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Reliability-Based Design Optimization for the Analysis of Vibro-Acoustic Problems

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Abstract

In this paper, we focus on the interaction fluid structure and specifically the vibro-acoustic coupling which is generally defined as the contact between bodies interacting according to the principles of continuum mechanics. For the coupling fluid structure finite elements models, the importance of the size reduction becomes obvious because the fluid freedom degrees will be added to those of the structure. A method of condensation will be used to reduce the size of the matrices. A numerical vibratory study is leaded on a three-dimensional structure immersed in water taking the acoustic aspect. In this context, we focused very specifically on a deterministic, stochastic and reliability analysis through numerical simulations in three-dimensional dynamic fluid-structure interaction problems.

Keywords: fluid-structure interaction, vibro-acoustic, numerical simulation, finite element method, reliability based design optimization.

1 Introduction

The comprehension of the mechanisms of interactions between a fluid and an elastic solid has a capital importance in several industrial applications. When a structure vibrates in the presence of a fluid, there is interaction between the natural waves of each media: the fluid flow generates a structural deformation and/or the movement of a solid causes the displacement of the fluid. These applications require an effective coupling. In [3, 5], we find many methods to resolve fluid-structure interaction problems. Furthermore, the dynamic analysis of the industrial systems is often expensive from the numerical (CPU) point of view. For the coupling fluid structure finite elements models, the importance of the size reduction becomes obvious because the fluid freedom degrees will be added to those of the structure.

One of the main hypothesizes in the study of mechanical systems is that the model is deterministic. That means that the parameters used in the model are constant. However the experimental works show the limitations of such assumption. This is because there are always differences between what we calculate and what we measure due mainly to the uncertainties in geometry, the material properties, the boundary conditions or the load, which has a considerable impact on the vibrating behavior of mechanical systems. This is why it is important to use numerical methods in order to take these uncertainties into count. In [6, 11], we find many approaches to treat mechanical systems with uncertain parameters.

We presented in this paper a stochastic numerical modal analysis of a solid 3D immersed in water to simulate the stochastic response, in medium frequencies in considering the random parameters. In the case of this problem, the presence of several parameters to random characters namely the Young's modulus of the structure and structure density and fluid density, which often show a great variability, which inevitably leads to a loss precision important. Better control of these parameters is thus based on the use of stochastic methods whose main objective is to improve the quality and the reinterpretation of results from simulations. To do this, a good understanding and formulation of the main phenomena involved in the coupling problem are needed.

The numerical approach has been to propose a finite element model of the structure coupled with the fluid and has validated the use of a general computer code for numerical modeling of problems coupled fluid / structure. The method is illustrated by an example of a solid 3D immersed in water with properties of both materials are random.

2 Fluid structure interaction

2.1 Introduction

In the context of this study of fluid-structure interaction, we focus on the vibro-acoustic problem where the fluid makes an elastic potential energy in contact with an elastic structure. The two media have their own degrees of freedom, and the coupled dynamic system is governed by the vibratory equations of the structure and the fluid coupled with each other. The numerical results are deduced from a finite element approach of the coupled problem with a non symmetric pressure/displacement formulation. These numerical techniques are based on a finite element discretization for solving the equations of problems fluid / structure interaction [12], these methods are applicable to general computer codes.

2.2 Modeling

The modelization of our problem is carried out by using a non-symmetric formulation: a displacement and a pressure (u, p) which presents the advantage of being easily treated by finite elements methods, because it leads to a representation incorporating only a one unknown by knot. This formulation presents in fact the interest of being easily manipulated from a IT (information technology) perspective. Moreover, the finite element codes which allow us to generate stiffness and mass matrices of coupled systems treat this type of formulation in a particularly efficient way, especially from a matrices conditioning.

2.3 Problem statement

We consider here the assumption of small perturbations and it is assumed that the structure is elastic it's characterized by the mechanical properties of materials that are Young's modulus E, the density ρ_s and the Poisson's ratio ν. The structure is immersed in fluid which is supposed to be perfect, homogeneous, linear and at rest (stagnant fluid), it's characterized by its density ρ_f and its sonic velocity c.

