Paper 227

©Civil-Comp Press, 2012
Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Formulation of Thin Plate Finite Elements using the Galerkin Boundary Integral Approach

M. Mazza and F. Mazza
Department of Structures
University of Calabria, Rende (Cosenza), Italy

Abstract

A Galerkin boundary integral approach is proposed to obtain a thin plate finite element without the typical drawbacks characterising the traditional finite elements. The usual polynomial interpolations, which ensure displacement continuity but do not always respect the interelement continuity, are replaced by a boundary integral description of the static and kinematic fields. The entries of the stiffness matrix are evaluated by a complex variable integration technique which provides compact analytical results easy to use in a computer code, improving the accuracy of the numerical results.

Keywords: thin plate, finite element, boundary integral equations, complex variables.

1 Introduction

The analysis of complex structural problems by standard numerical models, based on domain discretization, can imply high computation costs due to the large number of involved variables. In some contexts significant savings are possible using boundary elements and renouncing to the flexibility and simplicity which characterize most of finite elements. A further possibility consists of coupling finite elements with boundary elements, also if it compels to interface finite element computer codes with more complicated boundary element codes. Indeed boundary element models can be used in alternative way, directly designing finite elements ready to include in the library of the available computer codes. In this approach the boundary integral description of the domain fields replaces the usual polynomial interpolation of the same fields, avoiding the usual restrictions on number and location of interpolation nodes, element shapes and quality of approximations. So doing, the typical accuracy of the boundary element approximation is combined with the flexibility of the finite element methods, obtaining finite elements with flexible shapes and arbitrary number of nodes. So far this approach has been adopted only for the development of finite elements for plane elasticity problems [1], [2]. In the present work a new thin plate finite element is designed exploiting the Galerkin boundary integral approach as a discretization tool. More specifically the discrete forms of the weighted boundary integral equations associated with static and kinematic sources are used to evaluate the strain energy transferred on the boundary of the finite element. In this case the stiffness matrix descends from some influence matrices whose entries are obtained by the double (single) boundary integration of the product between a fundamental solution and two (one) polynomial shape functions. As the simultaneous presence of high order shape functions and articulated fundamental solutions makes difficult the accurate and efficient evaluation of the above mentioned boundary integrals, an analytical integration technique based on complex variables and specific integration rules is also used to minimize the computational costs [4]. The Gauss transformations, used to deactivate the singularities, allow to reduce the number of types of prime integrals, making easier the entire integration process. After a brief introductive description of the integral formulation used in the symmetric boundary element analysis of thin plates, the paper reports the procedure followed to define the strain energy in terms of boundary variables and to evaluate it by using the boundary integral equations weighted on the boundary of the finite element. The complex variable procedure used to evaluate the integral coefficients is then presented, discussing the main computational tasks which occur in the development of a computer code based on the proposed thin plate finite element.

2 Boundary integral formulations

A thin plate, having flexural rigidity D and defined over a domain Ω, delimited by the boundary Γ, is considered. In the standard boundary integral approch the bending of this structure, subjected to the transversal load p, is usually described by the equation

$$c\, f^{(i)} + \int_\Gamma (t_s^* w + m_s^* \theta - t\, w_s^* - m\, \theta_s^*) d\Gamma + \\ + \sum_{j=1}^{n_c} (R_s^{*\,(j)} w^{(j)} - R^{(j)} w_s^{*\,(j)}) - \int_\Omega p\, w_s^*\, d\Omega = 0 \qquad (1)$$

which defines the generic kynematic field $f \in \{w, \theta\}$ in terms of the static and kynematic boundary fields considering the fundamental solutions marked with an asterisk and associated to a unit point static source $s \in \{F, C\}$. In the above equation w is the transversal displacement, θ the normal slope, m the bending moment, t the equivalent shear and $R^{(j)} = m_t^{(j+)} - m_t^{((j-)}$ the corner reaction at the generic singular point j of the boundary Γ where the superscripts $(j+)$ and $(j-)$ mark, respectively, the sides forward and backward to the corner. The factor c distinguishes sources located on the boundary ($c = 1/2$) from sources inside the domain ($c = 1$).

The same problem can be also described by the boundary integral equation

$$c\,f^{(i)} + \int_\Gamma (t\,w_s^* + m\,\theta_s^* - t_s^* w - m_s^*\theta)d\Gamma + \\ + \sum_{j=1}^{n_c}(R^{(j)} w_s^{*\,(j)} - R_s^{*\,(j)} w^{(j)}) + \int_\Omega p w_s^*\,d\Omega = 0 \tag{2}$$

which defines the generic static field $f \in \{m, q, m_t\}$ in terms of the static and kynematic boundary fields considering the fundamental solutions marked with an asterisk and associated to a unit point kinematic source $s \in \{\Delta\theta, \Delta w, \Delta\theta_t\}$. The same equation (2) allows also the boundary integral description of both the equivalent shear t and the corner reaction $R^{(j)}$: the first one is obtained adding the tangent derivative of the boundary integral equation of twisting moment m_t (equation 2 written for $s = \Delta\theta$) to the boundary integral equation of shear q (equation 2 written for $s = \Delta w$); the second one descends from the boundary integral equation of twisting moment m_t written taking into account point sources $s = \Delta\theta_t^+$ and $s = \Delta\theta_t^-$ applied at the common end of two contiguous sides. Equations (1) or (2) are usually employed to develop boundary element models based on the collocation approach.

