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Efficient Calculation of the Added Mass Matrix for Vibration Analysis of Submerged Structures

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Abstract

This paper presents an efficient way to calculate the added mass matrix used to solve for the natural frequencies and modes of solids vibrating in an inviscid, incompressible infinite fluid. Finite element method is used to compute the vibration spectrum of just the dry structure, and boundary element method (BEM) is then applied to compute the frequencies and modes of the wetted structure. The BEM does not scale well and results in large computing cost. A reduction of the computational cost to compute the added mass was achieved using a coarse mesh in the BEM and subsequent interpolation to compute the pressure modes at the nodes of a fine mesh from the results of a coarse mesh. Although, damping can also influence the frequencies and modes of the submerged structure, the effects of damping are not taken into account in this work to reduce the cost of the computation time. Excitation of the structure by intermodal coupling is also not taken into account.

Keywords: added mass, wet modes, natural frequencies.

1 Introduction

The natural frequencies and modes of vibration of a structure are usually obtained from the solution of an eigenproblem involving the mass and stiffness matrices of the structure. In some approaches to obtain the natural frequencies of a structure submerged in fluid extra terms are added to the mass matrix. These terms are called the “added mass” matrix, which account for inertia changes associated with the motion of the fluid caused by the body acceleration of the structure. Similar definitions of “added mass” can be found in the literature as in the papers by Lin and Liao [1] and by Gassemi and Yari [2].

Several publications have defined different ways to obtain the added mass matrix or alternatively coupling matrices to solve fluid-structure-interaction (FSI) problems

[1-12]. Three interesting procedures to compute the added mass matrix are described in the work by Geers [3], Deruntz and Geers [4] and by Antoniadis and Kanarachos [5]. Geers [3] used doubly asymptotic approximations (DAA) to obtain the acoustic response of the structure. DAA are differential equations for boundary element analysis used to solve fluid-structure interaction and its formulation is based on the representation of the motion of the surface as a linear combination of orthogonal fluid boundary modes. This procedure avoids the use of the boundary element method, reducing the cost of the analysis. Another advantage of the method is that it is asymptotically exact for low and high frequencies. In the work by Deruntz and Geers [4] a two-dimensional mesh is built on the "wet" surface of the structure as a first step and then the boundary integral method (BIM) is used to discretize the fluid kinetic energy expression. This gives the added mass matrix directly from the matrices obtained in the BIM, providing the velocity of the structure and the fluid are considered to be equal and have common nodes. Antoniadis and Kanarachos [5] presented a general methodology to decouple the structure and fluid domains to solve for modal analysis. They used the common procedure based on the implementation of the finite element method (FEM) to compute the basis vectors of the structure and the corresponding 'mass' and 'stiffness' matrix terms of the 'dry' structure, while the BIM is used to solve a set of potential (Laplacian) problems for the fluid, using the previous basis vectors in the place of the structural displacement at the normal pressure derivative (Neumann) boundary condition on the fluid-structure interface. In addition, Antoniadis and Kanarachos used general coordinate transformations to simplify the expression of the stiffness and mass matrices of the structure. This is the procedure used in this work, with the difference that the pressure modes are first obtained using a coarse mesh and then the pressure mode values at the nodes of a finer mesh are obtained by interpolation. Antoniadis and Kanarachos reported that the FEM can be used instead of the BIM, which is the procedure used by Rajasankar *et al* [8] and more recent publications use the fast multiple boundary element method (FMBEM) such as [1,12].

In acoustic analysis using the FEM it is necessary to include a large amount of degrees of freedom to obtain accurate results. For instance, Jensen *et al* [13] recommend using at least 10 elements per wave length. In addition, it is often required to compute the response of a structure for stimuli at different frequencies. For these reasons, a method to reduce the cost of the simulations is highly desirable.

Analytical solutions for clamped plates have been published and will be used for comparison to evaluate the error of the methodology used in this work. Results of "dry" modes and "wet" modes of a cantilever plate will be presented. The method can be applied to structures with more complex geometry such as biological geometries derived from CT scans (bones plus surrounding water-saturated soft tissues).

2 Theoretical derivations

In this work, the FEM is used to obtain the mass and stiffness matrices, as well as the natural frequencies and the modes of the "dry" structure. In a second step the pressure modes, due to the interaction of the structure with a fluid, are computed using the boundary element method (BEM). Then the added mass matrix is obtained using the "dry" modes and pressure modes.