The structure occupies the area Ω_s, of Σ_s boarder, it's free from any exterior effort and blocked from one side Γ_s . The fluid that occupies the field Ω_f of border Σ_f. They are coupled through the interface noted $\Sigma = \Sigma_s \cap \Sigma_f$. n^s and n^f are respectively, the exterior normal to a solid area Ω_s and the exterior normal to a fluid areaΩ_f . The problem of the fluid/structure interaction is thus to resolve two problems simultaneously: The first problem concerns the structure which undergoes a pressure imposed by the fluid in the boarderΣ. The second one concerns the fluid which undergoes a field displacement u imposed by Σ interface.

With the previous hypotheses, the equations of the vibro-acoustic problem governing the movement of the coupled system in function of displacement u of the structure and the pressure p of fluid are:

$$\sigma_{ij,j}(u) + \omega^2 \rho_s u = 0 \qquad \text{in} \qquad \Omega_s. \tag{1}$$

$$\Delta p + \frac{\omega^2}{c^2} p = 0 \qquad \text{in} \qquad \Omega_f. \tag{2}$$

$$u = 0 \qquad \text{on} \qquad \Gamma_s. \tag{3}$$

$$\sigma_{ij}(u) n_j^s = p n_i^f \qquad \text{on} \qquad \Sigma. \tag{4}$$

$$\frac{\partial p}{\partial n^f} = \omega^2 \rho_f u n^f \qquad \text{on} \qquad \Sigma. \tag{5}$$

The angular frequency of vibration is denoted as ω. The linearized strain tensor is denoted as ϵ_{ij} and the corresponding stress tensor is denoted as σ_{ij}.

2.4 Variational formulation

By introducing the spaces of functions-test "sufficiently" regular and independent of time $C_u^* = \{u|u = 0 \text{ on } \Gamma_s\}$ et C_p^*, the variational formulation of coupled problem fluid-structure is to find $u \in C_u^*$ et $p \in C_p^*$ such as $\forall v \in C_u^*$ et $\forall q \in C_p^*$:
Taking into account the limits conditions (4) the variational formulation of the structure is obtained by writing for each field of virtual displacement v, c and a.

$$\int_{\Omega_s} \sigma_{ij}(u).\varepsilon_{ij}(v)dV - \omega^2 \int_{\Omega_s} \rho_s.u_i v_i dV = \int_{\Sigma} p.n_i.v_i.d\Sigma \qquad \forall v \tag{6}$$

The variational formulation is obtained for the field of pressure p by using the limits conditions (5) and what ever the virtual field pressure q statically admissible:

$$\int_{V_f} \frac{\partial p}{\partial x_i}.\frac{\partial q}{\partial x_i} dV - \omega^2 \int_{V_f} \frac{1}{c^2}.p.qdV = \omega^2.\rho_f \int_{\Sigma} u_i.n_i q.d\Sigma \qquad \forall q \tag{7}$$

The variational formulation of the system is the sum of the two variational equations (6) and (7):

$$\begin{aligned} F(u,p) &= \frac{1}{2}\int_{\Omega_s} \left(\sigma_{ij}(u)\varepsilon_{ij}(u) - \rho_s.\omega^2.(u,u)\right) dV - \int_{\Sigma} p.u.d\Sigma \\ &- \frac{1}{2\rho_f.\omega^2}.\int_{V_f} \left[\left(\frac{\partial p}{\partial x_i}, \frac{\partial p}{\partial x_i}\right) + k^2.p^2\right] dV \end{aligned} \tag{8}$$

2.5 Finite elements approximation

The interaction of the fluid and the structure at a mesh interface causes the acoustic pressure to exert a force applied to the structure and the structural motions produce an effective "fluid load." The governing finite element matrix equations then become:

- For the structure

$$[\mathbf{M}]\{\ddot{\mathbf{u}}\} + [\mathbf{K}]\{\mathbf{u}\} = [\mathbf{L}]\{\mathbf{P}\} \tag{9}$$