The weighted boundary integral equations used in the symmetric boundary element models can be obtained integrating on Γ the boundary integral equations (1) and (2) weighted by unit distributed sources F, C, $-\Delta\theta$, $-\Delta w$ and interpolating sources and boundary fields by identical shape functions according to the Galerkin approach. In the case of sources located on Γ the following discrete equations are obtained

$$\boldsymbol{A}\,\boldsymbol{x} = \boldsymbol{b} \tag{3}$$

with

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{W}_F^* & \boldsymbol{\Theta}_F^* & -\boldsymbol{M}_F^* & -\boldsymbol{T}_F^* & -\boldsymbol{R}_F^{*\,(j)} & \boldsymbol{W}_F^{*\,(j)} \\ \boldsymbol{W}_C^* & \boldsymbol{\Theta}_C^* & -\boldsymbol{M}_C^* & -\boldsymbol{T}_C^* & -\boldsymbol{R}_C^{*\,(j)} & \boldsymbol{W}_C^{*\,(j)} \\ -\boldsymbol{W}_{\Delta\theta}^* & -\boldsymbol{\Theta}_{\Delta\theta}^* & \boldsymbol{M}_{\Delta\theta}^* & \boldsymbol{T}_{\Delta\theta}^* & \boldsymbol{R}_{\Delta\theta}^{*\,(j)} & -\boldsymbol{W}_{\Delta\theta}^{*\,(j)} \\ -\boldsymbol{W}_{\Delta w}^* & -\boldsymbol{\Theta}_{\Delta w}^* & \boldsymbol{M}_{\Delta w}^* & \boldsymbol{T}_{\Delta w}^* & \boldsymbol{R}_{\Delta w}^{*\,(j)} & -\boldsymbol{W}_{\Delta w}^{*\,(j)} \end{bmatrix} \tag{4}$$

$$\boldsymbol{x} = \begin{bmatrix} \bar{\boldsymbol{t}} & \bar{\boldsymbol{m}} & \bar{\boldsymbol{\theta}} & \bar{\boldsymbol{w}} & \bar{\boldsymbol{w}}^{(j)} & \bar{\boldsymbol{R}}^{(j)} \end{bmatrix}^T \tag{5}$$

$$\boldsymbol{b} = \frac{1}{2}\left[\ \boldsymbol{\Psi}_{wt}\,\bar{\boldsymbol{w}} \quad \boldsymbol{\Psi}_{\theta m}\,\bar{\boldsymbol{\theta}} \quad \boldsymbol{\Psi}_{m\theta}\,\bar{\boldsymbol{m}} \quad \boldsymbol{\Psi}_{tw}\,\bar{\boldsymbol{t}}\ \right]^T \tag{6}$$

where the load p has been assumed equal to zero for the sake of brevity. When the boundary also includes singular points the above set of equations (3) has to be completed. To this end the boundary integral equations of transversal displacement and corner reaction at the different corners j, weighted by unit point sources $-\Delta\theta_t^{(i)}$ and $F^{(i)}$ applied at the corner points i, are considered, obtaining the discrete equations

$$\boldsymbol{A}_c\,\boldsymbol{x} = \boldsymbol{b}_c \tag{7}$$

where

$$\boldsymbol{A}_c = \begin{bmatrix} -\boldsymbol{W}^{*\,(i)}_{\Delta\theta_t} & -\boldsymbol{\Theta}^{*\,(i)}_{\Delta\theta_t} & \boldsymbol{M}^{*\,(i)}_{\Delta\theta_t} & \boldsymbol{T}^{*\,(i)}_{\Delta\theta_t} & \boldsymbol{R}^{*\,(ij)}_{\Delta\theta_t} & -\boldsymbol{W}^{*\,(ij)}_{\Delta\theta_t} \\ \boldsymbol{W}^{*\,(i)}_{F} & \boldsymbol{\Theta}^{*\,(i)}_{F} & \boldsymbol{M}^{*\,(i)}_{F} & -\boldsymbol{T}^{*\,(i)}_{F} & -\boldsymbol{R}^{*\,(ij)}_{F} & \boldsymbol{W}^{*\,(ij)}_{F} \end{bmatrix} \tag{8}$$

$$\boldsymbol{b}_c = \frac{1}{2}\begin{bmatrix} \bar{\boldsymbol{R}}^{(j)} & \bar{\boldsymbol{w}}^{(j)} \end{bmatrix}^T \tag{9}$$

The barred symbols appearing in equations (5), (6) and (9) are used to mark vectors collecting the interpolation parameters located at boundary regular points (see $\bar{\boldsymbol{t}}$) and at corner points (see $\bar{\boldsymbol{R}}^{(j)}$). The bold upper-case letters without superscripts (e.g. $\boldsymbol{W}^*_F$) appearing in (4) and (8) represent matrices collecting coefficients deriving from the double boundary integration of the product between two shape functions and a fundamental solution. In a similar way the bold upper-case letters with superscripts (e.g. $\boldsymbol{R}^{*\,(j)}_F$) represent vectors containing coefficients obtained by the single boundary integration of the product between a shape function and a fundamental solution. As for the matrices $\boldsymbol{\Psi}$ of vector (6), they contain the entries deriving from the single boundary integration of two shape functions. For instance some entries corresponding to the h^{th} (source) and k^{th} (field) interpolation parameters are given here

$$\boldsymbol{\Theta}^*_F[h,k] = \int_{\Gamma^{(h)}} \psi_t^{(h)} \int_{\Gamma^{(k)}} \psi_m^{(k)}\,\theta^*_F\;d\Gamma^{(k)}\,d\Gamma^{(h)} \tag{10}$$

