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A Mathematical Model of a Five Layer Sandwich Beam

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Abstract

The subject of this paper is an analysis of deflection of a five layer sandwich beam. The mechanical and physical properties vary through the thickness of the beam and depend on the material of each layer. Two faces of the beam are thin aluminium sheets and the core is made of aluminium foam. Between the faces and the core there are two thin binding glue layers. The main goal of the paper is to present a mathematical model of the five layer beam and to compare the results of the analyses obtained analytically and numerically.

Keywords: sandwich structure, deflection, metal foam, mathematical modelling, five layer beam, numerical model.

1 Introduction

Sandwich structures with a metal foam core are subject of contemporary studies. These structures are characterized by impact and heat resistance, acoustic and vibration reduction and easy assembly. Plantema [1] and Allen [2] described the bases of the theory of sandwich structures. Banhart [3] delivered manufacture, characterisation and application of cellular metals and metal foams. Noor *et al*. [4] and Vinson [5] presented strength and stability problems of sandwich structures. Wang *et al.* [6] described the structural response of clamped sandwich beams with aluminium foam core subjected to impact loading. Grigolyuk and Chulkov [7] provided the first hypothesis of cross section deformations of sandwich structures. Wang *et al*. [8] discussed the higher order hypotheses including shearing of beams and plates. He *et al.* [9] presented precise bending stress analysisof corrugated-core sandwich panels. Carrera [10] formulated the zig-zag hypotheses for multilayered plates. Iaccarino *et al.* [11] presented effect of a thin soft core on the bending behavior of a sandwich. Chakrabarti *et al.* [12] developed a new FE model based on higher order zig-zag theory for the static analysis of laminated sandwich beams with a soft core. Steeves *et al.* [13] and Qin *et al.* [14] presented analytical models of collapse mechanisms of sandwich beams under transverse force. Rakow and Wass [15] presented mechanical properties of an aluminium foam under shear. Birman [16] presented modeling and analysis of functionally graded materials. Magnucka-Blandzi and Magnucki [17] and Magnucki *et al.* [18] described strength and buckling problems of sandwich beams with a metal foam core. Zenkert [19] presented strength of sandwich beams with debondings in the interface between the face and core.

A simply supported five layer sandwich beam of the length L and the width b carries a concentrated force F as shown in Figure 1. The force F is located in the middle of the beam.

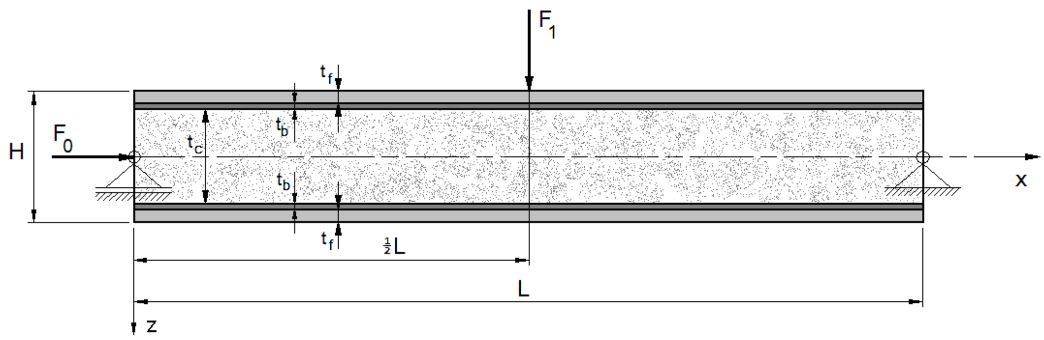

Figure 1. Scheme of the loaded beam.

2 Analytical analysis

The deformation of the flat cross section of the five layer beam is shown in Figure 2.

