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Numerical Analysis of Cable Structures

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Abstract

The paper presents a comparison of computational methods used in the static analysis of cable structures. Two different methods are used, the method of dynamic relaxation and the force density method. The dynamic relaxation method is presented in more detail. The sample design will be compared for accuracy, computational time and the conditions and speed of convergence of the methods used.

Keywords: cable structures, nonlinear analysis, dynamic relaxation, force density method.

1 Introduction

Cable - this structural element known since prehistoric times perhaps – was for many years used exclusively for mechanical engineering purposes. It has penetrated the civil engineering at the end of the 19th century thanks to the progress of the bridge engineering. [1]. Its excellent flexibility (and its related zero bending moments) and strength enable it to span very long distances with minimal crosssectional area. It becomes an excellent constructional element for overarching longspan capabilities. The inconvenience of the cables is their shape instability, especially in the case of the local loads. It is necessary to provide detailed analysis for varied using of the cable constructions.

The analysis of more complex cable structures is not easy. This is the geometrically nonlinear case the results can be obtained through the use of computer technology. Methods to solve these systems are many some are listed in [2, 3, 4, 5].

The aim of this paper is to compare the model example dynamic relaxation method and force density method. The cable element will be used to approximate the behavior of the cables [4]. Self-created scripts in MATLAB are used for both the calculation methods.

2 Single cable

A single cable must be analyzed before proceeded to analyze the entire cable structures. There are basically two approaches to this issue. The first approach approximates a single cable perfectly flexible cable element. The second approach consists in replacing the cables of several bar elements that carry only tension. In this paper only the first approach is considered.

2.1 Cable element

Problem of statics of flexible elastic cables on one field is the focus of the paper [1]. Further analysis of this work will be based primarily on papers [4, 5] and [6].



Figure 1: A cable element. Left – The symbols of geometry. Right – Tensile force T.

The basic assumption the analysis of flexible elastic cable is that the cable is regarded to be perfectly flexible and is devoid of any flexural rigidity. Homogeneous material with constant cross-section throughout its length is assumed. Elements have linear elastic stress-strain relationship, and their strains are small. Effect of temperature is not considered in this paper.

Load on cable, which must include at least self-weight, is distributed uniformly along the curve of the cable which is assumed to be a parabola. An internal force T in this paper will be a necessary implement. Its meaning and the difference from the normal force N is shown in Figure 1. Force T can be calculated for cable element iteratively of Equation (1).

$$g(T, r, l, c, s_0, Q) = \frac{l^2 T}{2rQ} \left[\ln \left(-\frac{2c}{l} + \frac{rQ}{lT} + \frac{\sqrt{b}}{lT} \right) - \ln \left(-\frac{2c}{l} - \frac{rQ}{lT} + \frac{\sqrt{a}}{lT} \right) \right] + \frac{c}{4rQ} \left(\sqrt{a} - \sqrt{b} \right) + \frac{1}{8T} \left(\sqrt{a} - \sqrt{b} \right) - s_0 - \frac{T}{EA} \left(\frac{l^2}{r} + \frac{c^2}{r} + \frac{Qr^2}{12T^2} \right) = 0 \quad (1)$$

Symbols are shown in Figure 1. Slack length s_0 is the un-elongated length of element, length *r* is distance between two end joints in the chord direction. Distance *l* is the horizontal distance between the two end joints and *c* is the vertical separation

between end joint *j* and end joint *i* (can be negative). Force *Q* is the resultant of the vertical uniform load *q* acting vertically the entire length of parabolic curved cable, while $Q = qs_0$. *E* is Young's modulus of elasticity and *A* is cross-sectional area for cables. For reasons of clarity calculation introduces two more substitutions: $a = Q^2r^2 + 4c^2t^2 + 4l^2T^2 + 4crQT$; $b = Q^2r^2 + 4c^2T^2 + 4l^2T^2 - 4crQT$.

