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The Spectral Method for Moving Load Analysis of Thin Plates

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Abstract

This paper shows an application of the spectral method in dynamics of structures for the special case of plate under the action of a moving load. The first part of the paper describes the spectral method formulated on the basis of matrix operators. The application in plate analysis is presented using an example. The other part elaborates on dynamic analysis and moving loads. An example illustrates the application of spectral matrix operators. The example includes Dirichlet and Neumann boundary conditions and complex loading since it is a simply supported - free plate under the action of a moving force. The boundary conditions have been imposed using Lagrange multipliers. The approach presented is general in its formulation and can be easily adapted for analysis of any other structure.

Keywords: dynamic structural analysis, plates, moving load, spectral method, matrix operators, strong formulation.

1 Introduction

Moving load is a very demanding task in structural analysis. It is a non-conservative problem in dynamic analysis of structures described as a system of differential equations in the case of discrete representation of structural masses, or a partial differential equation in the case of continuous mass representation. In order to be solved the system has to be discretized in space and time. Time discretization is usually based on finite difference technique leading to time integration scheme of the Newmark type (see [1]). However, there are some difficulties in the calculation of accelerations resulting from moving load and a modified Newmark could be advantageous [2]. For space discretization two choices are mostly used, depending of the formulation chosen. The most popular choice is the finite element discretization that requires weak (integral) formulation of the problem [3]. The strong form (partial differential equation) is usually discretized using finite differences. Accuracy of finite differences can be improved if spectral analysis is applied to the strong form (like Fourier transforms). In spite of their good properties spectral methods are just beginning to emerge in engineering applications. There is a similar situation regarding analysis of plates under the moving load; most authors prefer to model their structures as beams (e.g. [4], [5], [1]). However, there are formulations capable of dealing with 2 and 3 dimensional structures; quite a general formulation is presented in [6]. They work in convected coordinates and with weak formulation of the problem.

This paper is based on Chebyshev spectral method (see [7]). Unlike Fourier method there is no need for special representation (transformation) of loading. Mathematically Chebyshev spectral method has been somewhat modified so that it is completely formulated using matrix operators and its' use resembles stiffness matrix approach in structural analysis.

Boundary conditions are expressed through Lagrange multipliers and can be of Dirichlet or Neumann type (or any combination of them like in the plate example). That approach also allows for nonholonomic conditions (i.e. conditions involving velocity). They have not been treated in this paper but they could be important for moving load analysis [8]. The drawback of the method of Lagrange multipliers is increase in size but it is not so pronounced since spectral methods require modest number of equations. More important is possible creation of a stiff system for certain boundary conditions (like supported and free plate in the example at the end of the paper). The resulting system is so stiff that the penalty approach that works very well for beams could not be used. However, some simple numerical tricks were enough to allow the use of standard equation solvers for the system of equations expanded with the Lagrange multipliers.

2 Spectral Method Formulation

2.1 Spectral Matrix Operators

Spectral method has been chosen to replace the series expansion used in Fourier analysis solution of differential equations. Spectral methods offer high precision with minimal number of points used in spatial discretization.

In this work Chebyshev polynomials are chosen for spatial interpolation of the domain of the differential equation although there are other possibilities as well. Detailed description can be found in specialized literature (e.g. [9]). Here we will address some details specific to analysis of plates under the moving load.

Application of spectral method consists in solution of the strong formulation (differential equation) of structural (static or dynamic) problem. The method can be formulated using matrix differentiation operators

$$p_x = D_N \cdot p \qquad (1)$$

where p is a vector of discrete data of size N, p_x is its derivative and D_N is matrix differential operator, a square matrix of size [NxN]. Such matrix differential operator can be constructed for various methods (e.g. finite differences). Boundary conditions have to be incorporated in D_N.

Spectral methods produce full differentiation matrix, which means that all points are involved in getting the result. One may argue that that is the reason for high accuracy of spectral methods.

Normally, structural equations require higher derivatives and two dimensional modelling requires partial derivatives and modifications of matrix operators. In modelling of plates there is a need for second, third and fourth order derivatives in x and y directions. Higher order operators are simply produced using matrix multiplications while expansion in more than one direction can be obtained using Kronecker product of two matrices. In the case when there are N points in the x direction and M points in the y direction and numeration is consecutive in the x direction, differential operators are, respectively

$$D_x = I_M \oplus D_N \quad , \qquad D_y = D_M \oplus I_N \tag{2}$$

which are Kronecker products of unit matrices and differential operators. It is to be mentioned that the Kronecker product is not commutative. Also, if one matrix is a unit matrix, then the product simply rearranges and expands the original matrix. In our example [NxN] matrix times [MxM] matrix produces a matrix of the size [NMxNM].