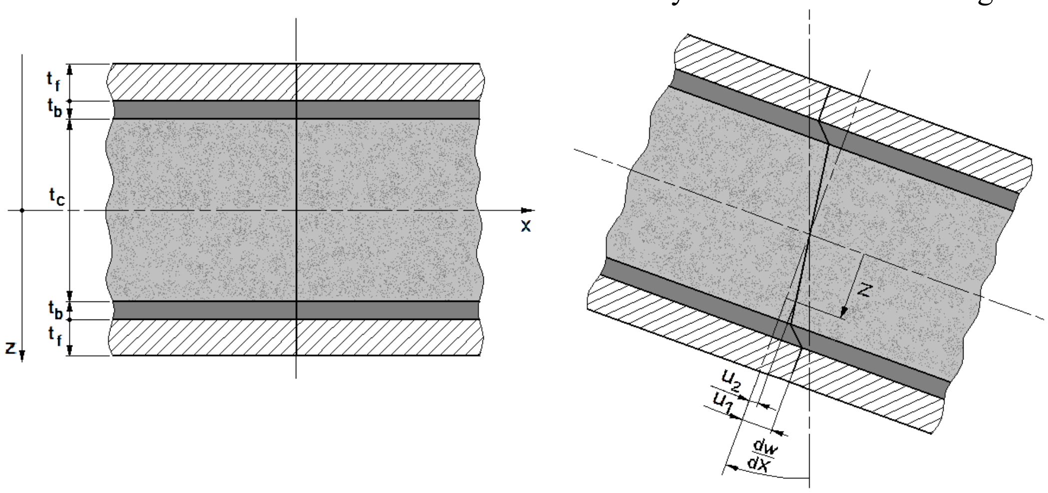

Figure 2. Scheme of displacements – the hypothesis for the beam.

The field of displacements is formulated as follows:

1. the upper face $\quad -\left(1/2+x_1+x_2\right) \le \zeta \le -\left(1/2+x_1\right)$

$$u(x,\zeta)=-t_c\left[\zeta\frac{dw}{dx}+\psi_1(x)\right],\tag{1}$$

2. the upper binding layer $\quad -(1/2+x_1)\le\zeta\le -1/2$

$$u(x,\zeta)=-t_c\left[\zeta\frac{dw}{dx}+\psi_2(x)-\frac{1}{x_1}\left(\zeta+\frac{1}{2}\right)(\psi_1(x)-\psi_2(x))\right],\tag{2}$$

3. the core $\quad -1/2\le\zeta\le 1/2$

$$u(x,\zeta)=-t_c\zeta\left[\frac{dw}{dx}-2\psi_2(x)\right],\tag{3}$$

4. the lower binding layer $\quad 1/2\le\zeta\le 1/2+x_1$

$$u(x,\zeta)=-t_c\left[\zeta\frac{dw}{dx}-\psi_2(x)-\frac{1}{x_1}\left(\zeta-\frac{1}{2}\right)(\psi_1(x)-\psi_2(x))\right],\tag{4}$$

5. the lower face $\quad 1/2+x_1\le\zeta\le 1/2+x_1+x_2$

$$u(x,\zeta)=-t_c\left[\zeta\frac{dw}{dx}-\psi_1(x)\right],\tag{5}$$

where:
$x_1=t_b/t_c$, $\quad x_2=t_f/t_c$, $\quad \zeta=z/t_c$, $\quad \psi_1(x)=u_1(x)/t_c$, $\quad \psi_2(x)=u_2(x)/t_c$.

Strains of the layers of the beam are defined by the geometric relationship in the following form:

1. the upper face

$$\varepsilon_x=-t_c\left[\zeta\frac{d^2w}{dx^2}+\frac{d\psi_1(x)}{dx}\right],\qquad \gamma_{xz}=0,\tag{6}$$

2. the upper binding layer

$$\varepsilon_x=-t_c\left[\zeta\frac{d^2w}{dx^2}+\frac{\psi_2(x)}{dx}-\frac{1}{x_1}\left(\zeta+\frac{1}{2}\right)\left(\frac{d\psi_1(x)}{dx}-\frac{d\psi_2(x)}{dx}\right)\right],$$

$$\gamma_{xz}=\frac{1}{x_1}[\psi_1(x)-\psi_2(x)],\tag{7}$$

3. the core

$$\varepsilon_x=-t_c\zeta\left[\frac{d^2w}{dx^2}-2\frac{d\psi_2(x)}{dx}\right],\qquad \gamma_{xz}=2\psi_2(x),\tag{8}$$

