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Paper 98

Free Vibration Analysis of Shallow Shells using the Superposition-Galerkin Method

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Abstract

A procedure to apply the Superposition-Galerkin method for the vibration analysis of a completely free doubly curved shallow shell is presented. To handle mixed derivatives in the equation of boundary conditions of the free edge, the sine series and the cosine series are used independently in the displacement functions of building blocks. The solution to each building block is obtained using the Galerkin method and expressed in term of its driving coefficients. The method is expected to work as the superimposed set of the building blocks can satisfy the free edge conditions. However, presence of extra natural frequencies proves to be a challenge. Currently work is in progress to eliminate the spurious modes.

Keywords: superposition method, Galerkin method, shallow shell, free vibration, natural frequencies.

1 Introduction

The Superposition method developed by Gorman has been applied firstly for the analysis of out-of-plane vibrations of plate [1] and afterwards for more complicated systems, such as orthotropic plates, plates with elastic supports, point supported plates, triangular and parallelogram plates, plates under in-plane forces, Mindlin plates and laminated plates, as well as in-plane vibrations of plates [2-4]. The Superposition method was also utilised for the analysis of open cylindrical shells [5]. It was also shown that this method was applicable for the determination of steady state response of plates [6] and transient response of a plate [7].

It is known that Gorman's Superposition method is very efficient and accurate for plate vibration analyses. In many cases the results from the Superposition method are the benchmarks for the natural frequencies. However, in some cases, generating the Levy-type solutions for building blocks could be difficult and the solutions will have several different forms depending on the roots of the governing differential equation[8]. To overcome this disadvantage, Gorman and Wing obtained solutions for the free vibration of the fully clamped orthotropic and Mindlin plates using the building blocks whose modes are obtained using the Galerkin method [8]. This alternative approach is called the Superposition-Galerkin method. Gorman also used the same method for the free vibration analysis of completely free orthotropic and Mindlin plates [9]. Recently the Superposition-Galerkin method was used to obtain the natural frequencies of doubly curved shallow shells with combinations of simply supported and clamped edges [10]. However, doubly curved shallow shells with free edges have not been investigated using the Superposition method.

We present here our efforts to demonstrate that the Superposition-Galerkin method is applicable for the doubly curved shallow shell with free edges if additional building blocks are employed together with those used in the case of the fully clamped shells. At the time of writing this paper, work is still in progress.

2 Analytical procedure

2.1 The governing differential equations

The governing equations based on Donnell theory for free vibration of thin shallow shells are given in reference [11] and the dimensionless form of the governing equations is expressed as follows.

$$
\begin{aligned}
&12\delta^2\frac{\partial^2 U}{\partial\xi^2}+12\bar{\nu}\frac{\delta^2}{\Phi^2}\frac{\partial^2 U}{\partial\eta^2}+12(\nu+\bar{\nu})\frac{\delta^2}{\Phi}\frac{\partial^2 V}{\partial\xi\partial\eta}+12\delta^2\alpha(1+\nu\mu)\frac{\partial W}{\partial\xi}+\Omega^2 U=0\\
&12(\nu+\bar{\nu})\frac{\delta^2}{\Phi}\frac{\partial^2 U}{\partial\xi\partial\eta}+12\bar{\nu}\delta^2\frac{\partial^2 V}{\partial\xi^2}+12\frac{\delta^2}{\Phi^2}\frac{\partial^2 V}{\partial\eta^2}+\frac{12\delta^2\alpha}{\Phi}(1+\mu)\frac{\partial W}{\partial\eta}+\Omega^2 V=0\\
&12\delta^2\alpha(1+\nu\mu)\frac{\partial U}{\partial\xi}+\frac{12\delta^2\alpha}{\Phi}(1+\mu)\frac{\partial V}{\partial\eta}+\left(\frac{\partial^4 W}{\partial\xi^4}+\frac{2}{\Phi^2}\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}+\frac{1}{\Phi^4}\frac{\partial^4 W}{\partial\eta^4}\right)\\
&\qquad+12\delta^2\alpha^2(1+2\nu\mu+\mu^2)W-\Omega^2 W=0
\end{aligned}
\tag{1}
$$

where α: curvature ratio, a/R_x, μ: Gaussian curvature, R_x/R_y, R_x, R_y: radius of curvature parallel to x axis and y axis, δ: thickness ratio, a/h, ν: Poisson's ratio of material and $\bar{\nu}$: $(1-\nu)/2$. The dimensionless membrane forces, N, bending moments, M and vertical edge reaction, Q are given in terms of displacements as Equations. (2) - (4).

