/r_2\right)-k_2^2/r_1r_2=0 \qquad (28)$$

where ω corresponds to the coupled natural frequencies of the system. However, substituting Eqs.(25) in Eqs.(20) gives

$$k_1/r_1=D^2+\omega_{i,1k}^2\gamma_1^2 \qquad \text{and} \qquad k_3/r_2=D^2+\omega_{i,2k}^2\gamma_2^2 \qquad (29a,b)$$

Substituting Eqs.(29) into Eq.(28) and re-arranging gives

$$\omega^4+b\omega^2+c=0 \qquad (30)$$

where

$$b=-(\omega_{i,1k}^2+\omega_{i,2k}^2+k_2/\gamma_1^2r_1+k_2/\gamma_2^2r_2) \qquad (31a)$$

and

$$c=\omega_{i,1k}^2\omega_{i,2k}^2+\omega_{i,1k}^2k_2/\gamma_2^2r_2+\omega_{i,2k}^2k_2/\gamma_1^2r_1 \qquad (31b)$$

After some manipulation, the discriminant of Eq.(30) can be written as

$$D^*=(\omega_{i,1k}^2-\omega_{i,2k}^2+k_2/\gamma_1^2r_1-k_2/\gamma_2^2r_2)^2+4k_2^2/\gamma_1^2r_1\gamma_2^2r_2 \qquad (32)$$

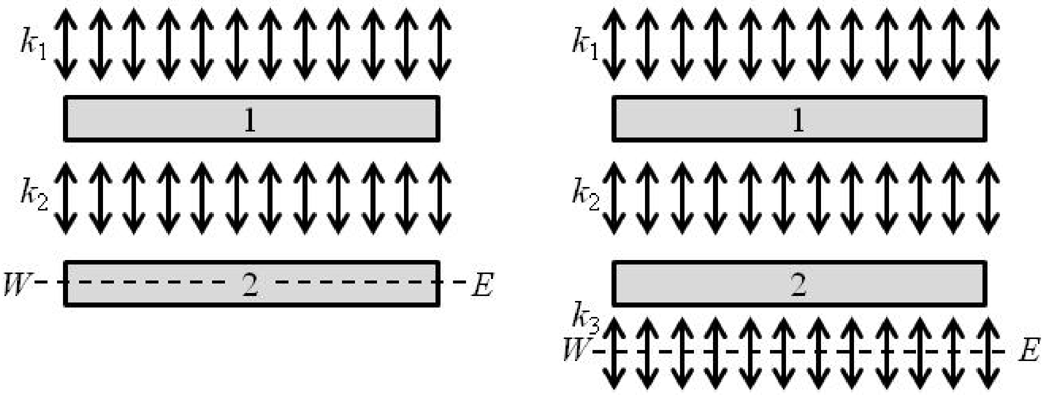
Since both D^* and c are clearly always positive, the roots of Eq.(30) are both real and positive. Finally, substituting Eqs.(22) and (27) into Eqs.(31) gives

$$\begin{aligned}2\omega_{i,jc}^2 &= \omega_{i,1}^2(\gamma_2^2+\gamma_1^2)/\gamma_2^2+(k_1+k_2)/\gamma_1^2r_1+(k_2+k_3)/\gamma_2^2r_2) \\ &\mp[(\omega_{i,1}^2(\gamma_2^2-\gamma_1^2)/\gamma_2^2+(k_1+k_2)/\gamma_1^2r_1-(k_2+k_3)/\gamma_2^2r_2)^2+4k_2^2/\gamma_1^2r_1\gamma_2^2r_2]^{1/2} \qquad (33)\end{aligned}$$

where $\omega_{i,jc}$ $\quad j=1,2 \quad i=1,\ 2,\ 3\ ....$ are the two coupled frequencies stemming from the i-th uncoupled frequency of one of the shear beams. Since Eqs.(23) and (24) relate the i-th uncoupled frequency to any other uncoupled frequency, every coupled frequency of the system can be developed from the knowledge of a single uncoupled frequency.

4 Alternative structural models

It is clear that the theory of the preceding sections can be applied to find the natural frequencies of single and double beam systems merely by assigning appropriate numerical values to the member and spring data. However, the same theory can be extended to cover three and four beam systems when they are symmetric about a parallel East-West axis, since in this case the motion can only be symmetric or anti-symmetric about the axis. The required results are thus obtained by considering only the half structure to the north of the axis, imposing firstly symmetric (S) and then anti-symmetric (A) boundary conditions in turn along the axis and in each case assigning appropriate values to certain parameters, as shown in Figure 3 and Table 2.



(a) (b)

Figure 3. Structure to the north of the East – West line of symmetry: (a) three beam system and (b) four beam system.

System	Mode	Parameter		
		k_3	m_2	r_2
3 beam	S	∞	0	∞
	A	0	m_2 / 2	r_2 / 2
4 beam	S	$2k_3$	original	original
	A	0	original	original

Table 2. Imposed parameter values for solving three and four beam symmetric systems using the theory of Section 2. All parameters retain their original values apart from those defined above.