2.2 Application to Thin Plate Equation

2.2.1 Boundary Conditions using Lagrange Multipliers

Only the simple boundary conditions could be incorporated into the matrix D_N (or D_M). For the simple support condition (homogeneous Dirichlet boundary condition) that consists of removing the first and the last rows and columns from the matrix. The resulting problem is reduced in size and homogenous condition is simply restored after the solution. Non-homogeneous boundary conditions could be obtained by introducing 1s on the diagonal and replacing the corresponding row and column with zeros. However, the Neumann type of boundary condition could scarcely be enforced directly into the matrix differential operator. Plate with two free ends requires more elaborated boundary conditions [10]. We require that the reaction and the moment along the free boundary *y* vanish

$$\frac{\partial^3}{\partial y^3}w + (2-\nu)\cdot\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial y}w = 0 \quad , \quad \frac{\partial^2}{\partial y^2}w + \nu\cdot\frac{\partial^2}{\partial x^2}w = 0 \tag{3}$$

These boundary conditions have to be translated into differential operators

$$\left(D_M{}^3 \oplus I_N\right) + (2-\nu)\cdot\left(D_M{}^2 \oplus I_N\right)\left(I_M \oplus D_N\right) = 0 \tag{4}$$

$$\left(D_M{}^2 \oplus I_N\right) + \nu\cdot\left(I_M \oplus D_N{}^2\right) = 0 \tag{5}$$

Operators in Equations (4) and (5) have full size of the problem but they do not act on all the points of the plate. The extra points are removed through extraction of only those rows that belong to the degrees of freedom where the desired boundary condition is present. In our example that leaves us with two matrix operators of the size [2NxNM]. Together with matrix operators resulting from the Dirichlet boundary conditions on the two supported sides of the plate they are assembled into constraint matrix C used in the Lagrange multiplier method

$$C = \begin{bmatrix} I_{PM} \\ I_{PN}\cdot\left[\left(D_M{}^3 \oplus I_N\right) + (2-\nu)\cdot\left(D_M{}^2 \oplus I_N\right)\left(I_M \oplus D_N\right)\right] \\ I_{PN}\cdot\left[\left(D_M{}^2 \oplus I_N\right) + \nu\cdot\left(I_M \oplus D_N{}^2\right)\right] \end{bmatrix} \tag{6}$$

I_{PN} and I_{PM} are purging matrices for extraction of N and M points respectively and C is of size [(2M+2N+2N)xNM].

2.2.2 Application to Moving Load Equation

By applying D'Alembert's principle differential equation describing two dimensional structural behaviour under the moving load is obtained

$$\rho\cdot\frac{\partial^2}{\partial t^2}u + \frac{1}{K}\cdot\left(\frac{\partial^4}{\partial x^4}u + 2\cdot\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}u + \frac{\partial^4}{\partial y^4}u\right) = p(t)\cdot\delta\left(x - v_x\cdot t\right)\cdot\delta\left(y - v_y\cdot t\right) \tag{7}$$

u=u(x,y,t) is displacement in space and time, ρ is surface density, K is plate stiffness, δ is Dirac function. Load description can be simplified if we assume that it moves along one axis only. Using spectral operator for space discretization and assuming load is moving along one coordinate only, we can write using sub matrices

$$\begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}\cdot\frac{\partial^2}{\partial t^2}\begin{pmatrix} u \\ \lambda \end{pmatrix} + \begin{pmatrix} \Delta\Delta & C^T \\ C & 0 \end{pmatrix}\cdot\begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} F(x,t) \\ 0 \end{pmatrix} \tag{8}$$

$$F(x,t) = p(t) \cdot \delta\left(x - v_x \cdot t\right) \tag{9}$$

M is mass matrix, 0s are zero matrices of the appropriate size, ΔΔ is spectral operator playing the role of the stiffness matrix, λ is vector of Lagrange multipliers, u is the displacement and F(x,t) is loading function. The main characteristic of the moving load problem described with Equations (8) and (9) is the right hand side. It is convenient to perform loading discretization prior to the time integration procedure (after time integration parameters, like Δt are set). After it has been discretized in space and time, solution of the dynamic equation can proceed using any time integration scheme. (Note: space discretization has to obey properties of the Dirac δ function. That results with a requirement of a constant force within one time increment). In the case of a constant moving force space and time discretization of the loading is presented in Figure 1. It is straightforward to implement a non-constant force (like in [4]) since there are no restrictions on the forcing function p(t).