$$
\begin{Bmatrix} N_\xi \\ N_\eta \\ N_{\xi\eta} \end{Bmatrix}=\begin{Bmatrix} \dfrac{\partial U}{\partial\xi}+\dfrac{\nu}{\phi}\dfrac{\partial V}{\partial\eta}+\beta(1+\gamma\nu)W \\ \dfrac{\partial U}{\partial\xi}+\phi\nu\dfrac{\partial V}{\partial\eta}+\beta\phi(\gamma+\nu)W \\ \dfrac{1-\nu}{2}\left(\dfrac{\partial U}{\partial\eta}+\phi\dfrac{\partial V}{\partial\xi}\right) \end{Bmatrix}
\tag{2}
$$

$$\begin{Bmatrix} M_\xi \\ M_\eta \\ M_{\xi\eta} \end{Bmatrix} = \begin{Bmatrix} -\left(\dfrac{\partial^2 W}{\partial \xi^2} + \dfrac{\nu}{\phi^2}\dfrac{\partial^2 W}{\partial \eta^2}\right) \\ -\left(\dfrac{\partial^2 W}{\partial \eta^2} + \nu\phi^2\dfrac{\partial^2 W}{\partial \xi^2}\right) \\ -(1-\nu)\dfrac{\partial^2 W}{\partial \xi \partial \eta} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} Q_\xi \\ Q_\eta \end{Bmatrix} = \begin{Bmatrix} -\left(\dfrac{\partial^3 W}{\partial \xi^3} + \dfrac{2-\nu}{\phi^2}\dfrac{\partial^3 W}{\partial \xi \partial \eta^2}\right) \\ -\left(\dfrac{\partial^3 W}{\partial \eta^3} + (2-\nu)\phi^2\dfrac{\partial^3 W}{\partial \eta \partial \xi^2}\right) \end{Bmatrix} \tag{4}$$

2.2 The Superposition-Galerkin method

The analysis of a fully clamped shallow shell on a rectangular planform using the Superposition method is described in reference [10]. It will be demonstrated that how the same method can be applied for the case of completely free shallow shells. For brevity, only the doubly anti-symmetric modes are considered. However, once it is shown to work for this case, the employed procedure can be easily extended for all mode families.

Eight building blocks shown in Figure1 are utilised for the analysis. In contrast to the fully clamped shells, it is difficult choose a series form of the displacement functions of the building block including a free edge, where each term in the series satisfies the governing equations and the boundary conditions of the free edge simultaneously since mix derivates appear in the equations [9]. In order to overcome this difficulty, additional building blocks are presented to the building blocks used for the fully clamped shallow shell. The first and second building blocks in Figure 1 are identical to those in reference [10]. The third and fourth building blocks differ from the first two building blocks only in the boundary conditions along the driving edges. The first and second building blocks are subjected to a distributed harmonic bending moment and in-plane normal force expressed by Equation. (5) at η = 1 respectively. The third and fourth are subjected to a distributed harmonic vertical edge reaction and in-plane tangential force expressed by Equation. (5) along their driving edges respectively. Out-of-plane displacements of the driving edges of first two building blocks are prohibited, while slope normal to the driving edges of third and fourth building blocks are prohibited. The boundary conditions of other edges are given by Equation (6).