$$\boldsymbol{W}^*_C[k,h] = \int_{\Gamma^{(k)}} \psi_m^{(k)} \int_{\Gamma^{(k)}} \psi_t^{(h)}\,w^*_C\;d\Gamma^{(h)}\,d\Gamma^{(k)} = \boldsymbol{\Theta}^*_F[h,k] \tag{11}$$

$$\boldsymbol{R}^{*\,(j)}_F[h] = \int_{\Gamma^{(h)}} \psi_t^{(h)}\,R^{*\,(j)}_F\,d\Gamma^{(h)} \tag{12}$$

$$\boldsymbol{W}^{*\,(i)}_{\Delta\theta_t}[k] = \int_{\Gamma^{(k)}} \psi_t^{(k)}\,w^{*\,(j)}_{\Delta\theta_t}\,d\Gamma^{(k)} \tag{13}$$

$$\boldsymbol{\Psi}_{wt}[h,k] = \int_{\Gamma^{(h)}} \psi_t^{(h)}\psi_w^{(k)}d\Gamma^{(h)} \tag{14}$$

Selecting the appropriate equations within the sets (3) and (7) on the basis of the boundary conditions a symmetric boundary system is obtained thanks to the presence of identical shape functions, chosen according to the Galerkin approach (i.e. $\psi_w = \psi_{\Delta w}$, $\psi_\theta = \psi_{\Delta\theta}$, $\psi_m = \psi_C$, $\psi_t = \psi_F$), and of reciprocal fundamental solutions

$$\theta^*_F = w^*_C \quad , \quad m^*_F = w^*_{\Delta\theta} \quad , \quad t^*_F = w^*_{\Delta w} \quad , \quad m^*_{t\,F} = w^*_{\Delta\theta_t} \tag{15}$$

$$m^*_C = \theta^*_{\Delta\theta} \quad , \quad t^*_C = \theta^*_{\Delta w} \quad , \quad m^*_{t\,C} = \theta^*_{\Delta\theta_t} \tag{16}$$

$$t^*_{\Delta\theta} = m^*_{\Delta w}\,, \quad m^*_{t\,\Delta\theta} = m^*_{\Delta\theta_t} \tag{17}$$

$$m^*_{t\,\Delta w} = t^*_{\Delta\theta_t} \tag{18}$$

3 Stiffness matrix by boundary integral equations

Conventional finite element models are developed by approximating the domain integral which defines the strain energy by shape functions which describe the involved mechanical fields. For thin plates the choice of these polynomial shape functions is quite problematic owing to the high order of continuity required along the interfaces between the finite elements. This trouble can be removed defining the strain energy in terms of boundary fields and then evaluating it by weighted boundary integral equations.

3.1 Strain energy in terms of boundary variables

The domain integral providing the bending strain energy of a thin plate finite element

$$\Phi = \frac{1}{2}\int_\Omega m_{ij}\, w_{,ij}\, d\Omega \tag{19}$$

can be defined on the boundary of the finite element by the Green formula as follows

$$\begin{aligned}\Phi &= \frac{1}{2}\left(\int_\Gamma m_{ij}n_j w_{,i}\, d\Gamma - \int_\Gamma m_{ij,j}n_j w\, d\Gamma + \int_\Omega m_{ij,ij} w\, d\Omega\right)\\ &= \frac{1}{2}\left(\int_\Gamma (m\theta + m_t\theta_t - tw)\, d\Gamma + \int_\Gamma m_{t,t} w\, d\Gamma - \int_\Omega pw\, d\Omega\right)\end{aligned} \tag{20}$$

Integrating by parts the second term on the right-hand side of the above equation

$$\begin{aligned}\int_\Gamma m_{t,t} w\, d\Gamma &= \sum_{j=1}^{n_c}(m_t^{(j^+)} - m_t^{(j^-)})w^{(j)} - \int_\Gamma m_t w_{,t}\, d\Gamma =\\ &= \sum_{j=1}^{n_c} R^{(j)}w^{(j)} - \int_\Gamma m_t\theta_t d\Gamma\end{aligned} \tag{21}$$

the energy (20) takes the form

$$\Phi = \frac{1}{2}\left(\int_\Gamma (m\,\theta - t\,w)d\Gamma + \sum_{j=1}^{n_c} R^{(j)}w^{(j)} - \int_\Omega pw d\Omega\right) \tag{22}$$

Interpolating the boundary fields by polynomial shape collected in the matrices $\boldsymbol{\Psi}_w$, $\boldsymbol{\Psi}_\theta$, $\boldsymbol{\Psi}_m$, $\boldsymbol{\Psi}_t$ and denoting the vectors of the interpolation parameters by $\bar{\boldsymbol{w}}, \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{m}}, \bar{\boldsymbol{t}}$ at regular boundary points and by $\bar{\boldsymbol{w}}^{(j)}$, $\bar{\boldsymbol{R}}^{(j)}$ at corner points, the above equation becomes

$$\Phi = \frac{1}{2}\int_\Gamma \begin{bmatrix}\bar{\boldsymbol{t}}\\ \bar{\boldsymbol{m}}\\ \bar{\boldsymbol{R}}^{(j)}\end{bmatrix}^T \begin{bmatrix}-\boldsymbol{\Psi}_t^T\boldsymbol{\Psi}_w & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Psi}_m^T\boldsymbol{\Psi}_\theta & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \boldsymbol{I}\end{bmatrix} \begin{bmatrix}\bar{\boldsymbol{w}}\\ \bar{\boldsymbol{\theta}}\\ \bar{\boldsymbol{w}}^{(j)}\end{bmatrix} d\Gamma = \frac{1}{2}\int_\Gamma \boldsymbol{t}^T\boldsymbol{\Psi}_{ut}\boldsymbol{u} \tag{23}$$

where $\boldsymbol{I}$ is the identity matrix and the load p has been assumed equal to zero.

