Abstract

The subject of this paper is an analysis of deflection of a five layer sandwich beam. The mechanical and physical properties vary through the thickness of the beam and depend on the material of each layer. Two faces of the beam are thin aluminium sheets and the core is made of aluminium foam. Between the faces and the core there are two thin binding glue layers. The main goal of the paper is to present a mathematical model of the five layer beam and to compare the results of the analyses obtained analytically and numerically.

Keywords: sandwich structure, deflection, metal foam, mathematical modelling, five layer beam, numerical model.

1 Introduction


A simply supported five layer sandwich beam of the length $L$ and the width $b$ carries a concentrated force $F$ as shown in Figure 1. The force $F$ is located in the middle of the beam.

![Figure 1. Scheme of the loaded beam.](image)

### 2 Analytical analysis

The deformation of the flat cross section of the five layer beam is shown in Figure 2.

![Figure 2. Scheme of displacements – the hypothesis for the beam.](image)

The field of displacements is formulated as follows:

1. the upper face $-(1/2 + x_1 + x_2) \leq \zeta \leq -(1/2 + x_1)$
\[ u(x, \zeta) = -t_e \left[ \zeta \frac{dw}{dx} + \psi_1(x) \right], \quad (1) \]

2. the upper binding layer \(- (1/2 + x_1) \leq \zeta \leq -1/2\)

\[ u(x, \zeta) = -t_e \left[ \zeta \frac{dw}{dx} + \psi_2(x) - \frac{1}{x_1} \left( \zeta - \frac{1}{2} \right) (\psi_1(x) - \psi_2(x)) \right], \quad (2) \]

3. the core \(-1/2 \leq \zeta \leq 1/2\)

\[ u(x, \zeta) = -t_e \zeta \left[ \frac{dw}{dx} - 2\psi_2(x) \right], \quad (3) \]

4. the lower binding layer \(1/2 \leq \zeta \leq 1/2 + x_1\)

\[ u(x, \zeta) = -t_e \left[ \zeta \frac{dw}{dx} - \psi_2(x) - \frac{1}{x_1} \left( \zeta - \frac{1}{2} \right) (\psi_1(x) - \psi_2(x)) \right], \quad (4) \]

5. the lower face \(1/2 + x_1 \leq \zeta \leq 1/2 + x_1 + x_2\)

\[ u(x, \zeta) = -t_e \left[ \zeta \frac{dw}{dx} - \psi_1(x) \right], \quad (5) \]

where:
\( x_1 = t_b / t_e, \quad x_2 = t_f / t_e, \quad \zeta = z / t_e, \quad \psi_1(x) = u_1(x) / t_e, \quad \psi_2(x) = u_2(x) / t_e. \)

Strains of the layers of the beam are defined by the geometric relationship in the following form:

1. the upper face
\[ \varepsilon_x = -t_e \left[ \zeta \frac{d^2w}{dx^2} + \frac{d\psi_1(x)}{dx} \right], \quad \gamma_{xz} = 0, \quad (6) \]

2. the upper binding layer
\[ \varepsilon_x = -t_e \left[ \zeta \frac{d^2w}{dx^2} + \frac{\psi_2(x)}{dx} - \frac{1}{x_1} \left( \zeta - \frac{1}{2} \right) \left( \frac{d\psi_1(x)}{dx} - \frac{d\psi_2(x)}{dx} \right) \right], \quad \gamma_{xz} = \frac{1}{x_1} [\psi_1(x) - \psi_2(x)], \quad (7) \]

3. the core
\[ \varepsilon_x = -t_e \zeta \left[ \frac{d^2w}{dx^2} - 2 \frac{d\psi_2(x)}{dx} \right], \quad \gamma_{xz} = 2\psi_2(x), \quad (8) \]

4. the lower binding layer
\[ \varepsilon_x = -t_e \zeta \left[ \frac{d^2w}{dx^2} - \frac{\psi_2(x)}{dx} - \frac{1}{x_1} \left( \zeta - \frac{1}{2} \right) \left( \frac{d\psi_1(x)}{dx} - \frac{d\psi_2(x)}{dx} \right) \right], \]
\[
\gamma_{sz} = \frac{1}{x_1} \left[ \psi_1(x) - \psi_2(x) \right],
\]

5. the lower face

\[
\varepsilon_s = -t_e \left[ \xi \frac{d^2w}{dx^2} - \frac{d\psi_1(x)}{dx} \right], \quad \gamma_{sz} = 0,
\]

The physical relationships, according to Hooke’s law, for individual layers are

\[
\sigma_s = E\varepsilon_s, \quad \tau_{sz} = G\gamma_{sz}.
\]