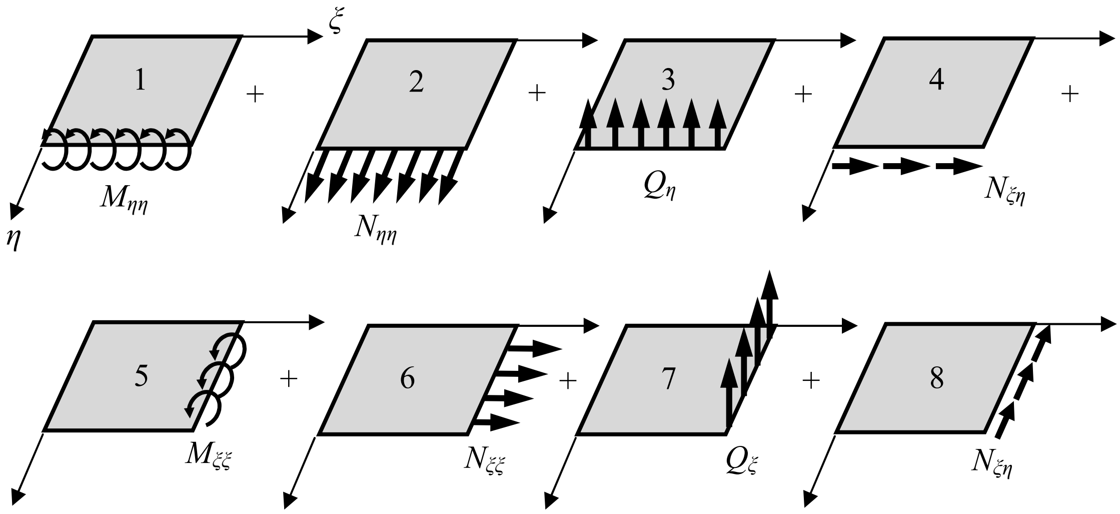

Figure1: Building blocks used for the free vibration analysis of a shell

$$\left.\begin{aligned} M_{\eta\eta} &= \sum_{m=1}^{k} E_m \sin m\pi\xi \\ N_{\eta\eta} &= \sum_{m=1}^{k} F_m \sin m\pi\xi \\ Q_{\eta} &= \sum_{m=1}^{k} G_m \sin m\pi\xi \\ N_{\xi\eta} &= \sum_{m=1}^{k} H_m \cos m\pi\xi \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} V = W = 0, N_{\xi\xi} = 0 \; and \; M_{\xi\xi} = 0 \quad (at \; \xi = 0 \; and \; 1) \\ U = W = 0, N_{\eta\eta} = 0 \; and \; M_{\eta\eta} = 0 \quad (at \; \eta = 0 \; and \; 1) \end{aligned}\right\} \tag{6}$$

The in-plane displacements *U* and *V*, and out-plane displacement *W* of the first to fourth building blocks are expressed by Equation (7) with Equations (8) - (11), respectively. It is noted that each trigonometric term in Equations (8) – (11) satisfies the boundary conditions given by Equation (6) and with the additional polynomial terms, these equations satisfy the boundary conditions given by Equation (5).

$$\left.\begin{aligned} U_m &= \sum_{m=1,2,3\cdots}^{k} X_m(\eta)\cos m\pi\xi \\ V_m &= \sum_{m=1,2,3\cdots}^{k} Y_m(\eta)\sin m\pi\xi \\ W_m &= \sum_{m=1,2,3\cdots}^{k} Z_m(\eta)\sin m\pi\xi \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} X_m(\eta) &= \sum_{i=1,2,3\cdots}^{K} A_{mi}\sin i\pi\eta \\ Y_m(\eta) &= \sum_{j=1,2,3\cdots}^{K} A_{mj}\cos(j-1)\pi\eta \\ Z_m(\eta) &= \sum_{l=1,2,3\cdots}^{K} A_{ml}\sin l\pi\eta + \frac{E_m}{6}(\eta-\eta^3) \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} X_m(\eta) &= \sum_{i=1,2,3\cdots}^{K} B_{mi}\sin i\pi\eta \\ Y_m(\eta) &= \sum_{j=1,2,3\cdots}^{K} B_{mj}\cos(j-1)\pi\eta + F_m\,\frac{\eta^2}{2} \\ Z_m(\eta) &= \sum_{l=1,2,3\cdots}^{K} B_{ml}\sin l\pi\eta \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} X_m(\eta) &= \sum_{i=1,3,5\cdots}^{K} C_{mi}\sin\frac{i\pi\eta}{2} \\ Y_m(\eta) &= \sum_{j=1,3,5\cdots}^{K} C_{mj}\cos\frac{j\pi\eta}{2} \\ Z_m(\eta) &= \sum_{l=1,3,5\cdots}^{K} C_{ml}\sin\frac{l\pi\eta}{2} + G_m\left(\frac{\eta^2}{12}-\frac{\eta^4}{24}\right) \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} X_m(\eta) &= \sum_{i=1,3,5\cdots}^{K} D_{mi} \sin\frac{i\pi\eta}{2} + \frac{2}{1-\nu} H_m \frac{\eta^2}{2} \\ Y_m(\eta) &= \sum_{j=1,3,5\cdots}^{K} D_{mj} \cos\frac{j\pi\eta}{2} \\ Z_m(\eta) &= \sum_{l=1,3,5\cdots}^{K} D_{ml} \sin\frac{l\pi\eta}{2} \end{aligned}\right\} \quad (11)$$

Equations.(7), and (8) are substituted into Equation (1) and differentiated term-by-term. This gives an algebraic equation relating the 3×*K* Fourier unknowns, A_{mi}, A_{mj}, A_{ml} and driving coefficient E_m. By following the Galerkin method, these equations are expanded in an appropriate trigonometric function of *K* term, which gives a set of 3×*K* simultaneous non-homogeneous algebraic equations [8, 9]. These Fourier unknowns are obtained by solving the algebraic equations, and thus a solution for the first building blocks is expressed in terms of E_m. Solutions for the second to fourth building blocks are obtained similarly and expressed in term of F_m, G_m, and H_m respectively.