5 Examples

Four examples are given that confirm the correctness and accuracy of the approach, while also giving an indication of its range of application.

5.1 Example 1

Consideration is given to the problem of two different, but parallel, taut strings of length 1 m that are linked by an elastic interface of stiffness k = 200 N/m per metre length. Five different combinations of mass/unit length and string tension, as defined in Table 3, are computed using Eq.(18) and the results compared with those of reference [6] in Table 4. See the Appendix for equivalent string parameters.

Data Set No.	Member properties			
	String 1		String 2	
	Mass/unit length kg/m	Tension N	Mass/unit length kg/m	Tension N
1	0.01	50.0	0.005	50.0
2	0.01	50.0	0.01	50.0
3	0.01	50.0	0.005	100.0
4	0.01	50.0	0.01	100.0
5	0.01	50.0	0.02	100.0

Table 3. Member data for the five string systems considered by Onisczcuk [6].

Data Set No.	Frequency No.							
	1		2		3		6	
	Ref. [6]	Eq.(18)	Ref. [6]	Eq.(18)	Ref. [6]	Eq.(18)	Ref. [6]	Eq.(18)
Synchronous								
1	243.5	243.465	462.4	462.355	680.0	680.011	1340.2	1340.18
2	221.1	222.144	444.3	444.288	666.4	666.432	1332.9	1332.86
3	254.4	254.392	465.0	464.853	680.8	680.838	1340.3	1340.29
4	249.5	249.520	464.1	464.097	680.6	680.613	1340.3	1340.26
5	222.1	222.144	444.3	444.288	666.4	666.432	1332.9	1332.86
Asynchronous								
1	385.7	385.706	662.1	662.121	964.4	964.356	1895.7	1895.65
2	298.9	298.911	487.2	487.229	695.8	695.796	1347.8	1347.79
3	492.0	491.960	911.5	911.522	1348.0	1348.01	2673.3	2673.25
4	354.7	354.660	645.6	645.593	953.5	953.500	1890.3	1890.31
5	281.7	281.688	476.9	476.857	688.6	688.573	1344.1	1344.07

Table 4. Comparison between the natural frequencies (rad/sec) given by Onisczcuk [6] and those from the theory presented for the data given in Table 3. Onisczcuk's result of 221.1 in column 2 should be 222.1 by comparison with the result of Data Set 5, since doubling one member's stiffness and mass leaves the frequency unaltered.

5.2 Example 2

This example compares Onisczcuk's results [6] for data set 3 of Table 3 above, with those obtained from the stiffness and relational matrices presented herein. The uncoupled results are included for hand confirmation.

Freq. No. i	Coupled frequencies (rad/sec)				Uncoupled frequencies (rad/sec)		
	Ref [6]	Eq.(18)	Eq.(33)		Eq.(20)		Eq.(22)
			$\omega_{i,1c}$	$\omega_{i,2c}$	$\omega_{i,1}$	$\omega_{i,2}$	$\omega_{i,2}$
1	254.4 S	254.392	254.392 (1)	491.960 (3)	222.144	444.288	444.288
2	465.0 S	464.853	464.853 (2)	911.522 (6)	444.288	888.577	888.577
3	492.0 A	491.960	680.838 (4)	1348.01 (9)	666.433	1332.87	1332.87
4	680.8 S	680.838	899.574 (5)	1788.47 (12)	888.577	1777.15	1777.15
5	899.6 S	899.574	1119.59 (7)	2230.48 (15)	1110.72	2221.44	2221.44
6	911.5 A	911.522	1340.29 (8)	2673.25 (18)	1332.87	2665.73	2665.73
7	1119.6 S	1119.59	1561.39 (10)	3116.46 (21)	1555.01	3110.02	3110.02
8	1340.3 S	1340.29	1782.75 (11)	3559.94 (24)	1777.15	3554.31	3554.31
9	1348.0 A	1348.01					
10	-	1561.39					
11	-	1782.75					
12	1788.5 A	1788.47					

Table 5. Comparison of results for data set 3 of Table 3 above. The modal shapes are indicated by S and A, where S = Synchronous and A = Asynchronous. The coupled frequency number is given in brackets and the table clearly shows that the eigenpairs in columns 3 and 4 do not yield sequential coupled frequencies.

5.3 Example 3

This example solves the symmetric three beam and four beam systems depicted in Figure 3. The beam and spring data for both systems are the same and given in Table 6. These values must be modified according to Table 2 to achieve the required results, which are given in Table 7.

Beams			Springs		
Length m	Mass / unit length kg/m	Shear rigidity N	k_1 N/m^2	k_2 N/m^2	k_3 N/m^2
8.0	90.0	6.0×10^4	1.0×10^2	1.0×10^3	1.0×10^4

Table 6. Beam and spring data for Example 3.