The main purpose of this work is to reduce the computational cost of the pressure mode calculations to obtain the natural frequencies of submerged structures of complex geometry such as ear bones of marine mammals. The present work follows the procedure given by Antoniadis and Kanarachos [5], but as mentioned earlier the pressure modes are obtained using a coarser mesh and then linear interpolation is used to obtain the pressure modes at the nodes of a finer mesh. The procedure in Antoniadis and Kanarachos [5] showed that the vibration problem of a submerged structure can be decoupled in two main subproblems as follows:

1. Compute the mass and stiffness matrices and solve the eigenproblem of the 'dry' structure using the FEM

$$[\mathbf{K}]\{\mathbf{c}\} - \omega^2[\mathbf{M}]\{\mathbf{c}\} = \{\mathbf{0}\}, \tag{1}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{M}$ is the mass matrix, ω is the circular frequency and $\mathbf{c}$ is a vector of unknown coefficients. The solution of the eigenproblem in Equation (1) gives the natural frequencies ω_i of the structure and the matrix $\mathbf{\Psi}$ which columns $\boldsymbol{\psi}_i$ are the corresponding ith orthogonal eigenvectors.

Then, the matrix $\mathbf{\Psi}$ and its transpose $\mathbf{\Psi}^{\mathrm{T}}$ are used to diagonalize the stiffness and mass matrices, providing the eigenvectors are $\mathbf{M}$-orthonormal

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{K}\mathbf{\Psi} = \Omega^2 \quad \text{and} \tag{2}$$

$$\mathbf{\Psi}^{\mathrm{T}}\mathbf{M}\mathbf{\Psi} = \mathbf{I}, \tag{3}$$

where Ω^2 is a diagonal matrix containing the eigenvalues of Eq. (1) which square root are the circular natural frequencies ω_i of the structure and $\mathbf{I}$ is the identity matrix. In the work by Antoniadis and Kanarachos [5] the stiffness, mass and added mass matrices are normalized in such a way that the stiffness matrix becomes an identity matrix. Following this normalization the m_{ii} terms of the diagonal mass matrix are equal to the inverse of the ith eigenvalues and the terms m_{ij}^{*} of the added mass terms are divided by the square root of the product of the ith and jth eigenvalues. This normalization step is skipped in the present work.

2. Solve the Laplacian problems with the boundary element method for the fluid using the "dry" eigenvectors to obtain the pressure modes.

The set of potential (Laplacian) problems as defined in [5] is:

for an inviscid fluid the potential (Laplace) equation for the fluid domain is

$$\Delta P = 0, \tag{4}$$

where P is the modal amplitude of the fluid pressure. The structure coupling effect at the common fluid-structure surface is

$$\partial p / \partial n = \omega^2 \rho_{\mathrm{F}} \mathbf{U} \cdot \mathbf{n}, \tag{5}$$

where $\partial p / \partial n$ is the normal derivative of the pressure, ρ_{F} is the density of the fluid, $\mathbf{U}$ is the modal amplitude of the structural displacements and $\mathbf{n}$ is the unit normal exterior to a boundary.

The solution of the Laplace equations using the boundary element method with N flat boundary elements E_{i} to discretize the surface of the structure D, together with the point collocation technique as described in the work by Pozrikidis [14] is used here. To compute the function f the following discretized integral equation is applied at the mid-point of each boundary element, denoted by $\mathbf{x}_j^M$, where $j=1,\ldots N$:

$$f\left(\mathbf{x}_j^M\right)=-2\sum_{i=1}^{N}\left(\frac{\partial f}{\partial n}\right)_i\int_{E_i}G\left(\mathbf{x},\mathbf{x}_j^M\right)dS(\mathbf{x})+2\sum_{i=1}^{N}f_i\int_{E_i}^{PV}\left[\mathbf{n}(\mathbf{x})\cdot\nabla G\left(\mathbf{x},\mathbf{x}_j^M\right)\right]dS(\mathbf{x}),\qquad(6)$$

where $\mathbf{x}$ is a vector defining the location of the variable "field point", $\mathbf{x}_0$ is the fixed location of the singular "point" and G is the free-space Green´s function in three dimension's for the Laplace equation

$$G(\mathbf{x},\mathbf{x}_0)=\frac{1}{4\pi r},\qquad(7)$$

where $r=\left|\mathbf{x}-\mathbf{x}_0\right|$.