4. the lower binding layer

$$\varepsilon_x=-t_c\left[\zeta\frac{d^2w}{dx^2}-\frac{\psi_2(x)}{dx}-\frac{1}{x_1}\left(\zeta-\frac{1}{2}\right)\left(\frac{d\psi_1(x)}{dx}-\frac{d\psi_2(x)}{dx}\right)\right],$$

$$\gamma_{xz} = \frac{1}{x_1}\left[\psi_1(x) - \psi_2(x)\right], \tag{9}$$

5. the lower face

$$\varepsilon_x = -t_c\left[\zeta \frac{d^2 w}{dx^2} - \frac{d\psi_1(x)}{dx}\right], \qquad \gamma_{xz} = 0, \tag{10}$$

The physical relationships, according to Hooke's law, for individual layers are

$$\sigma_x = E\varepsilon_x, \qquad \tau_{xz} = G\gamma_{xz}. \tag{11}$$

The bending moment of any cross section of the beam

$$\begin{aligned} M_b(x) = \int_A \sigma_x z dA = -bt_c^3 \Bigg\{ &\left(2E_f c_{2f} + 2E_b c_{2b} + \frac{1}{12}E_c\right)\frac{d^2 w}{dx^2} - \\ -&\left[E_f c_{1f} + E_b \frac{x_1}{6}(3+4x_1)\right]\frac{d\psi_1}{dx} - \left[\frac{1}{6}E_c + \frac{1}{6}E_b x_1(3+2x_1)\right]\frac{d\psi_2}{dx}\Bigg\}, \end{aligned} \tag{12}$$

where:

$c_{1b} = x_1(1+x_1)$, $c_{2b} = \frac{1}{12}x_1\left(3+6x_1+4x_1^2\right)$, $c_{1f} = x_2(1+2x_1+x_2)$,

$c_{2f} = \frac{1}{12}x_2\left[12x_1(1+x_1+x_2)+3+6x_2+4x_2^2\right]$.

The transverse force of any cross section of the beam

$$Q(x) = \int_A \tau_{xz} dA = 2bt_c\left[G_b\psi_1(x) + (G_c - G_b)\psi_2(x)\right], \tag{13}$$

where:

$$G_c = \frac{E_c}{2(1+\nu_c)}, \qquad G_b = \frac{E_b}{2(1+\nu_b)}.$$

2.1 Equations of equilibrium

The potential energy of the elastic strain of the beam is

$$U_\varepsilon = \frac{1}{2}\int_V (\varepsilon_x\sigma_x + \gamma_{xz}\tau_{xz})dV = \frac{1}{2}bt_c\int_0^L \left(f_{Ef} + f_{Eb} + f_{Ec}\right)dx, \tag{14}$$

where:

$$f_{Ef} = 2E_f t_c^2\left[c_{2f}\left(\frac{d^2 w}{dx^2}\right)^2 - c_{1f}\frac{d^2 w}{dx^2}\frac{d\psi_1}{dx} + x_2\left(\frac{d\psi_1}{dx}\right)^2\right],$$

$$f_{Eb} = 2E_b t_c^2 \left[c_{2b} \left(\frac{d^2 w}{dx^2} - \frac{1}{x_1}\frac{d\psi_1}{dx} + \frac{1}{x_1}\frac{d\psi_2}{dx} \right)^2 + \frac{c_{1b}}{2x_1}\left(\frac{d^2 w}{dx^2} - \frac{1}{x_1}\frac{d\psi_1}{dx} + \frac{1}{x_1}\frac{d\psi_2}{dx} \right) \times \right.$$

$$\left. \times \left(\frac{d\psi_1}{dx} - (1+2x_1)\frac{d\psi_2}{dx} \right) + \frac{1}{4x_1}\left(\frac{d\psi_1}{dx} - (1+2x_1)\frac{d\psi_2}{dx} \right)^2 \right] + \frac{2}{x_1} G_b [\psi_1(x) - \psi_2(x)]^2,$$

$$f_{Ec} = \frac{1}{12} E_c t_c^2 \left[\frac{d^2 w}{dx^2} - 2\frac{d\psi_2}{dx} \right]^2 + 4G_c \psi_2^2(x).$$