System Natural Frequencies (rad/sec)			
3 Beam		4 Beam	
Anti-symmetric	Symmetric	Anti-symmetric	Symmetric
5.14145 (1)	6.15827 (2)	5.12287 (1)	6.11289 (2)
7.70806 (3)	15.6058 (5)	6.96375 (3)	15.5879 (5)
15.2332 (4)	25.5886 (8)	15.2270 (4)	16.1119 (7)
16.2798 (6)	35.6598 (11)	15.9408 (6)	21.5687 (8)
25.3631 (7)	45.7612 (14)	25.3593 (9)	25.5777 (10)

Table 7. Natural frequencies (rad/sec) for the three and four beam systems shown in Figure 3, whose data are given in Table 6. The coupled frequency number is given in brackets.

5.4 Example 4

The final example considers the structure indicated in Figure 4. It comprises a piece-wise uniform, stepped shear beam that is suspended by a stepped elastic interface from a uniform taut string that also carries two point masses. The string is supported at each end by a single nodal spring and across its central portion by an elastic interface. The structure can be envisaged as three collinear segments, each of which can be modelled by an exact finite element developed from Eq.(18). Once the global dynamic stiffness matrix has been established, the nodal stiffnesses and point masses can be added in the usual way. The data for each segment are given in Table 8 and the first ten natural frequencies of the structure are given in Table 9.

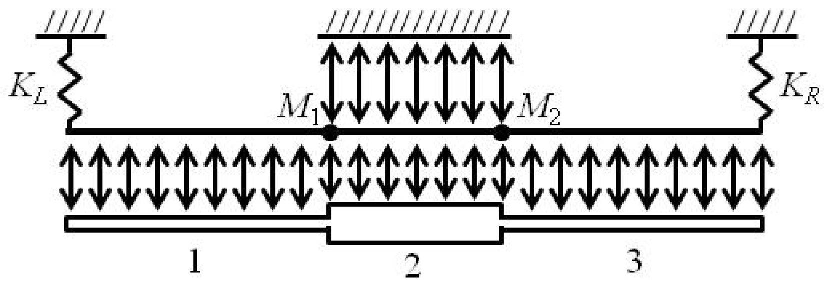


Figure 4. A contrived structure to indicate the range of application of the proposed theory.

Segment No.	Member Length m	String mass / length kg/m	String tension N	Beam mass / length kg/m	Shear rigidity N	k_1 N/m^2	k_2 N/m^2	k_3 N/m^2
1	3.0	10.0	2.0×10^3	60.0	5.0×10^4	0.0	1.0×10^3	0.0
2	2.0	10.0	2.0×10^3	90.0	8.0×10^4	2.0×10^3	5.0×10^2	0.0
3	3.0	10.0	2.0×10^3	70.0	6.0×10^4	0.0	1.0×10^3	0.0

Table 8. Member and elastic interface data for the structure shown in Figure 4. The values of K_L, K_R, M_1 and M_2 are 1.0×10^4 N/m, 2.0×10^4 N/m, 2.0×10^2 kg and 2.5×10^2 kg, respectively.

Natural frequency No.	1	2	3	4	5
Frequency (rad/sec)	2.26521	4.05080	4.42246	12.5842	17.3593
Natural frequency No.	6	7	8	9	10
Frequency (rad/sec)	17.7284	21.9757	27.4831	29.6695	30.4288

Table 9. The first ten natural frequencies (rad/sec) of the structure shown in Figure 4.

6 Summary and Conclusions

Consideration has been given to determining the exact natural frequencies of a class of structures comprising two parallel members with uniform distribution of mass and stiffness, which have independent properties and which are linked to each other, and possibly also to foundations, by uniformly distributed elastic interfaces of unequal stiffness.

The approach differs from all previous work in three distinctly different ways. Firstly, it is based on an exact dynamic stiffness approach. This is important because, at the heart of any exact solution procedure for this type of structure, it is necessary to solve a transcendental eigenvalue problem. Currently this can only be achieved exactly by using an exact dynamic stiffness formulation in conjunction with the Wittrick-Williams algorithm, in which the latter enables any required eigenvalue to be converged upon to any desired accuracy with the certain knowledge that none have been missed. Secondly, a comprehensive set of inter-relationships between the natural frequencies of the component elements comprising the structure have been formulated for the first time using a simple and novel procedure. Finally, the theory has been developed in the context of two linked shear beams. This has not been done previously and leads directly to the possibility of using well established simplification procedures that reduce multi-bay, multi-storey sway frames to equivalent one bay frames and then to simpler global models that can often retain sufficient accuracy for preliminary analysis and design procedures.

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Appendix

The foregoing theory governs the spring-linked motion of strings, shear beams and the axial and torsional motion of bars, or any appropriate combination of these, when the relevant member properties are selected from Table A below.

Member type	Motion	Member properties	
Shear beam	Lateral	m	$k'AG$
String	Lateral	m	T
Bar	Axial	m	EA
	torsional	$r_\theta m$	GJ

Table A. Corresponding member properties list. m is the mass / unit length, r_θ is the radius of gyration, T is the string tension and $k'AG$, EA and GJ are the shear, axial and torsional rigidities, respectively.