Paper 5

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Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Computational Non-Linear Buckling Analysis of an Elastically Restrained Steel Beam

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Abstract

This paper presents the computational method for the generalization of a buckling problem of the braced beam under different restraint stiffness and loading conditions. The method obtained in the paper is based on the solution of the appropriate differential equation. Through numerical examples it is demonstrated the accuracy and correctness of the proposed method and the comprehensive analysis of the deformation and flexural response of the considered structural element. The nonlinear solver of the commercial software ANSYS is used to obtain relative data and define the errors.

Keywords: continuous beam, elastic intermediate restraint, restraint stiffness, deformation response, non-linear buckling analysis.

1 Introduction

The objective of this paper is to present the method for predicting the deformation response of the braced structures composed of discrete additionally restraint members subjected simultaneously to axial force and bending moment.

The buckling analysis and design of elements considering axial compression as one of the effects makes it essentially a non-linear problem, in which axial force participation in flexural response need to be considered [1]. The non-linear analysis takes into account the effects of equilibrium in the deformed configuration, and the decrease of flexural stiffness due to the compressive axial force. The correct solution of the problem requires a non-linear analysis of the structure under the combined effect of bending and compression.

Thus, the solution of the buckling problem of the element with the intermediate elastic restraint under bending and compression is, therefore, the practical interest.

There are, however, solutions of the buckling problems for the laterally braced elements for some cases dispersed in the literature. Wang presents buckling capacities of braced columns [2] and comprehensive set of stability criteria for Euler columns with an intermediate elastic restraint [3]. Gambhir considers the case of Euler strut with an added elastic central support with potential energy approach [4]. Trahair uses differential equation method [5] for the solution of the same structural model. Saha develops the method to identify the buckling load of beam-column based on 'Multi-segment Integration technique' [10].

However, it has to be noted that, in the most performed studies, the primary objective of this paper have been treated either in the case when restrained stiffness approaches infinity [9] or lateral load vanishes [6,7,8].

The present paper discusses the non-linear buckling analysis procedures for both previously mentioned cases and their generalisation to the solution of the general problem, which is based, upon the solution of appropriate differential equation. The developed method is used to derive the governing equations and the governing parameter termed as *slenderness parameter*. The numerical example runs show the correctness of the computational method. Subsequent analysis with the commercial finite element software ANSYS using a nonlinear solver determinates the errors. The obtained errors are in the desirable limits of the accuracy. Furthermore, the paper presents a comprehensive analysis of considered buckling problem under variation of lateral load and restraint stiffness.

2 Description of the proposed computational method

The differential equation method for the solution of buckling problem and prediction of structural response of the considered structural element is used by implicating two boundary cases. First presents the solution of the structural model of the element with an intermediate inelastic restrained, subjected simultaneously to axial compression and bending moment. The second implicates the structural model of the compression element with an intermediate elastic restraint. Application of superposition procedure by using governing equations of boundary cases requires the solution of general problem.

2.1 Structural model

Consider the simple supported continuous beam of the length l with constant flexural rigidity EI subjected to a distributed lateral load q and an axial compression N_c shown in Figure 1.

An additional lateral restrained at the midspan is provided to prevent it from lateral deflection. The stiffness α of the restraint is defined by the restoring force F_{vb} acting on the restraint and thus the free movement of the member is restricted to greater or lesser extent δ.

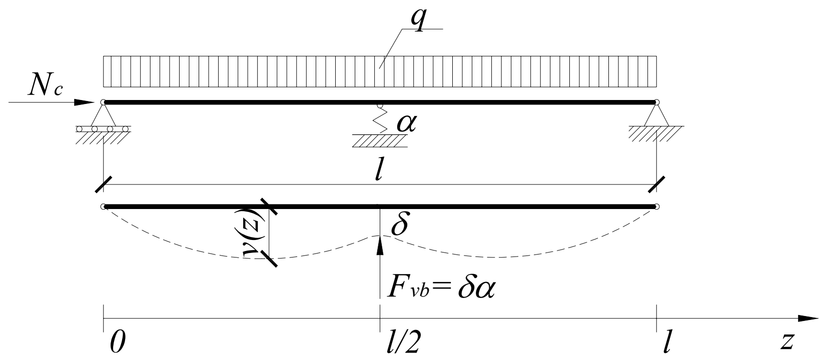

Figure 1: The structural model of the elastically restrained continuous beam

2.1.1 Simply supported continuous beam with inelastic intermediate restraint under interaction of bending and compression