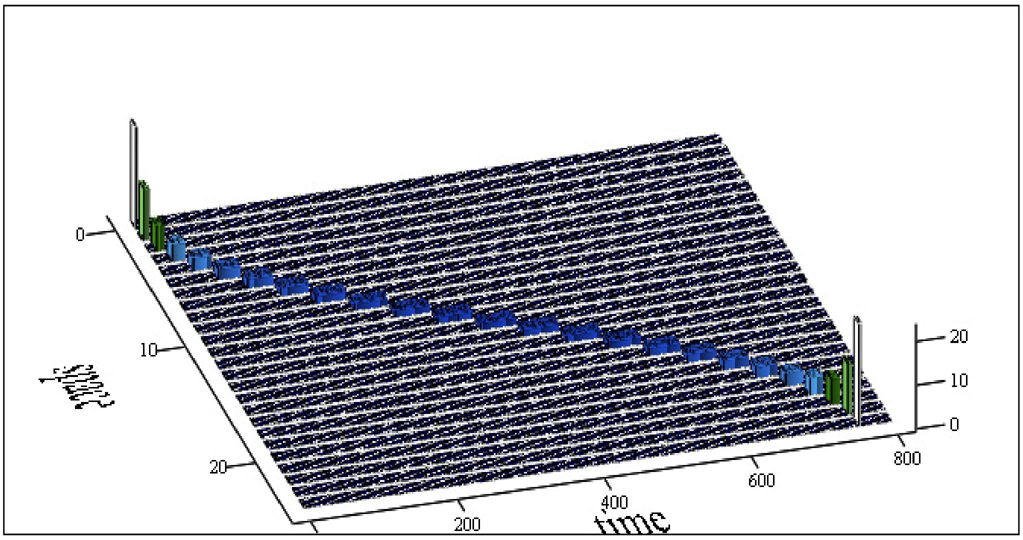

Figure 1: Space discretization of a constant moving force.

Note that due to the Chebyshev polynomials used in space discretization even the constant loading does not have equal amplitude in every time increment. It is straightforward to include other forms of moving load such as force changing in time etc. After the appropriate form for the right hand side (the loading) has been found, solution procedure can be applied.

In order to apply Newmark class of integration schemes the governing Equations (8) and (9) have to be rewritten in the incremental form

$$\begin{pmatrix} M & 0 \\ 0 & 0^{-20} \end{pmatrix} \cdot \frac{\partial^2}{\partial t^2} \begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix} + \begin{pmatrix} \Delta\Delta & C^T \\ C & 0^{-20} \end{pmatrix} \cdot \begin{pmatrix} \delta u \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} \delta F \\ 0 \end{pmatrix} \tag{10}$$

where δu is the displacement increment and δF is loading increment calculated from the discretized loading function. Mass and "stiffness" matrices could be rather stiff for some boundary conditions and introduction of 0^{-20} (very small number in place of 0s) can sometimes improve the behaviour of numerical procedures (especially eigenvalue analysis). Additionally $\Delta\Delta$ matrix can be stabilized by introduction of rigid body modes (see [11]). In our case numerical procedures in MathCAD 13 performed well with 0s but those in MathCAD 11 required that trick. In order to asses quality of the solution condition number for spectral operators have been calculated. Among many possibilities we have chosen L1 norm based on singular values of a matrix. In contrast to condition numbers based on other norms it can be evaluated for singular matrices as well. Actually, we are dealing with nearly singular matrices but due to limited calculation capabilities the practical performance can be similar to singular matrices. In Table 1 there are condition numbers for some examples. Beam is simply supported beam, plate 4 is plate simply supported on 4 sides (with different aspect ratios) and plate 2 is plate simply supported on 2 sides (which is numerically a much more difficult example due to boundary conditions on free ends).