3 Cable structures

Cable structures are formed of joining two or more cables together. The structure idealizes and decomposes the system elements (cables) into interconnected joints. Each joint can have space for up to three degrees of freedom (in the plane of the two). The joint in which only one cable element ends or begins is called outer joint and must be fixed. When we are dealing with an inner joint (connecting two or more cable elements) it can be a fixed joint or a free joint. Markings scheme is shown in Figure 2.



Figure 2: Schema of cable structure. The letters indicate the fixed joints. The numbers indicate the free joints.

Linear behavior of materials with large displacement is assumed for the cable structures. Cable systems represent geometrically nonlinear case, in which we are looking for equilibrium on deformed structure. It is thought that the deformation of the structure has no effect on the magnitude and direction of external forces. It is expected that the stiffness of the individual cables will change with the strain structure. Effect of temperature on the shape of the structure is not considered. Several methods exist to solve cable structures. Some are listed in [2, 3, 4]. In this work will be compared two selected methods: Dynamic relaxation method and force density method. The dynamic relaxation method is presented in a simplified theoretical explanation. Force density method is described for information only.

3.1 Dynamic relaxation method

Description of the method is based on publications [2, 3, 5].

3.1.1 Principle

Dynamic relaxation method is not used for dynamic analysis of structures, but uses a dynamic solution for fictitious damped structure to achieve a static solution. When calculating the response of the structure we do not prepare the stiffness matrix and therefore the dynamic relaxation is suitable for large scale nonlinear cases. The method is based on Newton's second laws of motion F = Ma. During the static analysis of construction the fictitious damping is used. The proportional, frequently critical damping factor is mostly applied. The basic unknowns form nodal velocity, which are calculated from nodal displacements. Speed of solution is dependent on a suitable distribution of mass structure in each node.

3.1.2 The governing equations

The second law of motion is reflected in the equation that describes the residual force at any time t in free joint i. The Equation (2) is entered in the x coordinate system (same relations can be naturally written for y and z coordinate directions).

$$R_{ix}^{t} = M_{ix} \cdot a_{ix}^{t} + C_{ix} \cdot v_{ix}^{t}$$
⁽²⁾

Where:

- R_{ix}^{t} is the residual force at joint *i* and at time *t*.
- M_{ix} is the fictitious mass at joint *i* and in the direction *x*.
- C_{ix} is the fictitious damping factor for joint *i* and in direction x
- v_{ix}^{t} is the velocity at joint *i* in the direction *x* and at time *t*.
- a_{ix}^{t} is the acceleration at joint *i* in the direction *x* and at time *t*.

3.1.3 Solution technique

Timeline can be divided into individual time points with a time step Δt . Compliance with the governing Equations (2) is required only at time points t, $t + \Delta t$, $t + 2\Delta t$, $t + 3\Delta t$, etc. During step Δt a linear change of velocity is assumed. Acceleration during step Δt is thus considered to be constant.

By substituting the above assumptions and adjusting Equation (2) the velocity can be expressed in a new time point $t + \Delta t/2$ from the velocity of the previous time point $t - \Delta t/2$ so:

$$v_{ix}^{(t+\Delta t/2)} = v_{ix}^{(t-\Delta t/2)} \frac{M_{ix} / \Delta t - C_{ix} / 2}{M_{ix} / \Delta t + C_{ix} / 2} + \frac{R_{ix}^{t}}{M_{ix} / \Delta t + C_{ix} / 2}$$
(3)

Current coordinates of the joint *i* at the time point $t + \Delta t$ can then be expressed as follows:

$$x_i^{(t+\Delta t)} = \Delta t \cdot v_{ix}^{(t+\Delta t/2)} \tag{4}$$

If the coordinates of unsupported nodes *i* are known, it is possible for each element *k* of Equation (1), to determine the internal forces $T_k^{(t+\Delta t)}$. By this all the forces entering the node *i* are clearly identified. From the imbalance (between external and internal forces) in node *i* we can calculate the residual strength for the corresponding node in time $t + \Delta t$.