Figure 2 presents structural model of a beam when the midspan lateral restraint of the stiffness α approaches infinity and becomes completely rigid. The applied distributed lateral load $q(z)$ causes deflection $v(z)$, which corresponds to the bending moment $-EI''(z)$, amplified by axial compression causing moment $N_c v(z)$, along the member. In general, the problem should be treated in two parts: one considering the beam to the left of central restraint ($0 \leq z \leq l/2$); and other to the right ($l/2 \leq z \leq l$). However, as the loading, boundary and continuity conditions are equal to both parts structural model can be simplified to one of the parts as shown in Figure 2.

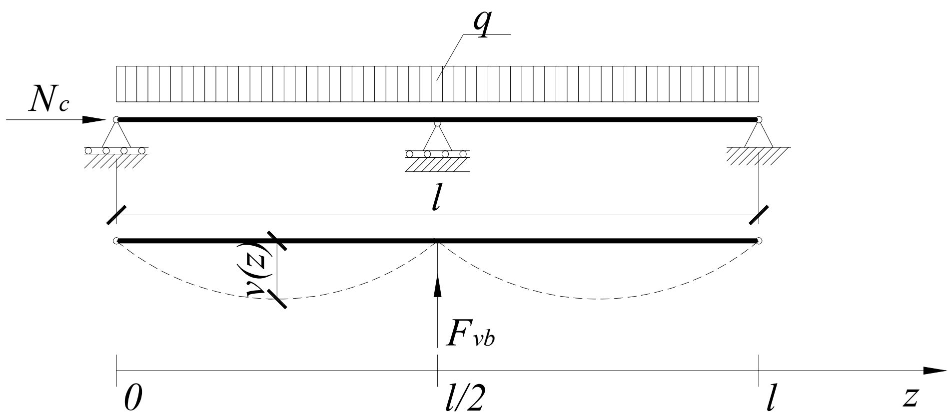

Figure 2: The structural model of the inelastically restrained continuous beam

The influence of the axial compression on the bending moment can be incorporated directly into governing differential equation [4]:

$$EI\left(\frac{d^4v}{dz^4}\right)-N_c\left(\frac{d^2v}{dz^2}\right)+q(z)=0 \tag{1}$$

Substituting $k^2=\frac{N_c}{EI}$ the equation reduces to:

$$\left(\frac{d^4v}{dz^4}\right)+k^2\left(\frac{d^2v}{dz^2}\right)=\frac{q(z)}{EI} \tag{2}$$

The general solution of Equation (1) has the form:

$$v(z)=C_1\sin kz+C_2\cos kz+C_3z+C_4+\frac{qz^2}{2EIk^2} \tag{3}$$

The integration constants C_1, C_2, C_3 and C_4 are determined from the prescribed boundary conditions: $v(0)=v(0)''=0$ and $v(l/2)=v(l/2)'=0$, where:

$$v'(z)=C_1k\cos kz-C_2\sin kz+C_3+\frac{qz}{EIk^2} \tag{4}$$

$$v''(z)=-C_1k^2\sin kz-C_2k^2\cos kz+\frac{q}{EIk^2} \tag{5}$$

Then the solution of Equation (3) which satisfies the boundary conditions is deflected shape:

$$v(z)=\frac{1}{N_c}\left[\frac{F_{vb}l}{2}\left(\frac{z}{l}-\frac{\sin kz}{kl\cos\frac{kl}{2}}\right)+\frac{ql^2}{(kl)^2}\left(tg\frac{kl}{2}\sin kz+\cos kz-1\right)-qz\left(\frac{l}{2}-\frac{z}{2}\right)\right] \tag{6}$$