	Beam	Plate 4 1:1	Plate 4 1:5	Plate 2 1:1	Plate 2 1:5
Dirichlet	N=49 $1.493*10^{10}$	N=961 $4.263*10^{8}$	N=713 $5.025*10^{7}$	n.a.	n.a.
singular	N=51 $7.761*10^{20}$	N=1089 $7.411*10^{19}$	N=825 $1.362*10^{20}$	N=1089 $3.797*10^{20}$	N=825 $1.053*10^{20}$
r.b.m.	N=51 $1.750*10^{20}$	N=1089 $1.633*10^{21}$	N=825 $4.480*10^{19}$	N=1089 $9.66*10^{19}$	N=825 $8.853*10^{19}$
Lagrange	N=55 $8.804*10^{15}$	N=1345 $1.226*10^{27}$	N=1049 $1.813*10^{27}$	N=1353 $7.234*10^{33}$	N=1057 $2.926*10^{32}$
Penalty	N=51 $1.235*10^{17}$	N=1089 $2.889*10^{16}$	N=825 $2.630*10^{16}$	N=1089 $5.194*10^{20}$	N=825 $4.393*10^{20}$

Table 1. Sizes and condition numbers of spectral stiffness matrices for various structures.

In the first column of Table 1 there is an explanation of boundary conditions: "Dirichlet" – homogenous boundary conditions imposed through removal of points where displacement values is known to be zero, "singular" – full sized matrix without any boundary conditions, it is singular, "r.b.m." – above matrix with removal of rigid body modes, not singular any more, "Lagrange" – r.b.m. matrix expanded with boundary conditions and lagrange multipliers, "Penalty" – r.b.m. matrix with boundary conditions imposed through penalty number (α from 10^8 to 10^{10}).

Note: condition number alone can not give the whole picture about matrix usability. It is evident from Table 1 that the removal of rigid body modes can worsen the condition number but at the same time matrix becomes non-singular and can be used in calculations. Also, introduction of lagrange multipliers seem to deteriorate the performance but it is only the multiplier part of the matrix that is badly conditioned, the rest behaves well and the results are quite acceptable. Introduction of techniques that would separate lagrange multipliers from the rest of the matrix would further confirm this statement. E.g. condition number for Plate 2 is smaller for penalty method then for lagrange multipliers method but penalty procedure gives completely wrong results in that example.

3 Examples

3.1 Static Plate Analysis

All examples are analysing the same plate: rectangular plate simply supported on two opposite sides and free on the other two; length = 10.0 m, width = 5.0 m, thickness = 0.10 m; material properties Y = 120000000, ν = 0.15. Various discretization resolutions have been applied, from N=12, M=6 to N=24, M=12 in steps of 4 points for length and 2 for width. That resulted in 91 to 325 equations for the plate (or (N+1)*(M+1)) and 80 to 152 equations for Lagrange multipliers (or 2(2N+2M+4)). Also, results for very narrow plate (width=0.25m and ν=0.0) have been compared with those for a beam for all three types of analysis; very good agreement has been observed but the results are not presented here.

Attention is needed in graphical presentation of results since they all come in Chebyshev coordinates and have to be mapped into regular rectangular coordinates.

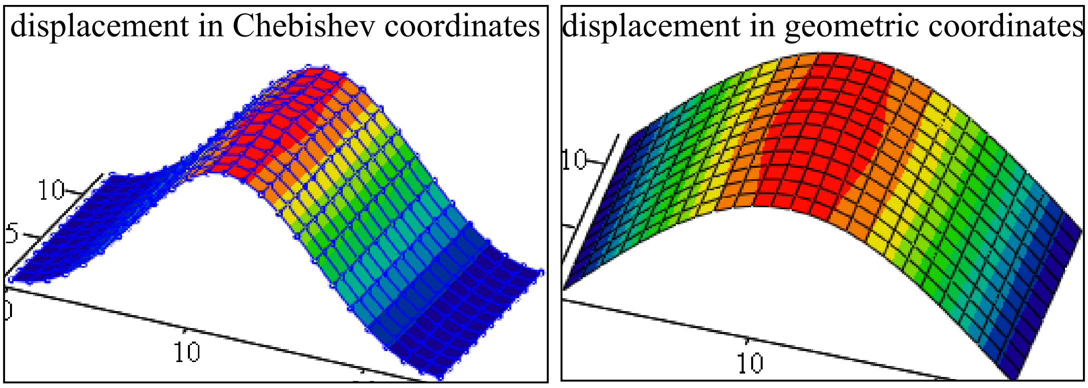

Figure 2: Deflections of plate supported at two sides.