The work of the external load is

$$W = \int_0^L q w dx + \frac{1}{2} F_0 \int_0^L \left(\frac{dw}{dx} \right)^2 dx. \tag{15}$$

The system of three partial differential equations obtained from the principle of stationary total potential energy $\delta(U_\varepsilon - W) = 0$, after integrating over the thickness of the beam and integrating by parts over the length of the beam, takes the following form:

$$\delta w) \quad \begin{aligned} & bt_c^3 \left\{ \left(2E_f c_{2f} + 2E_b c_{2b} + \frac{1}{12}E_c \right) \frac{d^4 w}{dx^4} - \left[E_f c_{1f} + \frac{1}{6}E_b x_1 (3+4x_1) \right] \frac{d^3\psi_1}{dx^3} + \right. \\ & \left. -\frac{1}{6}\left[E_c + E_b x_1 (3+2x_1) \right] \frac{d^3\psi_2}{dx^3} \right\} = q - F_0 \frac{d^2 w}{dx^2}, \end{aligned} \tag{16}$$

$$\delta\psi_1) \quad \begin{aligned} & \left[E_f c_{1f} + \frac{1}{6}E_b x_1 (3+4x_1) \right] \frac{d^3 w}{dx^3} - 2\left(E_f x_2 + \frac{1}{3}E_b x_1 \right) \frac{d^2\psi_1}{dx^2} - \frac{1}{3}E_b x_1 \frac{d^2\psi_2}{dx^2} \\ & + \frac{2G_b}{x_1 t_c^2} [\psi_1(x) - \psi_2(x)] = 0, \end{aligned} \tag{17}$$

$$\delta\psi_2) \quad \begin{aligned} & \frac{1}{6}\left[E_c + E_b x_1 (3+2x_1) \right] \frac{d^3 w}{dx^3} - \frac{1}{3}E_b x_1 \frac{d^2\psi_1}{dx^2} - \frac{1}{3}(E_c + 2E_b x_1) \frac{d^2\psi_2}{dx^2} + \\ & -\frac{2G_b}{x_1 t_c^2}\psi_1(x) + \frac{1}{t_c^2}\left(4G_c + \frac{2}{x_1}G_b \right)\psi_2(x) = 0. \end{aligned} \tag{18}$$

The first equation (16) of the system is equivalent to the bending moment (12). Therefore, for further analysis purpose the system of three Eqs. (12), (17) and (18) is applied.

2.2 Deflection of a beam

The simply supported sandwich beam is loaded by force F. The bending moment for this load case is written in the form $M_b(x) = \frac{1}{2}Fx$. After simply transformations of three equations (12), (17) and (18), two equations are obtained

$$\frac{d^4\psi_1}{dx^4}+k_2^2\frac{d^2\psi_1}{dx^2}-k_1\psi_1(x)=k_0Q(x)\,, \tag{19}$$

$$\psi_2(x)=\frac{t_c^2}{b_{02}}\left[b_f\frac{Q(x)}{bt_c^3}+b_{11}\frac{d^2\psi_1}{dx^2}-\frac{b_{01}}{t_c^2}\psi_1(x)\right], \tag{20}$$

where:

$$k_2^2=-\frac{B}{A},\quad k_1=-\frac{C}{A},\quad k_0=\frac{D}{A},\quad A=\frac{a_{12}a_{13}-a_{11}a_{23}}{a_{11}a_{12}}\frac{b_{11}}{b_{02}}t_c^2,$$