- For the fluid:

$$[\mathbf{E}]\{\ddot{\mathbf{P}}\} + [\mathbf{H}]\{\mathbf{P}\} = -\rho[\mathbf{L}]^{\mathbf{t}}\{\ddot{\mathbf{u}}\} \tag{10}$$

$[L]$ is a "coupling" matrix that represents the effective surface area associated with each node on the fluid-structure interface (FSI). Both the structural and fluid load quantities that are produced at the fluid-structure interface are functions of unknown nodal degrees of freedom. Placing these unknown "load" quantities on the left hand side of the equations and combining the two equations into a single equation produces the following:

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \rho_{\mathbf{f}}\mathbf{L}^{\mathbf{t}} & \mathbf{E} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{P}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & -\mathbf{L} \\ \mathbf{0} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{11}$$

we can still write as follows:

$$\left(-\omega^2\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \rho_\mathbf{f}\mathbf{L^t} & \mathbf{E}\end{bmatrix}+\begin{bmatrix}\mathbf{K} & -\mathbf{L}\\ \mathbf{0} & \mathbf{H}\end{bmatrix}\right)\begin{Bmatrix}\mathbf{u}(\omega)\\ \mathbf{p}(\omega)\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\end{Bmatrix} \tag{12}$$

The foregoing equation implies that nodes on a fluid-structure interface have both displacement and pressure degrees of freedom. The numerical technics based on a discretization of the type of finite elements allow us to resolve the equations of the problem of fluid/structure interaction (12), this methods is applicable with the codes of generalist calculation. In our work we were interested in the validation of the code of ANSYS calculation by implementing coupled calculation in elementary cases. We follow in the present article this measure of validation by proposing a comparison of the results within the framework of a numerical and experimental modal analysis of submerged structures. To determine the eigenfrequencies of the coupled system, the matrix must be symmetrical which is not the case. Therefore a symmetrization procedure such as Irons method will be used.

2.6 Condensation of coupled system

This method is a natural extension of the modal superposition method, largely used in dynamic study of structures. We start from the assumption that physique displacements can be described by the superposition of the first ($\mathbf{n_s}$) dry structure natural modes ($\{\mathbf{P}\} = \mathbf{0}$), and on the other hand that the fluid pressure field can be described by the superposition of the first pure ($\mathbf{n_f}$) acoustic modes ($\{\mathbf{u}\} = \{\ddot{\mathbf{u}}\} = \mathbf{0}$), such as:

$$\begin{aligned}\{\mathbf{u}\} &= [\phi_\mathbf{s}]\{\mathbf{q}\}\\ \{\mathbf{p}\} &= [\psi_\mathbf{f}]\{\mathbf{k}\}\end{aligned} \tag{13}$$

where:

- $[\phi_\mathbf{s}]$ is the matrix of the n_s dry structure first natural modes,
- $\{\mathbf{q}\}$ is the vector of modal displacements,
- $[\psi_\mathbf{f}]$ is the matrix of the nf acoustic first natural modes,
- $\{\mathbf{k}\}$ is the vector of modal pressures.

Usually, this method is used in the case of interaction between elastic structures and light fluids (gas), where the effects of added mass are not significant. In the case of interaction with dense fluids (liquids) where the dynamic characteristics are strongly modified by the added mass contribution, this method proves less powerful, because the boundary condition of fluid-structure interface given by equation (5), is not respected there. To solve this problem, we can superposed at the vectorial bases $[\phi_\mathbf{s}]$ and $[\psi_\mathbf{f}]$ a whole complementary Ritz vectors ($[\psi_\mathbf{r}]$) which will be used to assure the speeds continuity on $\mathbf{C}$ and consequently, to better representing fluid-structure interface condition. $[\psi_{\mathbf{r}}]$ is such as :

$$\{\mathbf{P}\} = [\psi_{\mathbf{f}}]\{\mathbf{k}\} + [\psi_{\mathbf{r}}]\{\mathbf{r}\} \tag{14}$$

where:

$$[\psi_{\mathbf{r}}] = \rho_{\mathbf{f}}[\mathbf{H}]^{-\mathbf{1}}[\mathbf{L}]^{\mathbf{t}}[\phi_{\mathbf{s}}] \tag{15}$$