$$R_{ix}^{(t+\Delta t)} = P_{ix} - \sum_{k} T_{k}^{(t+\Delta t)} \frac{x_{i} - x_{j}}{r_{k}}$$

$$R_{iy}^{(t+\Delta t)} = P_{iy} - \sum_{k} T_{k}^{(t+\Delta t)} \frac{y_{i} - y_{j}}{r_{k}}$$

$$R_{iz}^{(t+\Delta t)} = P_{iz} + \sum_{k} \frac{Q_{k}}{2} - \sum_{k} T_{k}^{(t+\Delta t)} \frac{z_{i} - z_{j}}{r_{k}}$$
(5)

where P_{ix} means the external loads at any free joint *i* in the direction *x*. Force Q_k is the total load *q* acting on element *k*. Index *k* means all elements entering into joint *i*. Index *j* means second endpoint on the element *k*.

The integration scheme is overlapping because the residual forces are calculated at the end of each time step and the velocities are calculated at half time step. It is clearly seen in Figure 3.



Figure 3: Timeline in dynamic relaxation

In order to even begin calculating, it is necessary to determine the velocity in time point $\Delta t/2$. Using the initial conditions for the time t = 0, when, the $v_{ix}^0 = 0$, is obtained:

$$v_{ix}^{(\Delta t/2)} = \frac{\Delta t}{2M_{ix}} R_{ix}^0 \tag{6}$$

where R_{ix}^{0} are residual forces at time t = 0.

It is also possible at each time point to calculate the kinetic energy throughout the structures.

$$U_{kin}^{(t+\Delta t/2)} = \sum_{i}^{n} \sum_{j}^{m} M_{ij} \left(v_{ij}^{(t+\Delta t/2)} \right)^{2}$$
(7)

where n is the number of joints and m is the number of dimension (3D or 2D).

3.1.4 Iterative algorithm

At the beginning of case the topology (shape of the structure, connection cables, location of supports and joints), parameters of cables (s_0, A, E) and external loads (q, P) are known. Initial coordinates of free joints are selected and value of the fictitious mass and value of the fictitious damping factor are set. Further, the length of the time step Δt is selected.

If the coordinates of nodes are known, it is possible for each element k to calculate the tensile force T_k from Equation (1), then to calculate the residual forces at each joint from Equations (5). Further, it is possible to calculate current velocities for each joint from Equation (3) and current kinetic energy of the cable structures from Equation (6).

If the residual forces at the joints and the kinetic energy of the structure are smaller than the required threshold, then the calculation is finished and the last calculated coordinates correspond with a deformed structure.

If the residual forces at the joints and the kinetic energy of the structure are greater than the required threshold, then using the Equation (4) we need to update the coordinates of all free joints and repeat the calculation.

3.1.5 Stability

The factors having an impact on the stability of the method and speed of convergence are:

- distribution of a fictitious mass of structure at joints
- using of a fictitious damping factor
- choice of time step Δt

As it is written in [2]: "Usually the time step is a fixed value and the other two factors are varied until stability of the iteration is achieved. If the time step Δt is too large or the masses are too small then instability of the iteration may occur and the analysis will not converge to an equilibrium state. Generally convergence may be achieved by reducing the time interval or increasing the fictitious masses."

The used damping factor partially affects the speed of convergence. If the structure is overdamped, convergence is relatively fast.

3.2 Force density method

Description of the method is based on publications [3, 4].

3.2.1 Principle

Basic equations form nodal equilibrium equations of free joints. The basic unknowns are the nodal coordinates of unsupported nodes. We are looking for such a state of unknown nodal coordinates, so that all internal forces entering the node are in equilibrium with the external nodal forces.

3.2.2 Governing equations



Figure 4: Forces at joint i

With the use of force *T* we can assemble the nodal equations of equilibrium for cable elements. It is seen in Figure 4. For each free joint *i* must apply:

$$\sum T_k \frac{x_i - x_j}{r_k} = P_{xi}$$

$$\sum T_k \frac{y_i - y_j}{r_k} = P_{yi}$$

$$\sum T_k \frac{z_i - z_j}{r_k} = P_{zi} + \sum \frac{Q_k}{2}$$
(8)

Symbols stay the same as in the previous chapter.