3.2 Strain energy by weighted boundary integral equations

The weighted boundary integral equations defined by (3) and (7) allow to evaluate the stiffness matrix of the thin plate finite element by using the equation (23) instead of the domain integral (19). To achieve this result the sets of equations (3) and (7) are premultiplied respectively for $\boldsymbol{y}^T$ and $\boldsymbol{y}_c^T$ defined as follows

$$\boldsymbol{y} = \left[\begin{array}{cccc} \bar{\boldsymbol{t}} & \bar{\boldsymbol{m}} & -\bar{\boldsymbol{\theta}} & -\bar{\boldsymbol{w}} \end{array} \right] \quad , \quad \boldsymbol{y}_c = \left[\begin{array}{cc} -\bar{\boldsymbol{w}}^{(j)} & \bar{\boldsymbol{R}}^{(j)} \end{array} \right] \tag{24}$$

So doing the following compact formulas are obtained

$$\boldsymbol{t}^T \boldsymbol{G}_{uu} \boldsymbol{t} - \boldsymbol{t}^T \boldsymbol{G}_{ut} \boldsymbol{u} = \boldsymbol{t}^T \boldsymbol{\Psi}_{ut} \boldsymbol{u} \tag{25}$$

$$\boldsymbol{u}^T \boldsymbol{G}_{tu} \boldsymbol{t} - \boldsymbol{u}^T \boldsymbol{G}_{tt} \boldsymbol{u} = \boldsymbol{u}^T \boldsymbol{\Psi}_{tu} \boldsymbol{t} \tag{26}$$

where

$$\boldsymbol{G}_{uu} = \begin{bmatrix} \boldsymbol{W}_F^* & \boldsymbol{\Theta}_F^* & \boldsymbol{W}_F^{*\,(j)} \\ \boldsymbol{W}_C^* & \boldsymbol{\Theta}_C^* & \boldsymbol{W}_C^{*\,(j)} \\ \boldsymbol{W}_F^{*\,(i)} & \boldsymbol{\Theta}_F^{*\,(i)} & \boldsymbol{W}_F^{*\,(ij)} \end{bmatrix} = \boldsymbol{G}_{uu}^T \tag{27}$$

$$\boldsymbol{G}_{tt} = \begin{bmatrix} \boldsymbol{T}_{\Delta w}^* & \boldsymbol{M}_{\Delta w}^* & \boldsymbol{R}_{\Delta w}^{*\,(j)} \\ \boldsymbol{T}_{\Delta \theta}^* & \boldsymbol{M}_{\Delta \theta}^* & \boldsymbol{R}_{\Delta \theta}^{*\,(j)} \\ \boldsymbol{T}_{\Delta \theta_t}^{*\,(i)} & \boldsymbol{M}_{\Delta \theta_t}^{*\,(i)} & \boldsymbol{R}_{\Delta \theta_t}^{*\,(ij)} \end{bmatrix} = \boldsymbol{G}_{tt}^T \tag{28}$$

$$\boldsymbol{G}_{ut} = \begin{bmatrix} \boldsymbol{T}_F^* & \boldsymbol{M}_F^* & \boldsymbol{R}_F^{*\,(j)} \\ \boldsymbol{T}_C^* & \boldsymbol{M}_C^* & \boldsymbol{R}_C^{*\,(j)} \\ \boldsymbol{T}_F^{*\,(i)} & \boldsymbol{M}_F^{*\,(i)} & \boldsymbol{R}_F^{*\,(ij)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_{\Delta w}^* & \boldsymbol{\Theta}_{\Delta w}^* & \boldsymbol{W}_{\Delta w}^{*\,(j)} \\ \boldsymbol{W}_{\Delta \theta}^* & \boldsymbol{\Theta}_{\Delta \theta}^* & \boldsymbol{W}_{\Delta \theta}^{*\,(j)} \\ \boldsymbol{W}_{\Delta \theta_t}^{*\,(i)} & \boldsymbol{\Theta}_{\Delta \theta_t}^{*\,(i)} & \boldsymbol{W}_{\Delta \theta_t}^{*\,(ij)} \end{bmatrix} = \boldsymbol{G}_{tu}^T \tag{29}$$

$$\boldsymbol{\Psi}_{ut} = \begin{bmatrix} \boldsymbol{\Psi}_{wt} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Psi}_{\theta m} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \qquad \boldsymbol{\Psi}_{tu} = \begin{bmatrix} \boldsymbol{\Psi}_{m\theta} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Psi}_{tw} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{30}$$

Adding member to member equations (25) and (26) the entries of $-\boldsymbol{t}^T \boldsymbol{G}_{ut}\boldsymbol{u}$ and $\boldsymbol{u}^T \boldsymbol{G}_{tu}\boldsymbol{t}$ cancel out each other to the presence of reciprocal fundamental solutions, identical shape functions and identical interpolation parameters. The above mentioned algebraic manipulation leads to the formula

$$\boldsymbol{t}^T \boldsymbol{G}_{uu} \boldsymbol{t} - \boldsymbol{u}^T \boldsymbol{G}_{tt} \boldsymbol{u} = 2\, \boldsymbol{t}^T \boldsymbol{\Psi}_{ut} \boldsymbol{u} \tag{31}$$