The bending moment of considered beam element is the solution of Equation (5) which satisfies the boundary conditions:

$$M(z)=\frac{ql^3}{(kl)^2}\left(tg\frac{kl}{2}\sin kz+\cos kz+1\right)-\frac{F_{vb}l}{2kl}\frac{\sin kz}{\cos\frac{kl}{2}} \tag{7}$$

Restoring force introduced by intermediate support can be obtained from:

$$F_{vb}=EIv'''=-EIC_1k^3\cos kz+EIC_2k^3\cos kz=EIk^3\left(C_2\cos kz-C_1\cos kz\right) \tag{8}$$

When the solution of Equation (8) satisfies the boundary conditions:

$$F_{vb} = 2\frac{ql}{kl}\left[\frac{1+\left(\frac{kl}{2}\right)^2/2-1/\cos\frac{kl}{2}}{\frac{kl}{2}-tg\frac{kl}{2}}\right] \tag{9}$$

2.2.1 Simply supported continuous beam with elastic intermediate restraint under compression

Figure 3 shows structural model of continuous simply supported beam with an elastic intermediate restraint, which prevents it from deflecting at its centre and introduce restoring force F_{vb}. The structural model presents the other boundary problem when distributed lateral load approaches zero and the member tends to become axially loaded strut. Because of continuity in terms of loading and boundary conditions, the structural model can be simplified to the part on the left from the intermediate support ($0 \le z \le l/2$).

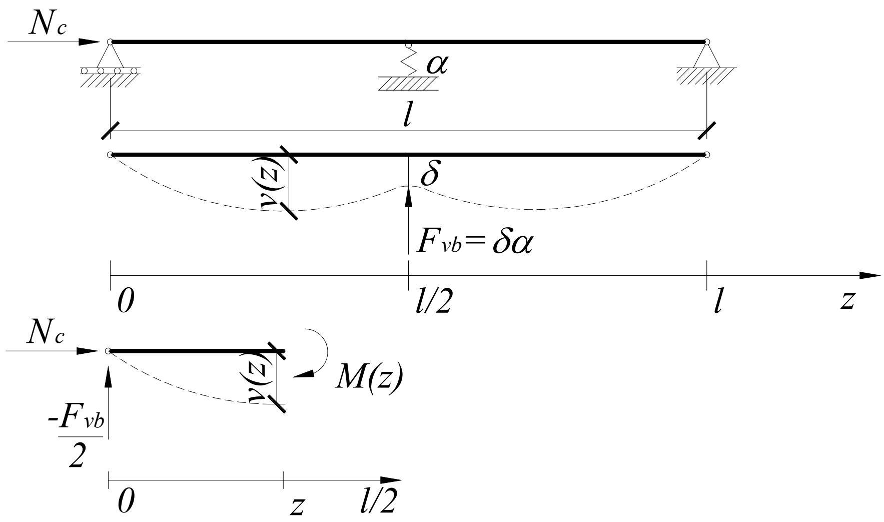

Figure 3: The structural model of the compression member with an elastic intermediate restraint

Moment equilibrium is defined by moment about an arbitrary point at a distance z as shown in Figure 3. When the axial load N_c is compressive the differential equilibrium equation of bending of the member is:

$$M(z) = \frac{F_{vb}z}{2} - N_c v(z) \tag{10}$$

This equation states that for equilibrium, the internal moment of resistance must exactly balance the external disturbing moment at any point along the length of the member.

With constant EI and substituting $k^2 = \frac{N_c}{EI}$ the Equation (10) reduces to:

$$v''(z) + k^2 v(z) = \frac{F_{vb} z}{2EI} \tag{11}$$

When this equation is satisfied at all points, the displacement position is one of equilibrium. The solution of Equation (11):

$$v(z) = C_1 \sin kz + C_2 \cos kz + \frac{F_{vb} z}{2EIk^2} \tag{12}$$

The boundary condition at the end support that $z = 0$ is satisfied when $v(z) = 0$ and at the intermediate restraint that $z = l/2$ is satisfied when:

$$v'(z) = C_1 k \cos kz - C_2 k \sin kz + \frac{F_{vb}}{2EIk^2} = 0 \tag{13}$$

The solution of the Equation (12) which satisfies the boundary conditions is given by:

$$v(z) = \frac{F_{vb} l}{2N_c} \left(\frac{z}{l} - \frac{\sin kz}{kl \cos kl} \right) \tag{14}$$