3 Reliability analysis

Physical tests or measures show that the mechanical properties, the geometrical characteristics of structure elements or applied loads could be random and follow statistical distributions. Thus leads to define a probabilistic model. In general, random variables give a good representation of structural stochastic parameters. Let $X = (X_1, X_2, \ldots, X_m)^t$ be the random vector of the probabilistic analysis. To preserve the integrity of the structure, the failure mode must be defined and the corresponding limit state function $G(X)$ established. The structure is situated in its safe domain D_s if $\{G(X) > 0\}$ and it is situated in its failure domain D_f if $\{G(X) \leq 0\}$. Then, the failure probability is:

$$\mathbf{P_f} = Prob(\mathbf{G}(\mathbf{X}) \leq \mathbf{0}) \tag{16}$$

Our purpose is the reliability analysis of a structure where a frictionless contact occurs between the two solids. In this situation, the analytical expressions of the limit state function G and its derivatives are often not available in function of the physical random variables $X_1, X_2, \ldots, X_m$. Then, it is only possible to obtain the failure probability under an implicit numerical form. The response surface methods have been widely developed in nonlinear reliability analysis. Several authors have proposed solutions to improve the accuracy of results, to decrease the number of necessary numerical calculations on FEM codes and to increase the robustness of the algorithms. In our nonlinear study, we propose an adaptive surface method coupled with the first order reliability method (FORM) [4, 13]. The sets of design points and the response surfaces are generated in the space of standard Gaussian variables. The scheme of the adaptive process is given as follows:

- $k = 1$, the generated set of points is a central composite design. Its center coordinates are the mean values of random variables. $d^{(1)}$ is a fixed real number and the distance from the central point to a 'corner' in the design is equal to $\sqrt{m d^{(1)}}$. So

$$\left\{ \begin{array}{lcll} u^{(k,1)} & = & (0, 0, \ldots, 0)^T & \\ u^{(k,r)} & = & (0, \ldots, \pm d^{(k)}, \ldots, 0)^T, & r = 2, \ldots, 2m+1 \\ u^{(k,r)} & = & (\pm d^{(k)}, \pm d^{(k)}, \ldots, \pm d^{(k)})^T, & r = 2m+2, \ldots, 2^m + 2m + 1 \end{array} \right. \tag{17}$$

- The response surface $\tilde{h}^{(k)}(u)$ is a second order polynomial with crossed terms:

$$\tilde{h}^{(k)}(u) = a_0 + \sum_{i=1}^{n} a_i u_i + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_i u_j \tag{18}$$

- The polynomial coefficients identification is done by the least square method

$$E^{(k)} = \sum_{r=1}^{p} w_i [\tilde{h}^{(k)}(u^{(k,r)}) - h(u^{(k,r)})]^2 \tag{19}$$

$$\frac{\partial E^{(k)}}{\partial a_i} = 0 \quad i = 0, \ldots, N_h \tag{20}$$

  $p = 2^m + 2m + 1$ and $N_h = (m+1)(m+2)/2$ is the number of coefficients of the function $\tilde{h}^{(k)}(u).w_i = 1$.

- The SQP optimization algorithm is used to compute the reliability index $\beta_{HL}^{(k)}$ and the design point $u^{(k,r)}$, solutions of the following minimization problem:

$$\beta_{HL}^{(k)} = \min \sqrt{u^t.u} \quad \text{subjected to: } \tilde{h}^{(k)} = 0 \tag{21}$$

- $k = k + 1$, generation of a new set of points. Its center is the point $u^{(k-1,r)}$ and the distance from the central point to a "corner" in the design is equal to $\sqrt{m}d^k$ with

$$d^{(k)} = \frac{d^{(k-1)}}{q} \tag{22}$$

  $q > 1$ is a real number which plays the role of a zoom factor.