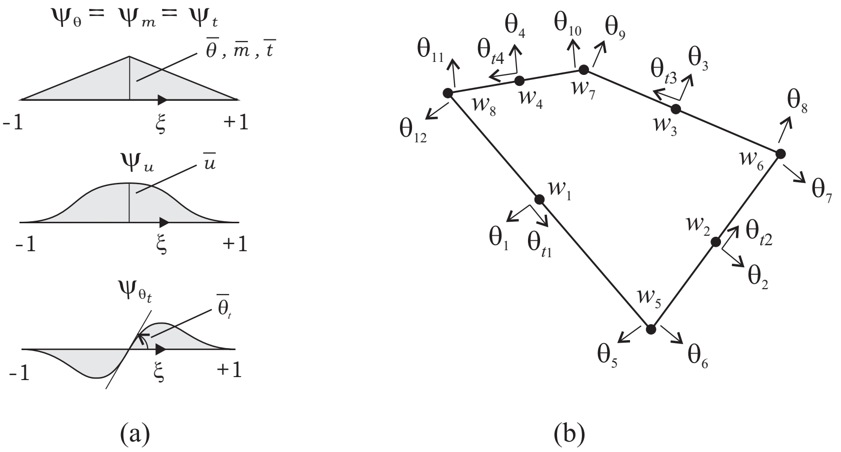
whose right-hand side is twice the bending strain energy (23). Expressing now $\boldsymbol{t}$ in terms of $\boldsymbol{u}$ by means of the equation (25)

$$\boldsymbol{t} = \boldsymbol{G}_{uu}^{-1}(\boldsymbol{G}_{ut} + \boldsymbol{\Psi}_{ut})\boldsymbol{u} \tag{32}$$

and replacing this result in equation (31), the bending strain energy takes the form

$$\Phi = \frac{1}{2}\boldsymbol{u}^T\left((\boldsymbol{G}_{ut} + \boldsymbol{\Psi}_{ut})^T\boldsymbol{G}_{uu}^{-1}(\boldsymbol{G}_{ut} + \boldsymbol{\Psi}_{ut}) - \boldsymbol{G}_{tt}\right)\boldsymbol{u} = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} \tag{33}$$

It is worth noting that the domain integrals of polynomial shape functions, required by the traditional finite element approaches, are replaced by double (single) boundary integrals of products of two (one) shape functions and one fundamental solution and by single boundary integrals of products of two shape functions. This feature allows to develop triangular, quadrilateral or polygonal thin plate finite element in the same way without changing shape functions to ensure the continuity of the mechanical along the interfaces of the finite elements. More specifically the equivalent shear, the bending moment and the normal slope are interpolated by C^0 linear shape functions while the transversal displacement is described combining the contributions of pure transversal displacement and tangent rotation described by two different C^1 cubic shape functions (Figure 1). Each functions is hat-defined on a support made of two contiguous boundary elements. So doing the degree of freedom of each intermediate node of the proposed finite element are u, θ and θ_t while the degree of freedom of each corner node are the normal slopes on the two sides merging at the same corner.



(a) (b)

Figure 1: Shape functions and interpolation parameters.

4 Efficient evaluation of the boundary integrals

The singular behaviours and the articulated expressions of the fundamental solutions involved in the boundary integrals providing the entries of the matrices $\boldsymbol{G}_{uu}$, $\boldsymbol{G}_{tt}$, $\boldsymbol{G}_{ut}$ influence the performances of the proposed thin plate finite element. Standard quadrature rules are usually considered the simplest and efficient tool to evaluate the boundary integrals when the singularity of the fundamental solution is not active. However these techniques become cumbersome for large problems and unreliable in case of singular or nearly singular situations. Analytical integration techniques in the real plane allow these drawbacks to be overcome, saving computer time and obtaining accurate numerical results but paying the penalty for heavy algebraic manipulations in the presence of generically oriented integration domains and high order interpolation functions. Moreover special integration techniques have to be used in order to compute the boundary integrals when the singular behaviour of the fundamental solution is actived [3]. The complex variable integration technique proposed in [4] represents an efficient tool to obtain the exact evaluation of the boundary integrals avoiding all the above mentioned drawbacks.

The complex variable technique can be directly applied without regularizing the kernels when their orders of singularity do not exceed $\mathrm{O}(1/r)$ (see matrix $\boldsymbol{G}_{uu}$). To this end the fundamental solutions are represented in the complex plane defining the components of the unit vectors n and ν, normal to the boundary elements including the field point $x = (x_1, x_2)$ and the source point $\xi = (\xi_1, \xi_2)$, as follows

$$m_1 = \frac{m + \bar{m}}{2} \quad , \quad m_2 = \frac{m - \bar{m}}{2} \qquad m \in \{n, \nu\} \tag{34}$$

and replacing the real expression of the distance $r = |x - \xi|$ and its derivatives r_1, r_2 with the corresponding complex formulas

$$r = |z| \quad , \quad r_1 = \frac{z + \bar{z}}{2|z|} \quad , \quad r_2 = \frac{z - \bar{z}}{2\,i\,|z|} \tag{35}$$

being $z = x + i\xi$ and $\bar{z} = x - i\xi$. At the same time the complex variable description of the shape functions is obtained expressing the local abscissa ℓ and λ by the formulas

$$\begin{aligned} \ell &= i\,a(b + 2\Re(\bar{\nu}z)) \\ \lambda &= i\,a(c + 2\Re(\bar{n}z)) \end{aligned} \tag{36}$$

being

$$\begin{aligned} a &= 1/\left(\nu\bar{n} - n\bar{\nu}\right) \\ b &= 2\left(\Im(\nu)h_2 - \Re(\nu)h_1\right) \\ c &= 2\left(\Im(n)h_2 - \Re(n)h_1\right) \end{aligned} \quad , \quad \begin{aligned} h_1 &= x_1^A - \xi_1^A \\ h_2 &= x_2^C - \xi_2^C \end{aligned} \tag{37}$$