Since the displacement at the intermediate restraint $\delta = v(l/2)$, it follows that:

$$\delta = \frac{F_{vb} l}{2N_c} \left[\frac{\frac{kl}{2} - tg \frac{kl}{2}}{kl} \right] \tag{15}$$

By using this obtain expression for the displacement at the restraint and considering the relation between it and restoring force, the following equation can be obtained:

$$F_{vb} = 2N_c \frac{\delta}{l} \frac{kl}{\left(\frac{kl}{2} - tg \frac{kl}{2} \right)} \tag{16}$$

2.2 Governing equations

In previous sections, the application of the method to obtain structural response by non-linear buckling analysis of the continuous simply supported and additionally elastically restraint beam subjected simultaneously to bending moment and axial compression is extrapolated in to two boundary problems. For this extrapolation, it is assumed that either the stiffness α of elastic intermediate restraint approaches infinity or distributed lateral load $q(z)$ approaches zero. The general structural response of the structural member shown in Figure 1 and described in Section 2.1 can be analysed by considering the differential equilibrium Equation 1. The solution of this which satisfies the boundary conditions $v(0)=v'(0)=0$, $v(l/2)=\delta$ and $v'(l/2)=0$ is:

$$v(z)=\frac{1}{N_c}\left[\frac{1}{2}F_{vb}l\left(\frac{z}{l}-\frac{\sin kz}{kl\cos\frac{kl}{2}}\right)+\frac{ql^2}{(kl)^2}\left(tg\frac{kl}{2}\sin kz+\cos kz-1\right)-qz\left(\frac{l}{2}-\frac{z}{2}\right)\right] \quad (17)$$

in which F_{vb} is the restoring force at the restraint and may be obtained as the solution of:

$$F_{vb}=EIv'''(z) \quad (18)$$

whence:

$$F_{vb}=2\left[\frac{ql}{kl}\frac{\left(1+\left(\frac{kl}{2}\right)^2/2-1/\cos\frac{kl}{2}\right)}{\left(\frac{kl}{2}-tg\frac{kl}{2}\right)}+N_c\frac{\delta}{l}\frac{kl}{\left(\frac{kl}{2}-tg\frac{kl}{2}\right)}\right] \quad (19)$$

Obtained expression is generalization of the boundary solutions described in Sections 2.1.1 and 2.1.2 and express possibility of superposition.

If the restoring force is expressed in terms of an elastic restraint stiffness α and central deflection δ as $F_{vb}=\alpha\delta$ and substituted into Equation (19) central deflection of the beam is given by:

$$\delta=\frac{ql^4}{EI(kl)^4}\frac{\left(1+\left(\frac{kl}{2}\right)^2/2-1/\cos\left(\frac{kl}{2}\right)\right)}{\left(\frac{\alpha l^3}{2EI}\left(\frac{\frac{kl}{2}-tg\frac{kl}{2}}{(kl)^3}\right)-1\right)} \quad (20)$$

Alternatively, Equation (19) can be rearranged to express the relationship between the restoring force and the restraint stiffness as:

$$F_{vb} = 2\frac{ql}{kl}\frac{\left(1+\left(\frac{kl}{2}\right)^2/2-1/\cos\frac{kl}{2}\right)}{\left(\frac{kl}{2}-tg\frac{kl}{2}\right)}\times\left[1+\frac{1}{\left[\frac{\alpha l^3}{2EI}\left(\frac{\frac{kl}{2}-tg\frac{kl}{2}}{(kl)^3}\right)-1\right]}\right] \tag{21}$$

The bending moment due to both lateral load and axial compression can be obtained by:

$$M(z) = \frac{ql^2}{(kl)^2}\left(tg\frac{kl}{2}\sin kz+\cos kz-1\right)-\frac{F_{vb}l}{2kl}\frac{\sin kz}{\cos\frac{kl}{2}} \tag{22}$$