The bending moment of any cross section of the beam

\[
M_b(x) = \int_A \sigma_s zdA = -bt_1^2 \left\{ \left( 2E_f c_{z_f} + 2E_b c_{z_b} + \frac{1}{12} E_c \right) \frac{d^2w}{dx^2} - \left[ E_f c_{z_f} + E_b \frac{x_1}{6}(3 + 4x_1) \right] \frac{d\psi_1}{dx} - \left[ \frac{1}{6} E_c + \frac{1}{6} E_b x_1(3 + 2x_1) \right] \frac{d\psi_2}{dx} \right\},
\]

where:

\[c_{z_b} = x_1(1 + x_1), \quad c_{z_b} = \frac{1}{12} x_1(3 + 6x_1 + 4x_1^2), \quad c_{z_f} = x_2(1 + 2x_1 + x_2), \]

\[c_{z_f} = \frac{1}{12} x_2 \left[ 12x_1(1 + x_1 + x_2) + 3 + 6x_2 + 4x_2^2 \right].\]

The transverse force of any cross section of the beam

\[
Q(x) = \int_A \tau_{sz} dA = 2bt_1 \left[ G_b \psi_1(x) + (G_c - G_b) \psi_2(x) \right],
\]

where:

\[G_c = \frac{E_c}{2(1 + \nu_c)}, \quad G_b = \frac{E_b}{2(1 + \nu_b)}.\]

### 2.1 Equations of equilibrium

The potential energy of the elastic strain of the beam is

\[
U_e = \frac{1}{2} \int_r \left( \varepsilon_s \sigma_s + \gamma_{sz} \tau_{sz} \right) dV = \frac{1}{2} bt_1 \left[ \int_0^l \left( f_{E_f} + f_{E_b} + f_{E_c} \right) dx \right],
\]

where:

\[f_{E_f} = 2E_f t_1^2 \left[ c_{z_f} \left( \frac{d^2w}{dx^2} \right)^2 - c_{z_f} \frac{d^2w}{dx^2} \frac{d\psi_1}{dx} + x_2 \left( \frac{d\psi_1}{dx} \right)^2 \right],\]
The work of the external load is
\[ W = \int_0^l qwdx + \frac{1}{2} F_0 \int_0^l \left( \frac{dw}{dx} \right)^2 dx. \] (15)

The system of three partial differential equations obtained from the principle of stationary total potential energy \( \delta (U_e - W) = 0 \), after integrating over the thickness of the beam and integrating by parts over the length of the beam, takes the following form:

\[ \delta w \]
\[ b t_c \left( 2 E_f c_{z_f} + 2 E_b c_{z_b} + \frac{1}{12} E_c \right) \frac{d^4 w}{dx^4} - \left[ E_f c_{1_f} + \frac{1}{6} E_b x_1 (3 + 4 x_1) \right] \frac{d^2 \psi_1}{dx^2} + \]
\[ -\frac{1}{6} \left[ E_c + E_b x_1 (3 + 2 x_1) \right] \frac{d^3 \psi_2}{dx^3} = q - F_0 \frac{d^2 w}{dx^2}, \] (16)

\[ \delta \psi_1 \]
\[ \left[ E_f c_{1_f} + \frac{1}{6} E_b x_1 (3 + 4 x_1) \right] \frac{d^3 w}{dx^3} - 2 \left( E_f x_2 + \frac{1}{3} E_b x_1 \right) \frac{d^2 \psi_1}{dx^2} - \frac{1}{3} E_b x_1 \frac{d^2 \psi_2}{dx^2} + \]
\[ + \frac{2 G_b}{x_1 t_c} [\psi_1(x) - \psi_2(x)] = 0, \] (17)

\[ \delta \psi_2 \]
\[ \frac{1}{6} \left[ E_c + E_b x_1 (3 + 2 x_1) \right] \frac{d^3 w}{dx^3} - \frac{1}{3} E_b x_1 \frac{d^2 \psi_1}{dx^2} - \frac{1}{3} (E_c + 2 E_b x_1) \frac{d^2 \psi_2}{dx^2} + \]
\[ - \frac{2 G_b}{x_1 t_c} \psi_1(x) + \frac{1}{t_c} \left( 4 G_c + \frac{2 G_b}{x_1} \right) \psi_2(x) = 0. \] (18)

The first equation (16) of the system is equivalent to the bending moment (12). Therefore, for further analysis purpose the system of three Eqs. (12), (17) and (18) is applied.