where (x_1^A, x_2^A) and $(\xi_1^C - \xi_2^C)$ are the global cartesian coordinates of the initial ends A and C of the two boundary elements including the field point and source points. The above procedure allows to rewrite the double and single boundary integrals corresponding to the generic h^{th} and k^{th} interpolation parameters as follows

$$\int_{\Gamma_\lambda^{(h)}} \psi^{(h)}[\lambda] \int_{\Gamma_\ell^{(k)}} \psi^{(k)}[\ell] f^*[\ell,\lambda] d\ell d\lambda = \sum_{i=1}^{m_1} \int_{\Gamma_\lambda^{(h)}} \int_{\Gamma_\ell^{(k)}} 2\Re\{\, g_i[z,\bar{z}] \,\} \, d\ell d\lambda \quad (38)$$

$$\int_{\Gamma_\lambda^{(h)}} \psi^{(h)}[\lambda] f^*[\ell,\lambda] d\lambda = \sum_{i=1}^{m_2} \int_{\Gamma_\lambda^{(h)}} 2\Re\{\, g_i[z,\bar{z}] \,\} \, d\lambda \quad (39)$$

$$\int_{\Gamma_\ell^{(k)}} \psi^{(k)}[\ell] f^*[\ell,\lambda] d\ell = \sum_{i=1}^{m_3} \int_{\Gamma_\ell^{(k)}} 2\Re\{\, g_i[z,\bar{z}] \,\} \, d\ell \quad (40)$$

where $d\ell = dz/in$ and $d\lambda = -dz/i\nu$. Finally the integration of each complex function $g_i[z,\bar{z}]$ on the boundary Γ_ℓ or Γ_λ is performed by using the specific rule

$$\int_\Gamma z^p h[\bar{z}] d\Gamma = \sum_{j=0}^{p} \beta_j \frac{d^j}{dz^j}(z^p) \int_{j+1} h[\bar{z}] d\bar{z} \quad (41)$$

being

$$\beta_j = \begin{cases} i\, n^{2j+1} & \text{for } \Gamma = \Gamma_\ell \\ -i\, \nu^{2j+1} & \text{for } \Gamma = \Gamma_\lambda \end{cases} \quad , \quad \frac{d^0}{dz^0}(z^p) = z^p \quad (42)$$

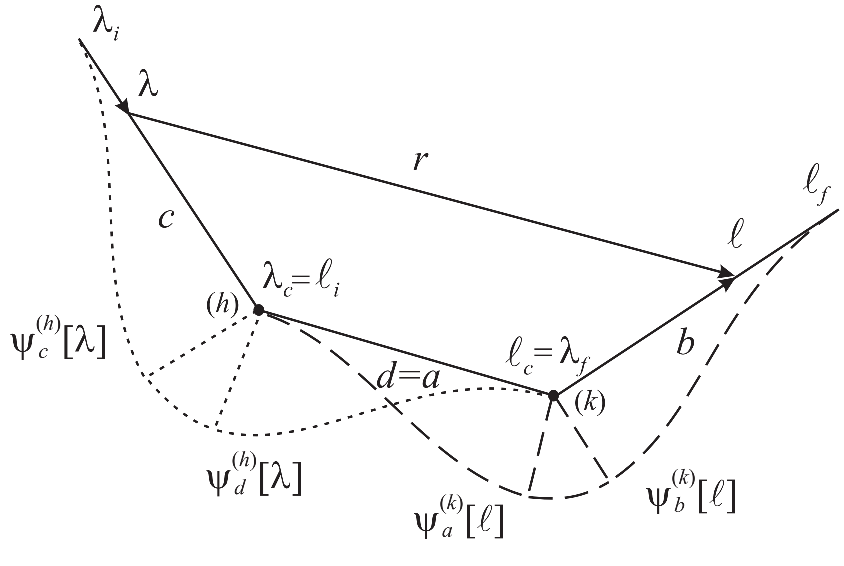


Figure 2: Entry $\boldsymbol{G}_{tt}[u^{(h)}, u^{(k)}]$.

In presence of orders of singularity greater than $\mathrm{O}(1/r)$ (see matrix $\boldsymbol{G}_{tt}$ and some submatrices of $\boldsymbol{G}_{ut}$), the complex variable integration technique has to include a regularization procedure based on integration by parts. As an example, the regularization of the entry $\boldsymbol{G}_{tt}[u^{(h)}, u^{(k)}]$, corresponding to the $u^{(h)}$ and $u^{(k)}$ interpolation parameters of the source and field distributions (Figure 2), is considered below

$$
\begin{aligned}
\boldsymbol{G}_{tt}[u^{(h)}, u^{(k)}] &= \int_{\Gamma_\lambda^{(h)}} \psi_u^{(h)}[\lambda] \int_{\Gamma_\ell^{(k)}} \psi_u^{(k)}[\ell]\, t^*_{\Delta w}\, d\Gamma_\ell^{(k)}\, d\Gamma_\lambda^{(h)} \\
&+ \int_{\Gamma_\lambda^{(h)}} \psi_u^{(h)}[\lambda]\, R^{*\,(k)}_{\Delta w}\, d\Gamma_\lambda^{(h)} + \int_{\Gamma_\ell^{(k)}} \psi_u^{(k)}[\ell]\, t^*_{\Delta\theta_t^{(h)}}\, d\Gamma_\ell^{(k)} + R^{*\,(k)}_{\Delta\theta_t^{(h)}} \quad (43)
\end{aligned}
$$

In this expression $\psi_u[\ell]$ and $\psi_u[\lambda]$ are cubic shape functions, each hat-defined on two contiguous boundary elements forming a support, while the orders of singularity are equal to $\mathrm{O}(1/r^4)$ for $t^*_{\Delta w}$, $\mathrm{O}(1/r^2)$ for $R^*_{\Delta\theta_t}$, $\mathrm{O}(1/r^3)$ for $R^*_{\Delta w}$ and $t^*_{\Delta\theta_t}$.