2.2.1 Governing parameters and variables

It can be seen that in all governing equation of the structural response of considered structural element is an undetermined variable kl. This variable is assumed as the governing parameter of buckling analysis of considered structural model and is termed as *slenderness parameter*. The slenderness parameter relates to the axial load, the flexural rigidity and the length of the element and can be expressed as:

$$kl = l\sqrt{\frac{N_c}{EI}} \tag{23}$$

The deformation response of simply supported and additionally elastically restraint continuous beam under compression and bending can be expressed with the variation either of the displacement or the bending moment with the slenderness parameter. To obtain correct solution, the minimum value of the slenderness parameter must be set greater than zero.

The invariable lateral load $q(z)$ applied on an axially compressed element causes initial eccentricity which increase the stress, but it does not affects the elastic stability of the system [4]. Whereas in case of elastic restraint, it results in a deflection dependent lateral load and a restoring force as the deflection takes place. Thus the additional elastic restraint affects the elastic stability and even small restraint stiffness may cause a considerable increase in elastic stability of the member [4]. The limiting stiffness α_L of the restrained using structural model described in Section 2.1.2 when distributed lateral load vanishes can be determined. Due to Trahair [5] when the restraint stiffness exceeds:

$$\alpha_L = 16\pi^2 EI/l^3 \tag{24}$$

member buckles in the antisymmetrical second mode.
In general case when distributed lateral load $q(z)$ applied without reference to its value when the restraint stiffness exceeds the minimum value given by Equation (24) it cause significant changes of the deflected shape of the member.

3 Numerical example

The numerical analysis was carried out for the buckling problem of the laterally braced element using the structural model described in Section 2.1. The 7530 mm^2 cross-section continuous beam of 10,0 m length and 1,157e+13 Nmm^2 constant flexural rigidity of Figure 1 is simply supported at each end and has a central brace which prevents lateral deflections. The elastic restraint at the centre has limiting value of 1876N/mm. By considering the variation of the distributed lateral load from 2,5 Nmm to 20 Nmm and the elastic restraint stiffness from 0,25 to 20,0 the structural response of the beam is shown in Figures 4 to 7. Analysis is performed increasing slenderness parameter from 0,5 by the increment of 0,25.
Subsequently the calculation model is used for the analysis with the nonlinear solver of the commercial finite element software ANSYS.

Table 1 shows the randomly chosen numerical values of the displacements, bending moments and restoring forces from the obtained data. This table defines the errors of the proposed computational method to ANSYS results. The errors vary in the desirable limits, and the proposed computational method can be declared as accurate and correct.

Computational method/ANSYS						
q=1 Nmm and α/α_L= 0,25				q=10 Nmm and α/α_L=1,0		
kl	3.00	3.14	3.50	3.00	3.14	3.50
Restoring force F_{vb}, kN Errors	5,73/5,73 -0,10%	6,37/6,36 -0,11%	9,12/9,11 -0,16%	61,8/61,8 -0,10%	63,6/63,5 -0,11%	69,1/69,0 -0,14%
q=5 Nmm and α/α_L= 0,5				q=5 Nmm and α/α_L=0,75		
Bending Moment $M(z=l/4)$, kNm Errors	35,0/35,0 0,00%	37,0/37,0 0,00%	43,8/43,8 0,00%	26,6/26,6 0,02%	27,7/27,7 0,02%	31,0/31,0 0,03%
q=5 Nmm and α/α_L= 0,75				q=10 Nmm and α/α_L=1,0		
Displacement $v(z=l/4)$, mm Errors	17,4/17,4 -0,08%	18,0/18,0 -0,09%	21,1/20,1 -0,11%	27,0/27,0 -0,08%	27,8/27,8 -0,09%	30,3/30,2 -0,11%
δ, mm Errors	22,4/22,4 -0,10%	23,2/23,2 -0,10%	25,9/25,9 -0,14%	33,8/33,8 -0,10%	34,9/34,8 -0,11%	37,9/37,8 -0,14%

Table 1: Numerical values and errors

By considering the relative stiffness of the intermediate response equal to $\alpha/\alpha_L = 1$ the variations of the displacement $v(l/4)$ and the disturbing bending moment $M(l/4)$ with the slenderness parameter kl are shown in Figure 4 and Figure 5 respectively. The relationship between slenderness parameter and either displacement or bending moment is non-linear and approaches the value of the slenderness parameter equal to 6,0. Furthermore, the Figure 4 and Figure 5 show the increase of the deformation and the flexural response respectively with the increase of the lateral distributed load.