The solution for other building blocks will be obtained in similar manner by interchanging the axes, i.e. interchanging η and ξ as well as *X* and *Y* in Equations (7) – (11). In addition, the subscript "*m*" should be changed to "*n*" to distinguish the displacement functions

2.3 Composing Eigen matrix

Once the solutions for all building blocks are obtained, these building blocks are superimposed so as to satisfy the boundary conditions of the original free shell. In other words, the membrane forces, bending moments and edge reactions of the superimposed set of building blocks should vanish. The boundary conditions of the free shell are expressed as follows.

$$\left.\begin{matrix} N_{\xi\xi}=0, & N_{\xi\eta}=0, & M_{\xi\xi}=0, & Q_\xi=0 & (at\ \xi=1) \\ N_{\eta\eta}=0, & N_{\xi\eta}=0, & M_{\eta\eta}=0, & Q_\eta=0 & (at\ \eta=1) \end{matrix}\right\} \quad (12)$$

The natural frequencies are determined by searching for the Ω values for which the determinant vanishes as described in references [1, 2].

The sum of contributions of the building blocks to the in-plane normal and tangential forces, the bending moment and the vertical edge reaction at the edge is expanded in an appropriate trigonometric series [1, 2] and this yields a set of 8*k* homogeneous algebraic equations relating 8*k* driving coefficients *E*, *F*, *G*,and *H*'s, which can be expressed in matrix form as Equation. (13):

$$\mathbf{A}\begin{Bmatrix}\mathbf{E}\\ \mathbf{F}\\ \mathbf{G}\\ \mathbf{H}\end{Bmatrix} = \mathbf{0} \tag{13}$$

where **A** is $8k\times 8k$ matrix, $\begin{Bmatrix}\mathbf{E}\\ \mathbf{F}\\ \mathbf{G}\\ \mathbf{H}\end{Bmatrix}$ is $8k\times 1$ column vector of driving coefficients.

A schematic representation of the matrix **A** of Equation. (13) when k=3 is given in Figure 2. The dots in the figure depict non-zero components. The matrix is divided into 8×8 segments, and each column and row of the segment denote the building block and its contribution to the boundary condition at the edge respectively. For example, the third line of segments shows the contribution to the vertical edge reaction on the edge η = 1. The harmonic vertical edge force is applied along the driven edge of the third building block. However, there are some residual vertical reaction forces from other building blocks. All building blocks should complement each other and the sum of vertical edge reaction forces should be disappeared when they are superimposed.

The procedure for the analysis of the doubly anti-symmetric modes are described above, however, the same procedure can be easily extended to the analysis for all mode families by adding building blocks. Another four building blocks will be added to impose the membrane forces, bending moment and edge reaction force along the edge η = 0 as these forces, which are expressed by Equation (5), are applied the edge η = 1. The solutions for those building blocks can be generated from the first four building blocks by simply replacing η to $1-\eta$ changing subscripts from m to p in Equations (7) – (11). Similarly, four additional building blocks are used to impose the forces along the edge ξ = 0. The solutions for these building blocks are generated by interchanging ξ in the fifth and eighth building block solution to $1-\xi$, and changing subscripts from n to q. The solution for Y_p, and X_q should be preceded with negative sign, i.e. Equations. (14) and (15).

$$Y_p(\eta) = -Y_m(1-\eta) \tag{14}$$

$$X_q(\eta) = -X_n(1-\eta) \tag{15}$$

3 Natural frequency parameters

The non-dimensional natural frequency parameters given by $\Omega = \omega a^2\sqrt{\rho/D}$, are determined by searching for the Ω values for which the determinant of the eigenmatrix vanishes by trial and error. To avoid missing coincident modes, the W-W algorithm [12-13] is also used during the computation. Once the Ω values are found, the driving coefficients are found by substituting into Equation (13) and these give the natural modes.