The equilibrium equations for whole system can be expressed in matrix form as:

$$\mathbf{C}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{t} = \mathbf{p} + \left|\mathbf{C}\right|^{\mathrm{T}}\frac{\mathbf{q}}{2} \tag{9}$$

where C is "branch-node matrix". Matrix U is the diagonal matrix where the elements are made up of the coordinate differences between two end joints of

elements. Vector \mathbf{p} is made up of nodal load. Vector \mathbf{q} is made up of total load acting on cable, \mathbf{R} is diagonal matrix of the chord distance between the two end joint of each element. Vector \mathbf{t} is made up of internal forces *T*.

It is clear that U, R and t of Equation (9) are functions of the unknown vector of nodal coordinates x. If substitution $\mathbf{f}(\mathbf{x}) = \mathbf{C}^{\mathrm{T}}\mathbf{U}\mathbf{R}^{-1}\mathbf{t}$ is performed and also $\mathbf{p}' = \mathbf{p} + |\mathbf{C}|^{\mathrm{T}}\mathbf{q}/2$, Equation (9) can be expressed in a simplified form as follows:

$$\mathbf{f}(\mathbf{x}) = \mathbf{p}' \tag{10}$$

Expression of Equation (10) represent the basic governing equations for the equilibrium system consisting of a cable elements, these equations are clearly nonlinear in relation to \mathbf{x} , the coordinate vector.

3.2.3 Solution technique

The basic governing Equation (10) can be solved by employing an iterative algorithm based on Newton's method for solving nonlinear problems. We prescribe value of $\mathbf{x} = \mathbf{x}_0$ and take Taylor's expansion of $\mathbf{f}(\mathbf{x})$ at \mathbf{x}_0 , and only members of the first order should be considered.

$$\mathbf{f}(\mathbf{x}_0) + \left[\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}_0} \Delta \mathbf{x} = \mathbf{p}'$$
(11)

Derivation of Equation (11) can be derived analytically or computed numerically. Increment vector coordinates $\Delta \mathbf{x}$ can be easily computed.

$$\Delta \mathbf{x} = \left[\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}_0}^{-1} \left(\mathbf{p}' - \mathbf{f}(\mathbf{x}_0)\right)$$
(12)

The coordinate vector **x** is updated after calculating the increment vector $\Delta \mathbf{x}$ and a new iteration step is continued again from Equation (9). The calculation cycle continues as long as it displacements in the vector $\Delta \mathbf{x}$ falls below a predetermined limit value Δs_{lim} . The accuracy of calculation can be affected by the value Δs_{lim} .

4 Example

To compare both methods we chose example given in [4].

The geometric configuration is shown in Figure 5. It is a suspended cable ring with axi-symmetry. The system consists of 16 cables connected to 16 joints (of which 8 are fixed) with an inner radius of 35 m and outer radius of 75 m. The structures have 8 radial cables and 8 tangential cables. All cables have the same cross-sectional area $A = 1,96344 \cdot 10^{-3} \text{ m}^2$ and the same Young's modulus E = 170 GPa. The slack length

 s_0 of all radial cables is 40 m, and that of all ring cables is 32 m. Any external node loads acts on the structure. On each cable acts an uniform load $q = 1,5105 \cdot 10^{-1} \text{ kN/m}.$



Figure 5: Schema of suspended cable ring

Self-created scripts in MATLAB 7.11.0 (2010b) was used for both the calculation methods. The calculation was carried out on the computer ACER Aspire 3694WLMi, Intel Celeron M processor 440 (1,86 GHz, H33 MHz FSB, 1 MB L2 cache), memory RAM 512MB DDR2.

Coordinates z of the unsupported nodes were always set to zero. The accuracy of both methods was set up so that coordinate of points were calculated with an accuracy of millimeters. Speeds, progressions and number of iterative steps were different in both methods. Joint coordinates at the final equilibrium state were same in both cases (see Table 1) and correspond to the results in [4].