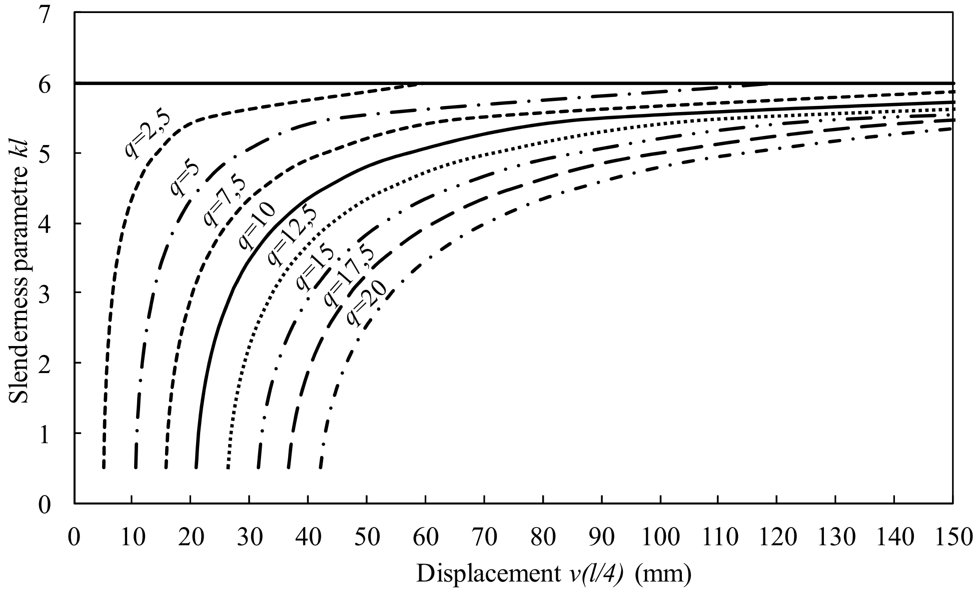

Figure 4: The variation of the deformation response with the slenderness parameter