Equation (43) can be regularized carrying out a different number of integration by parts depending on the fundamental solution taken into account. More specifically four integrations by parts of $t^*_{\Delta w}$, two on the support $\Gamma_\lambda^{(h)} = (c, d)$ of the cubic shape function $\psi_u^{(h)}$ and two on the support $\Gamma_\ell^{(k)} = (a, b)$ of the cubic shape function $\psi_u^{(k)}$ have to be considered while two integration by parts on $\Gamma_\lambda^{(h)}$ and $\Gamma_\ell^{(k)}$ are necessary for the kernels $m^*_{t\,\Delta w}$ and $t^*_{\Delta\theta_t}$, respectively. It is worth noting that the integration by parts of the similar coefficients of the fundamental solutions, marked by equal capital letters

$$
t^*_{\Delta w} = \underbrace{-2\Re\left[\frac{n^3\nu}{z^4}\right](6c_1k_2k_3)}_{A.1}\ \underbrace{-2\Re\left[\frac{n\nu^3}{z^4}\right](6c_1k_2k_3)}_{B.1}\ \underbrace{-2\Re\left[\left(\frac{1}{n^3\nu^3}\right)\frac{z}{\bar{z}^5}\right]\left(24c_1k_2^2\right)}_{C.1} \quad (44)
$$

$$
m^*_{t\,\Delta w} = \underbrace{-2\Re\left[\frac{n^2\nu}{z^3}\right](2\,i\,c_1k_2k_3)}_{A.2}\ \underbrace{+2\Re\left[\left(\frac{1}{n^2\nu^3}\right)\frac{z}{\bar{z}^4}\right]\left(6\,i\,c_1k_2^2\right)}_{C.2} \quad (45)
$$

$$
t^*_{\Delta\theta_t} = \underbrace{+2\Re\left[\frac{n\nu^2}{z^3}\right](2\,i\,c_1k_2k_3)}_{B.2}\ \underbrace{-2\Re\left[\left(\frac{1}{n^3\nu^2}\right)\frac{z}{\bar{z}^4}\right]\left(6\,i\,c_1k_2^2\right)}_{C.3} \quad (46)
$$

$$
m^*_{t\,\Delta\theta_t} = \underbrace{-2\Re\left[\left(\frac{1}{n^2\nu^2}\right)\frac{z}{\bar{z}^3}\right]\left(2c_1k_2^2\right)}_{C.4} \quad (47)
$$

provide singular boundary coefficients which cancel out each other. This result is attained thanks to the zero values of shape functions and their derivatives at the ends of the integration domains and thanks to the continuity of the same functions on the integration domains (Figure 2). In the above expressions n and ν denote the unit vectors normal to the boundary elements of the supports $\Gamma_\ell^{(k)}$ and $\Gamma_\lambda^{(h)}$, z and $\bar{z}$ are the complex and conjugate complex variables connecting source and field points while c_1, k_1, k_2 and k_3 are constant coefficients of the fundamental solutions defined as

$$
c_1 = 1/(16\pi) \quad , \quad k_1 = (1+\mu) \quad , \quad k_2 = (-1+\mu) \quad , \quad k_3 = (-5+\mu) \quad (48)
$$

being μ the Poisson coefficient. After the regularization the entry (43) takes the form

$$\boldsymbol{G}_{tt,\,r}[u^{(h)},u^{(k)}]=\int_{\Gamma^{(h)}_{\lambda}}\frac{d^2\psi_u^{(h)}[\lambda]}{d\lambda^2}\int_{\Gamma^{(k)}_{\ell}}\frac{d^2\psi_u^{(k)}[\ell]}{d\ell^2}\,t^{*\,\ell\ell\lambda\lambda}d\ell d\lambda \tag{49}$$

where the regularized fundamental solution

$$t^{*\,\ell\ell\lambda\lambda}=2\,\Re\,c_1\left[k_2\left(\left(\frac{1}{2}-\frac{3n^2}{2\nu^2}\right)+k_3\left(\frac{n^2}{\nu}+\frac{\nu^2}{n}\right)\right)\ln(z)+k_2\left(\frac{1}{2\nu^2}\right)\frac{z}{\bar{z}}\right] \tag{50}$$

exhibits only a weak singularity $\mathrm{O}(\ln r)$. More specifically the expression (49) becomes more compact than the relationship (43) as the two linear shape functions $d^2\psi_u[\lambda]/d\lambda^2$, $d^2\psi_u[\ell]/d\ell^2$ and the regularized fundamental solution $t^{*\,\ell\ell\lambda\lambda}$ replace the two cubic shape functions $\psi_u[\lambda]$ and $\psi_u[\ell]$ and the four singular fundamental solutions $t^*_{\Delta w}$, $m^*_{t\,\Delta w}$, $t^*_{\Delta\theta_t}$ and $m^*_{t,\Delta\theta_t}$. Similar regularizations are possible for the other entries of the matrices $\boldsymbol{G}_{tt}$ and $\boldsymbol{G}_{ut}=\boldsymbol{G}^T_{tu}$ which are now replaced by