$$B=\frac{a_{11}a_{22}-a_{12}^2}{a_{11}a_{12}}+\frac{a_{12}a_{13}-a_{11}a_{23}}{a_{11}a_{12}}\frac{b_{01}}{b_{02}}+\frac{b_{11}}{b_{02}}\frac{2G_b}{a_{12}x_1},\qquad C=\left(1+\frac{b_{01}}{b_{02}}\right)\frac{2G_b}{a_{12}x_1t_c^2},$$

$$D=\frac{1}{bt_c^3}\left(\frac{1}{a_{11}}+\frac{b_f}{b_{02}}\frac{2G_b}{a_{12}x_1}\right),\qquad b_{11}=\frac{a_{11}a_{22}-a_{12}^2}{a_{12}a_{13}-a_{11}a_{23}}-\frac{a_{12}a_{31}-a_{11}a_{32}}{a_{11}a_{33}-a_{13}a_{31}},$$

$$b_{01}=\left(\frac{a_{11}}{a_{12}a_{13}-a_{11}a_{23}}-\frac{a_{11}}{a_{11}a_{33}-a_{13}a_{31}}\right)\frac{2G_b}{x_1},$$

$$b_{02}=\left(\frac{a_{11}}{a_{11}a_{33}-a_{13}a_{31}}-\frac{a_{11}}{a_{12}a_{13}-a_{11}a_{23}}\right)\frac{2G_b}{x_1}+\frac{a_{11}}{a_{11}a_{33}-a_{13}a_{31}}4G_c,$$

$$b_f=\left(\frac{a_{12}}{a_{12}a_{13}-a_{11}a_{23}}+\frac{a_{31}}{a_{11}a_{33}-a_{13}a_{31}}\right)\frac{2G_b}{x_1},\qquad a_{11}=2E_fc_{2f}+2E_bc_{2b}+\frac{1}{12}E_c,$$

$$a_{12}=E_fc_{1f}+\frac{1}{6}E_bx_1\left(3+4x_1\right),\qquad a_{13}=\frac{1}{6}\left[E_c+E_bx_1\left(3+2x_1\right)\right],\qquad a_{21}=a_{12},$$

$$a_{22}=2\left(E_fx_2+\frac{1}{3}E_bx_1\right),\qquad a_{23}=\frac{1}{3}E_bx_1,\qquad a_{31}=a_{13},\quad a_{32}=a_{23},$$

$$a_{33}=\frac{1}{3}\left(E_c+2E_bx_1\right).$$

The two unknown functions are assumed forms of the first word of the Fourier series

$$\psi_1(x)=\psi_{11}\cos\frac{\pi x}{L},\qquad Q(x)=f_{q1}\cos\frac{\pi x}{L}. \tag{21}$$

The equations (19) and (20) are approximately solved by means of the Bubnov-Galerkin method. After simply transformations of this equations and equation (12), the deflection is obtained

$$w\left(\frac{L}{2}\right)=\frac{1}{a_{11}}\left[a_{12}\frac{L}{\pi}\psi_{11}+a_{13}\frac{L}{\pi}\frac{t_c^2}{b_{02}}\left(b_f\frac{f_{q1}}{bt_c^3}-b_{11}\frac{\pi^2}{L^2}\psi_{11}-\frac{b_{01}}{t_c^2}\psi_{11}\right)+\frac{2FL^3}{96bt_c^3}\right]. \tag{22}$$

The examples of deflections for the middle of the beam depending on Young modulus and thickness of the binding layers are shown in Table 1. The parameters of the beam are: $t_f = 1mm$, $H = 20mm$, $E_f = 65600MPa$, $E_c = 1200MPa$, $L = 100mm$, $b = 50mm$, $\nu_c = \nu_b = 0.3$, $F_1 = 1kN$.

$t_b[mm]$ \ $E_b[MPa]$	50	100	500	1000	1500
0.1	0.07537	0.07156	0.06842	0.06801	0.06787
0.2	0.08264	0.07511	0.06887	0.06806	0.06776
0.3	0.08979	0.07866	0.06928	0.06811	0.06766
0.4	0.09685	0.08220	0.07003	0.06816	0.06756
0.5	0.10383	0.08573	0.07036	0.06820	0.06746

Table 1. Deflections of a beam.