Coor. [m]	Number of joint							
	1	2	3	4	5	6	7	8
x	41,649	29,451	0	-29,451	-41,649	-29,451	0	29,451
у	0	29,451	41,649	29,451	0	-29,540	-41,649	-29,451
Z	21,713	21,713	21,713	21,713	21,713	21,713	21,713	21,713

Table 1: Joint coordinates at the final equilibrium state.

5 Conclusions

Different variants of time steps Δt , the fictitious mass M and the fictitious damping C factor have been chosen. The most interesting results are given in Table 2. Figure 6 shows iteration progress of coordinates x at joint 2 for the first setting ($\Delta t = 0.3$ s, M = 10 t, C = 2500 ts⁻¹). The quickest solution (while meeting the required accuracy) was arriving after 103 iterations. The coordinate progress at joint 2 in this configuration ($\Delta t = 2$ s, M = 220 t, C = 28 ts⁻¹) is shown in Figure 7. The progress of residual forces is shown in Figure 8. Figure 9 shows progress of kinetic energy of structure for the same choice of parameters.

	1		1	1		1			
Δt	M _{ix}	M _{iz}	C _{ix}	C_{iz}	Accuracy R _{zi}	Count of iteration	Time of solution	Coordin joint	nates of 2 [m]
[s]	[t]	[t]	$[t.s^{-1}]$	$[t.s^{-1}]$	[kN]	-	[min]	х	Z
0,3	10	10	2500	2,5	0,005	5416	12:15	29,451	21,713
2	1000	1000	250	250	0,005	580	1:10	29,540	21,712
2	1000	1000	50	50	0,001	218	0:25	29,541	21,713
2	1000	1000	45	45	0,001	unstable	-	-	-
2	220	220	28	28	0,001	103	0:13	29,541	21,713
2	220	220	28	28	0,1	72	0:09	29,542	21,720
2	200	200	28	28	0,001	unstable	-	_	-

Table 2: Dynamic relaxation. Records of the calculation. The values M_{iy} and C_{iy} are the same as for the *x* direction.



Figure 6: Dynamic relaxation. Left – The coordination x of joint 2. Right – The coordination z of joint 2 (first 100 iterations). Parameters are set: $\Delta t = 0,3$ s, M = 10 t, $C_{ix} = 2500$ ts⁻¹, $C_{iy} = 2500$ ts⁻¹, $C_{iz} = 2,5$ ts⁻¹.



Figure 7: Dynamic relaxation. Left – The coordination x of joint 2. Right – The coordination z of joint 2. Parameters are set: $\Delta t = 2$ s, M = 220 t, C = 28 ts⁻¹.



Figure 8: Dynamic relaxation. Left – The residual force in direct x of joint 2. Right – The residual force in direct z of joint 2. Parameters are set: $\Delta t = 2$ s, M = 220 t, C = 28 ts⁻¹.



Figure 9: Dynamic relaxation. The kinetic energy of structure. Parameters are set: $\Delta t = 2 \text{ s}, M = 220 \text{ t}, C = 28 \text{ ts}^{-1}.$

The solution in force density method was found after 204 iterations, respectively after 109 iterations (with less accuracy). Details of the calculation are recorded in Table 3. The coordinate progress at joint 2 is shown in Figure 10.

Accuracy $\Delta s_{\rm lim}$	Count of iteration	Time of solution	Coordinates of joint 2 [m]		
[m]	-	[min]	Х	Z	
0,00001	204	0:18	29,451	21,713	
0,001	109	0:11	29,457	21,700	

Table 3: Force density method. Records of the calculation.



Figure 10: Force density method. Left – The coordination x of joint 2. Right – The coordination z of joint 2.

The method of dynamic relaxation in this structure has a lower number of iterations and faster time of solution at the same accuracy than the force density method. It is clear that the calculation time can be affected by the used scripts may be written effectively. However a lower number of iterations shows that the method of dynamic relaxation has considerable potential for numerical calculations of cable structures. The disadvantage of dynamic relaxation is the correct setting of parameters and appropriate fictional setting time step. However, some procedures for the selection of these parameters are already at present devised [2]. The calculations can be quite effective.

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