Repeat $(13) - (17)$ until a test of convergence on $\beta_{HL}^{(k)}$ stops the iterative algorithm. Then the failure probability is evaluated by the first order reliability method

$$P_f \approx \Phi(-\beta_{HL}) \tag{23}$$

$u = (u_1, u_2, \ldots, u_m)^T$ is a realization of the random vector U. $\tilde{h}^{(k)}(u)$ is the approximated limit state function in the space of standard Gaussian variables. U is the image of X by the probabilistic transformation and Φ is the standard normal distribution function. This iterative scheme is particularly efficient. The adaptive central composite designs give a very good representation of the random variables domain. The second order polynomial and the least square method assure a good compromise between the computational effort and the approximation accuracy of the real limit state function $h(u)$. The number of necessary calculations is reasonable and depends on the number of variables. The SQP algorithm is robust and efficient for this application in nonlinear finite-element reliability analysis. For more details, the interested reader can review the different works of [1, 2, 7] and .

4 Numerical results

The reliability analysis methodology used in the numerical example integrate a set of reliability analysis tools(based on FORM and SORM)developed under MATLAB with finite element analysis of vibro-acoustic problem using this programming environment(for the first numerical application) and using the commercial software ANSYS (for the second numerical application). In the numerical applications, the reliability analysis takes into account the deterministic and stochastic analysis. The numerical results are deduced from a finite element approach of the coupled problem with a non symmetric pressure/displacement formulation. In a previous work, we were interested in validating the computer code ANSYS implementing calculations coupled to a simple case for a plate immersed in water [8]. We continue in this article this validation approach by proposing a correlation ANSYS/MATLAB as part of the numerical and experimental modal analysis of a solid 3D immersed in water.

4.1 Deterministic case

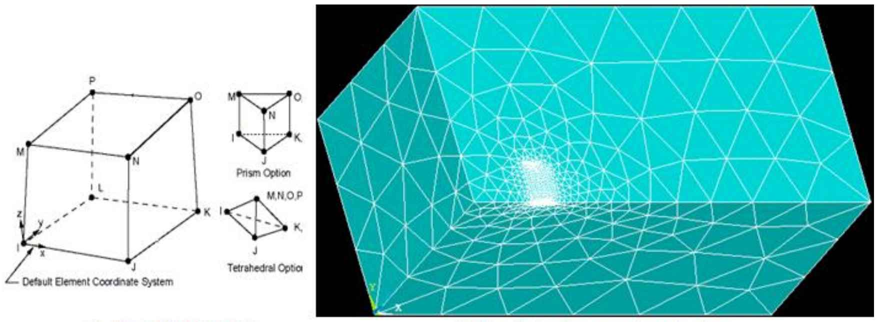
We begin by the validation of our fluid-structure interaction in the deterministic case. The numerical study is simple example considered in this section which consists of a solid 3D coupled with a compressible fluid which was modeled using MATLAB code. This application aims at illustrating the methodology proposed in a deterministic analysis. Geometrical and material properties are:

- For the structure: density = $7860\ kg.m^{-3}$; Young's modulus = 2.1×10^{11} Pa ; Poisson's ratio = 0.3 ; Length= 2 m ; Width = 1 m ; Height = 0.2 m.

- For the fluid: density = $1000\ kg.m^{-3}$; Speed of sound = $1500\ m.s^{-1}$; Length = 20 m ; Width = 10 m; Height = 10 m.

For the finite elements calculation: SOLID45 is used for the 3-D modeling of solid, the element is defined by eight nodes having three degrees of freedom at each node: translations in the nodal x, y, and z directions. FLUID30 is used for modeling the fluid medium and the interface in fluid/structure interaction problems. Typical applications include sound wave propagation and submerged structure dynamics. The governing equation for acoustics, namely the 3-D wave equation, has been discretized taking into account the coupling of acoustic pressure and structural motion at the interface. The element has eight corner nodes with four degrees of freedom per node: translations in the nodal x, y and z directions and pressure. The translations, however, are applicable only at nodes that are on the interface.