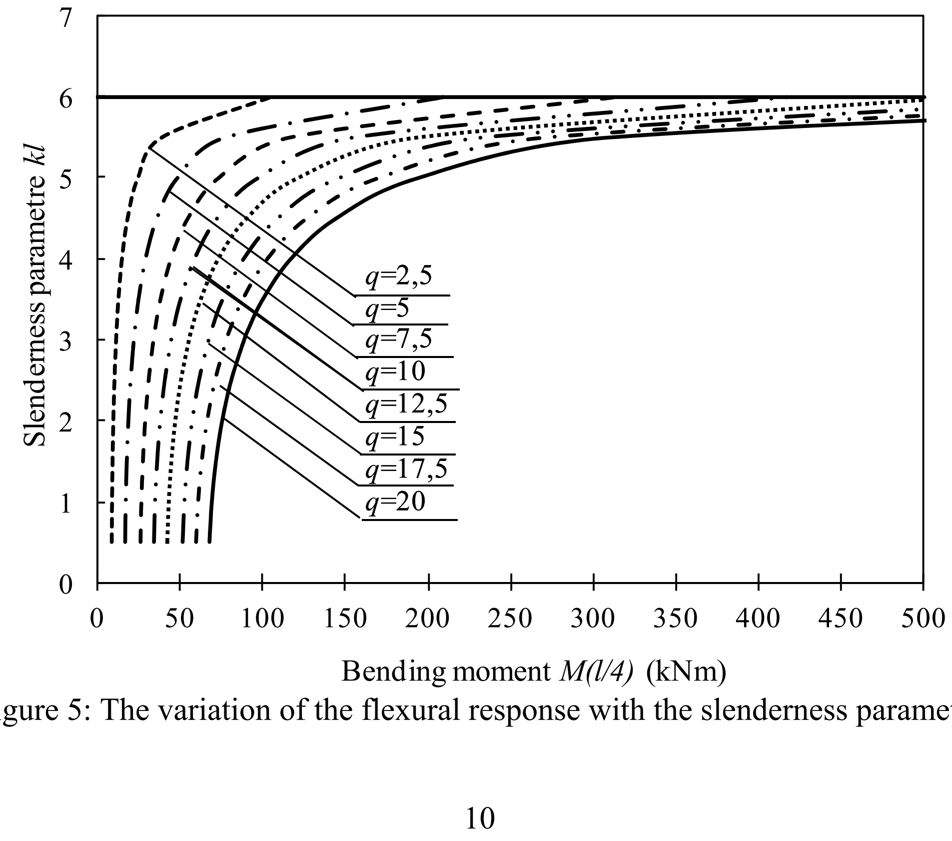

Figure 5: The variation of the flexural response with the slenderness parameter

Figure 6 shows the variation of the relative displacement $\delta/v(l/4)$ with the slenderness parameter for the different relative stiffness $\alpha/\alpha_L$of the intermediate restraint. The variation was obtained for the different values of the slenderness parameter and the constant lateral distributed load of 10 Nmm. Between the values of the relative restraint stiffness of 2 to 4 for any value of the slenderness parameter the ratio of the restrained displacement and the displacement at the hogging part of the deflection shape becomes less than 1,0. The value of the displacements ratio equal to 1,0 was assumed as the limiting value for the significant changes of the deflection shape of the element. When the ration of the displacements is greater than 1,0 the restrained deflection shape of the element approaches the unrestrained deflection shape. When the relative displacement exceeds 1,0 and decreases the deflection of the element approaches the braced deflection shape.

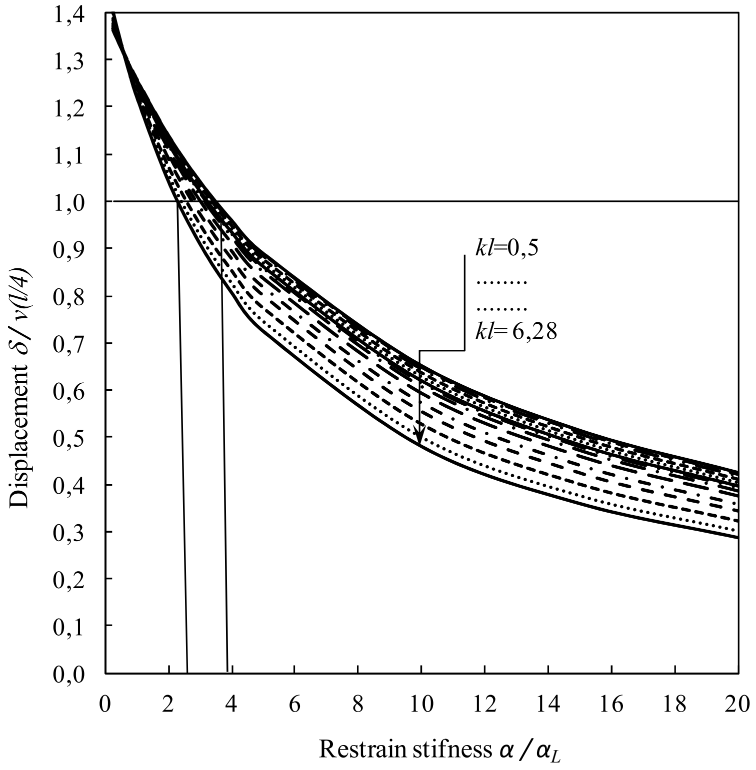

Figure 6: The variation of the relative displacement with the relative restraint stiffness