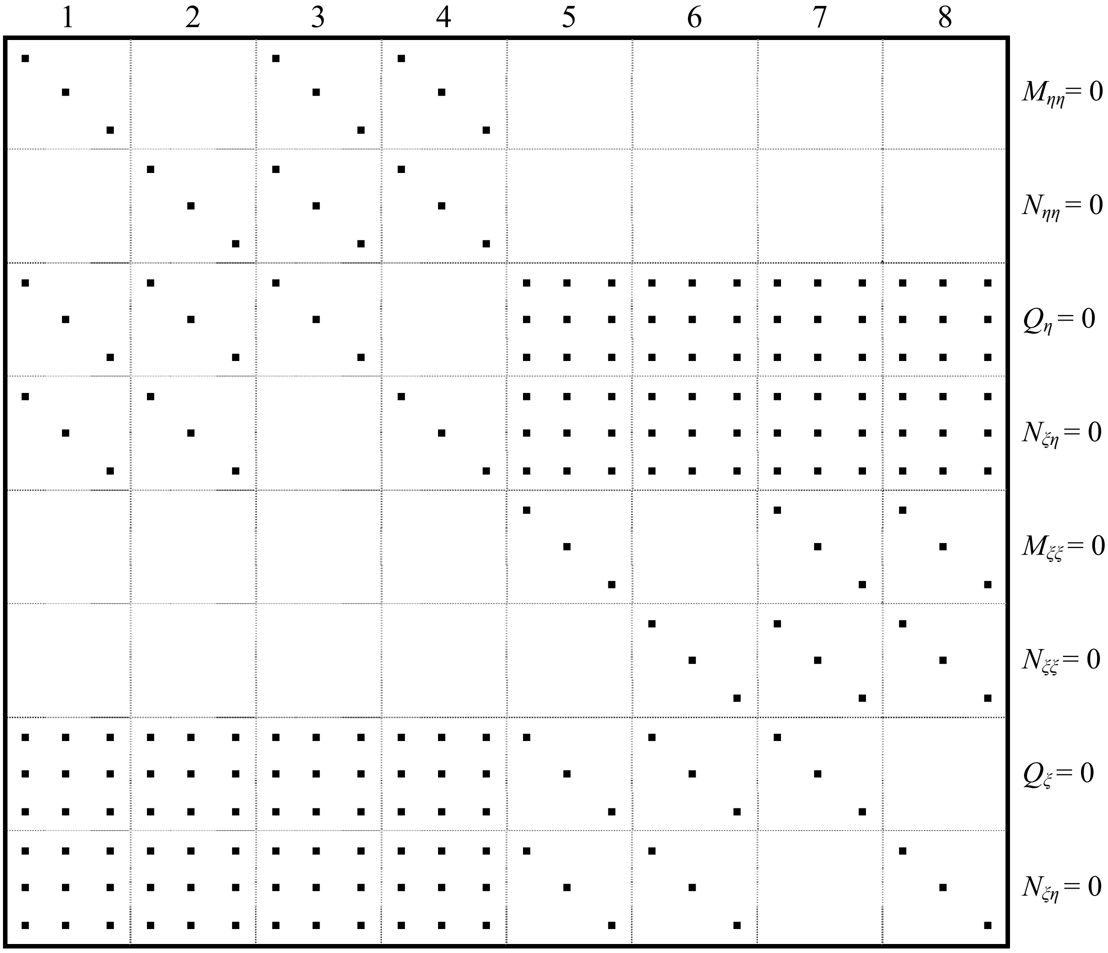

Figure2: A schematic representation of matrix A for $k = 3$.

The Superposition-Galerkin Method requires two numbers for the series summations. One is the number of driving coefficients, which is “k” in Equation (7) and the other is the number of terms in the series expansions in Equations. (8) - (11), which is “K”. It was found that there is no change in fourth digit is after using more than five terms of “k”. However, more than 200 terms of “K” is needed to obtain a converged result.

In computation, extra roots for the natural frequency parameters were also gathered in addition to the wanted natural frequencies. By carefully inspecting the determinant vs frequency parameter plot, some of those unwanted natural frequency parameters are eliminated, however, at the time of writing this paper excluding all of the false roots still remains a challenge. The method is expected to work because the series used are complete and satisfy the governing PDE. It offers the potential to satisfy the edge conditions.

4 Conclusion

A procedure to apply the Superposition-Galerkin method to the vibration analysis of a completely free doubly curved shallow shell is presented. To handle mixed derivatives in the equation of boundary conditions of the free edge, the sine series and the cosine series are used independently in the displacement functions of building blocks. The solution to each building block is obtained using the Galerkin method and expressed in term of its driving coefficients. The building blocks are superimposed to complement each other and satisfy the free edge conditions, which is no net forces or moments at the edge. It is advantageous to use the sine and cosine series because of their simplicity and orthogonality. However, the presence of some extra roots for the natural frequency parameter which are not the actual solution poses a challenge. Currently work is in progress to identify and eliminate these false roots.

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