Figure 3 presents deflections of the plate in global coordinates. Width of the plate is chosen such that the mid point deflection is just beginning to visibly vary in the y direction (in order to have interesting pictures since parametric analysis is not the subject of this paper). More pronounced deflection in the perpendicular direction could be obtained by reducing the Poisson or further increasing the width.

3.2 Plate under Moving Load

This is example of the same plate under action of the moving load. The load moves along the middle line of the plate and changes in space and time as presented in Fig. 3b. Analysis in time domain is performed for Δt=0.005 sec. and 800 time steps (total time of the analysis is 2 sec.). Analysis is extended in time for 400 cycles after the load leaves the plate (1 second).

Time analysis has been performed with a variation of the Newmark integration method (impulse acceleration method from [2]). This calculation was performed without structural damping but it can be easily added. Figure 3 presents result of the analysis. Interpolation points between Chebyshev nodes have been introduced for easier evaluation of results and for better graphical presentation. Deflection in time of the middle line is compared with the exact solution that exists for beam and excellent agreement was found.

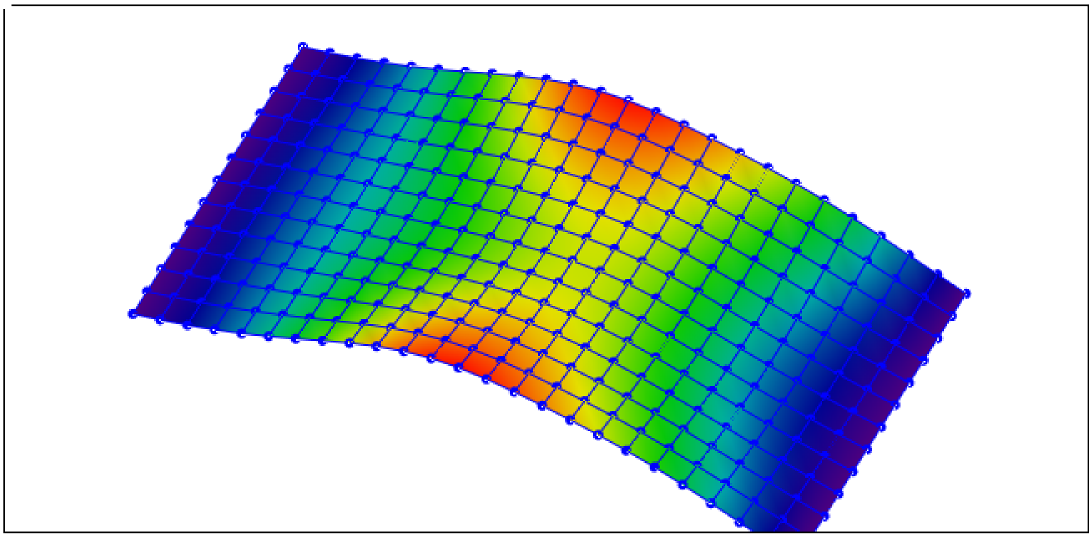

Figure 3: Displacement in time of the mid-line of plate under the moving load.

Displacement presented in Figure 3 looks very realistic and one can even observe upward movement of sides of the plate ("wing flapping") that exists in suficiently wide plates.

4 Conclusions

In this paper discretization in space is obtained by the Chebyshev spectral method that is implemented in form of a matrix differential operator. Resulting discretization in its form resembles the stiffness matrix and time discretization can be performed using any suitable method. The spectral differential matrix operator is fast and simple to construct; it is sufficient to construct one dimensional operator and expand it into two or more dimensions using the formalism of matrix Kronecker product.

The resulting matrix is rather dense and small in size since spectral methods achieve high accuracy with a modest number of points. Boundary conditions can be treated in several ways but Lagrange multipliers offer the most general approach suitable even for the most complicated boundary conditions. Several methods have been tried out and the condition numbers of the resulting matrix operators are presented. Penalty method requires special mentioning since it has the best condition number but fails in complicated cases where the number of constraints is large (compared to size of the stiffness operator). That remains the subject of further investigation.

The proposed procedure has been tried on a dynamic example of moving load analysis with very good results and realistic behaviour of plate under the moving load. The matrix operator formalism of the spectral method produces accurate results while retaining the small size of the problem. Hence it is very suitable for integration of all strong forms of engineering problems.

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