This sample problem demonstrates the use of FLUID30 and SOLID45 to predict the acoustic standing wave pattern of a solid submerged in fluid. Figure 1 shows a diagram of the types elements used in this study and the finite elements discretization of this immersed structure.



a) The element type b) The finite elements discretization

Figure 1: Element type and finite elements discretization of this immersed structure.

The founding results in the immersed structure and the comparison results between the ANSYS results and MATLAB one are given in table1. The adopted vibro-acoustic model gives a good results looking the ANSYS one.

Deterministic case	ANSYS	MATLAB
R_1	23.395	24.14
R_2	67.646	67.89
R_3	91.626	90.73
R_4	131.76	131.02
R_5	210.458	212.23

Table 1: The first 5 frequencies of submerged structure (modal synthesis).

There is a substantial drop in natural frequencies of the structure after its immersion in the fluid which changes the vibration behavior of the structure.

4.2 Probabilistic case

The choice of the random variables is all time a central point. This work is the continuity of other works done inside the LMR team. In the three dimensions case, the following variables are taking as random one (see table 2).

Parameters	Means	Standard deviation	Distribution
Young modulus (Pa)	2.1×10^{11}	0.05×10^{11}	Gaussian(μ, σ)
Density of structure (Kg/m^3)	7860	250	Uniform(a,b)
Density of the fluid (Kg/m^3)	1000	40	Uniform(a,b)

Table 2: Moments of the parameters of the problem and Distribution laws.

The stochastic calculation was carried out using probabilistic design system of the ANSYS and MATLAB code. This tool is based on a calculation with Monte Carlo simulation (for 100 samples) and the response surface method (for 40 samples). The table 3 shows means and standard deviations of the natural frequencies and the finding results using the structure immersed.

Modes	Deterministic case	M C	RSM	FORM	SORM	Standard deviation
R_1	23.39	21.21	22.14	22.78	22.78	2.63
R_2	67.64	70.51	68.13	67.04	67.04	3.24
R_3	91.62	90.98	91.01	91.11	91.11	4.15
R_4	131.76	129.23	130.87	131.52	131.52	6.07
R_5	210.45	215.43	211.99	212.62	212.62	10.68

Table 3: Means and standard deviations of the natural frequencies for the immersed structure.

4.3 Reliability Study

The table 4 summarizes the design parameters and their statistical moments considered in the immersed structure for this example. The study of reliability analysis is based on a single state limit function which considers the first natural frequency R_1 coupled systems, such as:
$G(E, \rho_s, \rho_f) = R_1 - R_0$ for immersed structure, In this implicit function of the design variables, $R_0 = 24Hz$. The Table 4 show a comparison among the results obtained from FORM and SORM methods. The results are considered satisfactory and demonstrate the applicability of these techniques in solving reliability problems.

Parameters	FORM	SORM
Young's modulus (Pa)	2.165×10^{11}	$2.165.10^{11}$
Density of the structure (Kg/m^3)	7389.76	7389.76
Density of the fluid (Kg/m^3)	947.93	947.93
Reliability index β	2.83	3.41
Probability $P_f(^0/_0)$	0.92	0.13

Table 4: Design parameters and their statistical moments considered in the immersed structure.

5 Conclusion

In this paper we focus on stochastic simulation of problems of fluid structure interaction. Specifically we have developed stochastic methods for a vibro-acoustic problem.

We have proposed a modal method combined to response surface method for the resolution of the great size stochastic fluid-structure interaction problems. The developed methodology integrates finite element and reliability analysis which enable to assess the reliability index in coupled fluid-structure systems.

The results obtained in the case of a solid three-dimensional coupled to a fluid show the validity and the potential of the proposed method. This study allows a comparison of the numerical results with the FORM and SORM approach, and as such allowed the validation of the method used in the numerical modelling of the dynamic behaviour of an immersed three-dimensional structure. Regarding the performance of analysis tools, it can be concluded that the methodology presented is able to handle implicit limit state functions based on numerical models of fluid-structure interaction problems.

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