The variation of the dimensionless bending moment with the relative restraint stiffness for the different values of the slenderness parameter and the distributed lateral load of 10 Nmm is shown in Figure 7. The ratio of the bending moments at the restraint and at the centre of the midspan respectively was taken to present the changes of the bending moment distribution relative to the restraint stiffness. When the ration of the restraint stiffness exceeds 1,0 the bending moment changes the sign. The change of the sign from positive to negative presents the change of the bending moment at the restraint from disturbing to restoring. It can be seen that the variations of the greater slenderness parameter vanishes from the positive part of the chart this corresponds to the constant restoring moment at the support. The intersection of the graphs at the particular values of the relative restraint stiffness and the ratio of the bending moment shows the change of the acceleration of the restoring moment increase for the less values of the slenderness parameter. Furthermore the ration of the bending moment grater that -1,0 presents the extreme values of the bending moment shifting from hogging to sagging.

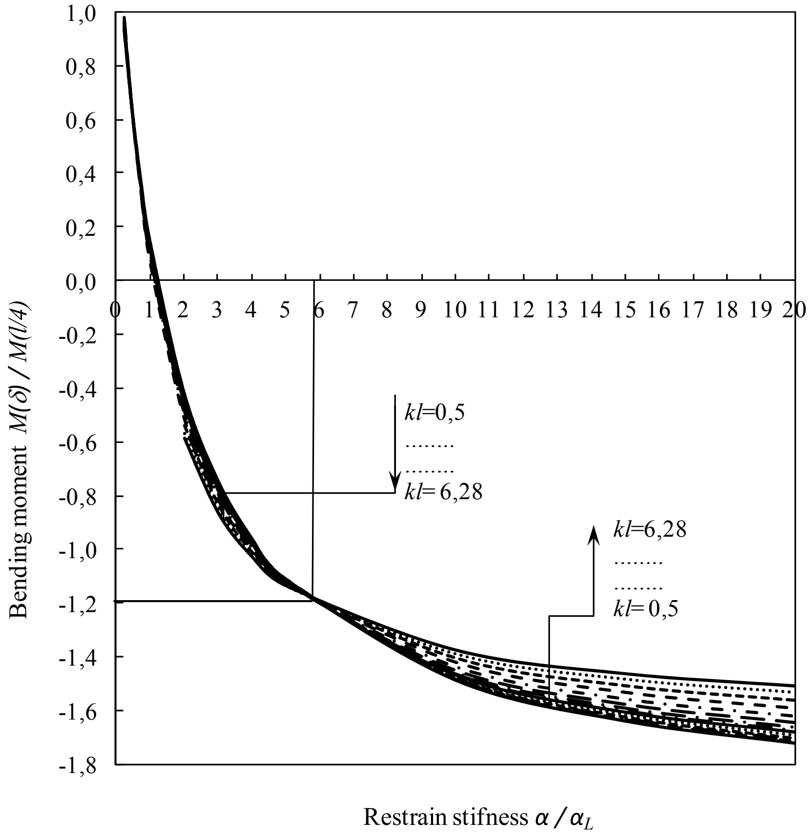

Figure 7: The variation of the relative bending moment with the relative restraint stiffness

4 Conclusions

The errors presented in Table 1 show that developed method for the non-linear buckling problem of the structural model described in Section 2.1 is accurate and correct enough.

The governing equations derived in Section 2.2 present the possibility of the appropriate superposition procedure using the boundary solutions discussed in Sections 2.1.1 and 2.1.2. The boundary solutions introduce buckling problems of the braced structures dealt by the research in the papers on this subject.

The parameter governing the structural response of the element considered is defined and appropriate relationships are presented. The influence of the additional restraint stiffness on the deflection shape of the structural element is discussed, and the limiting value is given. It has to be noted that the provided elastic restraint results in a deflection dependant lateral load and increases its influence on the elastic stability of the considered element. The numerical analysis shows the relationship between the limiting value of the restraint stiffness and the lateral load.

The numerical analysis was carried out to show that the comprehensive analysis of the deformation and flexural response of the simply supported and laterally braced beam under interaction of bending and compression can be performed using the proposed computational method. The different values of the lateral load were applied to illustrate the decrease of the elastic stability of the braced structural element with the increase of the lateral load. Similarly, the same analysis with the different values of the restraint stiffness was carried out.

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