$$\boldsymbol{G}_{tt,\,r}=\begin{bmatrix}\boldsymbol{T}^*_{\Delta w,\,r} & \boldsymbol{M}^*_{\Delta w,\,r}\\ \boldsymbol{T}^*_{\Delta\theta,\,r} & \boldsymbol{M}^*_{\Delta\theta,\,r}\end{bmatrix}\quad,\qquad \boldsymbol{G}_{ut,\,r}=\begin{bmatrix}\boldsymbol{T}^*_{F,\,r} & \boldsymbol{M}^*_{F}\\ \boldsymbol{T}^*_{C,\,r} & \boldsymbol{M}^*_{C}\\ \boldsymbol{T}^{*\,(i)}_{F,\,r} & \boldsymbol{M}^{*\,(i)}_{F}\end{bmatrix}=\boldsymbol{G}^T_{tu,\,r} \tag{51}$$

where the terms without the subscript r were not subjected to regularization.

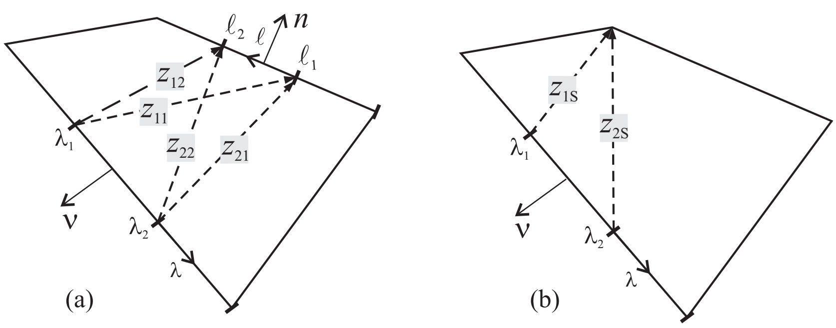


Figure 3: Complex bounds for double (a) and single (b) integrations.

The boundary integrations of the entries of (27) and (51) are then carried out by applying the rule (42) without specifying the integration bounds which are added only when the stiffness matrix of each finite element of the mesh is evaluated. This feature provides compact analytical formulas which can be quite easily included in a computer code. For instance, the final value of the entry $\boldsymbol{G}_{tt,\,r}[u^{(h)},u^{(k)}]$ is obtained specifying the real location of the boundary elements $p\in(c,d)$ of the support $\Gamma^{(h)}_{\lambda}$ and $q\in(a,b)$ of the support $\Gamma^{(k)}_{\ell}$ in order to evaluate the contributions g_{ac}, g_{ad}, g_{bc} and g_{bd}. The real extremes λ_1, λ_2 and ℓ_1, ℓ_2 of the two boundary elements p and q are replaced by the complex quantities z_{11}, z_{22}, z_{12}, z_{21} and conjugate complex $\bar{z}_{11}$, $\bar{z}_{22}$, $\bar{z}_{12}$, $\bar{z}_{21}$ which denote the distances (Figure 3) between the ends of the considered boundary elements taken into account

$$\boldsymbol{G}_{tt,r}[u^{(h)}, u^{(k)}] = \sum_{p=c,d} \sum_{q=a,b} (g_{pq}[z_{11}, \bar{z}_{11}] - g_{pq}[z_{12}, \bar{z}_{12}] - g_{pq}[z_{21}, \bar{z}_{21}] + g_{pq}[z_{22}, \bar{z}_{22}]) \tag{52}$$

In a similar way the final value of an entry of the submatrix $\boldsymbol{M}^{*(i)}_{F,r}$, belonging to the matrix $\boldsymbol{G}_{ut,r}$, is evaluated by the formula

$$\boldsymbol{G}_{ut,r}[u^{(h)}, \theta^{(i)}] = \sum_{p=c,d} (g_p[z_{1S}, \bar{z}_{1S}] - g_p[z_{2S}, \bar{z}_{2S}]) \tag{53}$$

where the complex quantities z_{1S}, z_{2S} represent the distances between the corner point S and the ends of the boundary elements of the support.

5 Conclusions

A new thin plate finite element has been proposed by using the Galerkin boundary integral approach to define its stiffness matrix. The domain integrals of polynomial shape functions, required by the traditional finite element approaches, have been replaced by boundary integrals of products of shape functions and fundamental solutions and by single boundary integrals of products of shape functions. This feature allows to develop general shape finite elements, able to represent the mechanical behaviour of irregular domains and large regions, in the same way without changing shape functions to ensure the continuity of the mechanical along the interfaces of the finite elements. To ensure the accurate evaluation of the stiffness entries a complex variable integration technique including a regularization procedure has been adopted. It allows the compact and analytical evaluation of all the boundary prime integrals which can be then collected in a database, easily to store in a computer code. The integration bounds are specified only when the stiffness matrix of each finite element is evaluated. The development of a computer code based on the proposed finite element is in progress.

References

[1] V.E. Bulgakov, M.V. Bulgakova, “Multinode finite element based on boundary integral equations”, Int. J. Numer. Meth. Engng., 43, 533–548, 1998.

[2] M. Aristodemo, L. Leone, M. Mazza, “Energy based boundary elements”, Meccanica, 36, 463–477, 2001.

[3] S. Holzer, ‘How to deal with hypersingular integrals in the symmetric BEM”, Comm. Numer. Methods Engin., 9, 219–232, 1993.

[4] M. Mazza, M. Aristodemo, “Costruzione efficiente di modelli BEM simmetrici di lastre di Kirchhoff”, GIMC 2008 – XVII Convegno Italiano di Meccanica Computazionale.