3 Numerical analysis

The numerical model FEM of the five layer sandwich beam was built with use of 3D brick elements for the core and two binding layers. The faces was modelled with the use of 2D shell elements. Between particular layers the tie conditions have been imposed. Because of the symmetry of a model only the quarter of the beam has been modelled.

Figure 3 shows a model and graphical visualisation of the distribution of the deflections of a beam.

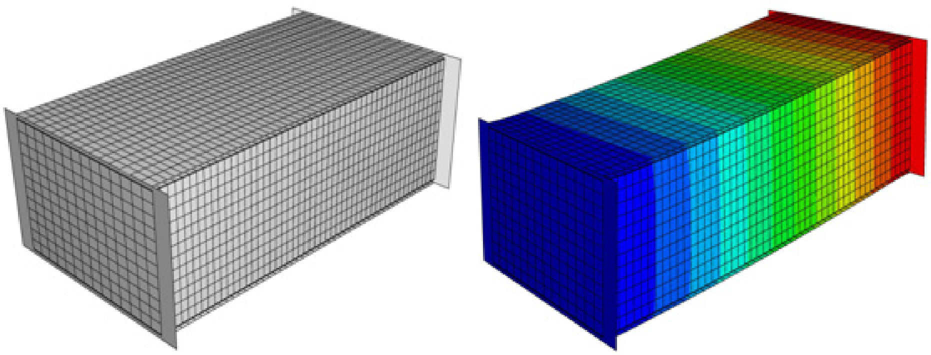

Figure 3. Numerical model of the beam.

The static analysis has been performed and the deflections has been obtained. The dimensions and the material properties were the same as in the analytical analysis. The comparison of the results obtained in the analytical and numerical (FEM) analysis is shown in Figure 4.

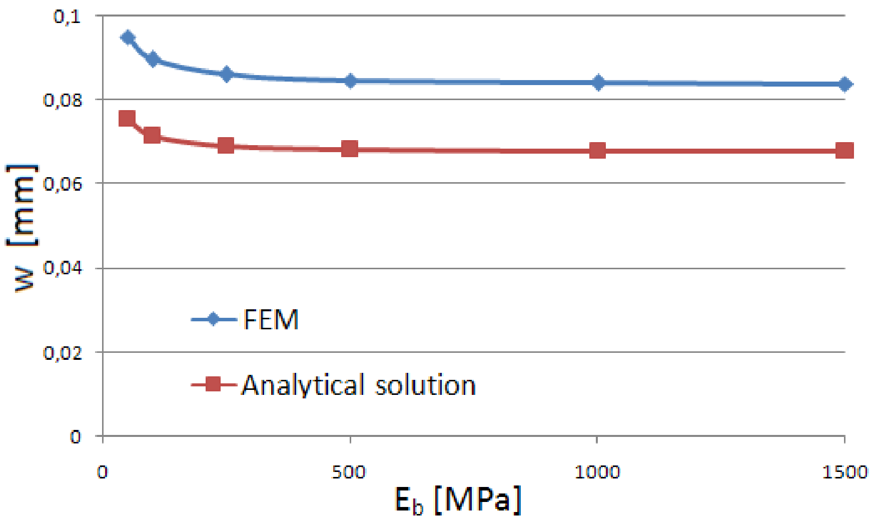

Figure 4 – The comparison of the results obtained analytically and numerically

The difference between results obtained numerically and analytically is about 20%.

4 Conclusions

In this paper a mathematical model of a five layer beam was presented. The faces are glued to the core with thin binding layers. The glue is treated as a separate layer. The influence of the binding layer thickness and properties on the deflection of the beam under bending was analysed. The results obtained from the FE analysis have been compared with those given by the analytical model proposed in the paper. A poor agreement can be seen between these two analyses (20%). Probably it is because of a too large approximation of the functions (21). An increase of degrees of freedom for these functions should reduce the difference to a few percent.

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