### 2.2 Deflection of a beam

The simply supported sandwich beam is loaded by force \( F \). The bending moment for this load case is written in the form \( M_e(x) = \frac{1}{2} F x \). After simply transformations of three equations (12), (17) and (18), two equations are obtained
\[ \frac{d^4 \psi_1}{dx^4} + k_2^2 \frac{d^2 \psi_1}{dx^2} - k_1 \psi_1(x) = k_0 Q(x), \]  

(19)

\[ \psi_2(x) = \frac{t^2}{b_{02}} \left[ b_j \frac{Q(x)}{bt_c^2} + b_{01} \frac{d^2 \psi_1}{dx^2} - t^2_c \psi_1(x) \right], \]  

(20)

where:

\[ k_2^2 = -\frac{B}{A}, \quad k_1 = -\frac{C}{A}, \quad k_0 = \frac{D}{A}, \quad A = \frac{a_{12}a_{33} - a_{11}a_{23} b_{11}}{a_{11}a_{12}} t_c^2, \]

\[ B = \frac{a_{12}a_{33} - a_{11}a_{23}}{a_{11}a_{12}} + \frac{a_{12}a_{23} - a_{11}a_{33}}{a_{11}a_{12}} b_{01} + \frac{2G_b}{b_{02} a_{12} x_1}, \quad C = \frac{b_{01}}{b_{02}} \frac{2G_b}{a_{12} x_c}, \]

\[ D = \frac{1}{b_{t_c}^2} \left( \frac{1}{b_{01}} + \frac{b_f}{b_{02}} 2G_b \right), \quad b_{01} = \frac{a_{11}a_{12} - a_{11} a_{13} a_{33} - a_{12} a_{13} a_{23} - a_{13} a_{13} a_{33}}{2G_b}, \]

\[ b_{02} = \frac{a_{11}a_{12} - a_{11} a_{13} a_{33} - a_{12} a_{13} a_{23} - a_{13} a_{13} a_{33}}{2G_b} + \frac{a_{11}}{a_{11} a_{33} - a_{13} a_{33}}, \]

\[ b_j = \frac{a_{11}a_{12} - a_{11} a_{13} a_{33} - a_{12} a_{13} a_{23} - a_{13} a_{13} a_{33}}{2G_b}, \quad a_{11} = 2E_c c_{2f} + 2E_b c_{2b} + \frac{1}{12} E_c, \]

\[ a_{12} = E_f c_{1f} + \frac{1}{6} E_b x_1 (3 + 4x_1), \quad a_{13} = \frac{1}{6} [E_c + E_b x_1 (3 + 2x_1)], \quad a_{21} = a_{12}, \]

\[ a_{22} = 2 \left( E_f x_2 + \frac{1}{3} E_b x_1 \right), \quad a_{33} = \frac{1}{3} E_c x_1, \quad a_{31} = a_{13}, \quad a_{32} = a_{23}, \]

\[ a_{33} = \frac{1}{3} (E_c + 2E_b x_1). \]

The two unknown functions are assumed forms of the first word of the Fourier series

\[ \psi_1(x) = \psi_{11} \cos \frac{\pi x}{L}, \quad Q(x) = f_{q1} \cos \frac{\pi x}{L}. \]  

(21)

The equations (19) and (20) are approximately solved by means of the Bubnov-Galerkin method. After simply transformations of this equations and equation (12), the deflection is obtained

\[ w \left( \frac{L}{2} \right) = \frac{1}{a_{11}} \left[ a_{12} \frac{L}{\pi} \psi_{11} + a_{13} \frac{L}{\pi} t_c^2 b_j \frac{f_{q1}}{bt_c^2} - b_{01} \frac{L^2}{L} \psi_{11} - \frac{b_{01}}{t_c^2} \psi_{11} \right] + \frac{2FL^3}{96bt_c^3}. \]  

(22)

The examples of deflections for the middle of the beam depending on Young modulus and thickness of the binding layers are shown in Table 1. The parameters
The dimensions and the material properties were the same as in the analytical analysis. The comparison of the results obtained in the analytical and numerical (FEM) analysis is shown in Figure 4.
The difference between results obtained numerically and analytically is about 20%.

4 Conclusions

In this paper a mathematical model of a five layer beam was presented. The faces are glued to the core with thin binding layers. The glue is treated as a separate layer. The influence of the binding layer thickness and properties on the deflection of the beam under bending was analysed. The results obtained from the FE analysis have been compared with those given by the analytical model proposed in the paper. A poor agreement can be seen between these two analyses (20%). Probably it is because of a too large approximation of the functions (21). An increase of degrees of freedom for these functions should reduce the difference to a few percent.

Acknowledgements

The studies are supported by the Ministry of Sciences and Higher Education in Poland – Grant No. DS-MK 21-388/2011.

References