The integrals of the terms on the right hand side of Eq. (6) are called the single-layer and double-layer integrals respectively. Setting $f\left(\mathbf{x}_j^M\right)=f_j$ and $f_i=\delta_{ij}f_j$ and rearranging

$$\left(A_{ij}-\frac{1}{2}\delta_{ij}\right)f_i=B_{ij}\left(\frac{\partial f}{\partial n}\right)_i,\qquad(8)$$

where δ_{ij} is the Kronecker's delta and the coefficient matrices A_{ij} and B_{ij} are

$$A_{ij}\equiv\int_{E_i}^{PV}\left[\mathbf{n}(\mathbf{x})\cdot\nabla G\left(\mathbf{x},\mathbf{x}_j^M\right)\right]dS(\mathbf{x})\qquad(9)$$

$$B_{ij}\equiv\int_{E_i}G\left(\mathbf{x},\mathbf{x}_j^M\right)dS(\mathbf{x})\qquad(10)$$

where PV denotes the principal-value integral, which applies only to the elements that share the evaluation point $\mathbf{x}_0$.

In this work quadrilateral boundary elements discretize the surface of the structure. The source strength of the potential is assumed to be uniform in each panel. To evaluate the coefficients A_{ij} and B_{ij} in Equations (9,10) it is necessary to rely on numerical integration. The coefficients of A_{ij} exhibit a singularity $1/r$, whilst the coefficients of B_{ij} exhibit a singularity of $1/r^3$ which are characteristic of the Green's functions of the Laplace's equation in three dimensions.

For a flat element $\nabla G\left(\mathbf{x},\mathbf{x}_j^M\right)=0$, because the normal $\mathbf{n}(\mathbf{x})$ and the vector $\left|\mathbf{x}-\mathbf{x}_0\right|$ are perpendicular, thus

$$\mathbf{n}(\mathbf{x})\cdot\nabla G\left(\mathbf{x},\mathbf{x}_j^M\right)=\mathbf{n}(\mathbf{x})\cdot(\mathbf{x}-\mathbf{x}_0)\frac{1}{4\pi r^3}=0\qquad(11)$$

This reduces the computational cost and the coefficients of A_{ij} for flat boundary

elements have zero elements for $i = j$. Furthermore, in the case of the plate all A_{ij} elements are zero as all elements lie on the same plane.
The following schemes were selected to perform the single-layer integrals

$$A_{ij} = \begin{cases} 0 & i = j \\ w\nabla G \det \mathbf{J} & i \neq j \end{cases}, \tag{12}$$

where $\det \mathbf{J}$ is the determinant of the Jacobian.
The following schemes were selected to perform the double-layer integrals

$$B_{ij} = \begin{cases} 7.050989 wG \det \mathbf{J} & i = j \\ wG \det \mathbf{J} & i \neq j \end{cases} \tag{13}$$

The integration schemes applied to A_{ij} and B_{ij} for $i \neq j$ correspond to the Gauss-Legendre quadrature in two dimensions [15] and one point of integration was used to reduce the computation cost. The integration scheme applied to B_{ij} for $i = j$ was performed according to the numerical integration given by Kwak in [9] using two integration points. After computing the coefficients A_{ij} and B_{ij} it is possible to solve the Laplace problem, which gives a function f that corresponds to the pressure modes of the structure evaluated at the center of the elements. The pressure modes were then evaluated at the nodes of a finer mesh using the following equation [14]:

$$f_i = \frac{\sum_{j=1}^{ne_i} f_j^E A_j^E}{ne_i \sum_{j=1}^{ne_i} A_j^E}, \tag{14}$$

where ne_i is the number of elements sharing the ith node, f_j^E are the values of the function on the elements that share that node, A_j^E are the areas of the jth elements and f_i is the value of the function at the ith node.
After this it is possible to compute the terms of the added mass matrix $\mathbf{M}^*$ as follows:

$$\mathbf{M}^*_{ij} = \int_D \rho_{\mathrm{F}} \boldsymbol{\psi}_i \cdot f_j \mathbf{n} dD \tag{15}$$

and the natural frequencies of the wet structure are obtained solving the eigenproblem

$$\left[\boldsymbol{\Omega}^2\right]\{\mathbf{c}\} - \omega^2\left[\mathbf{I} + \mathbf{M}^*\right]\{\mathbf{c}\} = \{\mathbf{0}\}, \tag{16}$$

while the wet modes of vibration are obtained multiplying the eigenvectors of Equations (1) and (16).

3 Examples

The solution of a cantilever plate and a free plate were obtained using the free software FAESOR by P. Krysl, available at http://hogwarts.ucsd.edu/~pkrysl/. It is also possible to obtain the solution of structures with more complex geometries such as marine mammal ear bones building the finite element mesh with the free software iso2mesh available at http://iso2mesh.sourceforge.net/cgi-bin/index.cgi [16].

a) Cantilever square plate

This example shows the results of the natural frequencies and modes of vibration of a plate in cantilever submerged in water. The plate has dimensions a=10 m and b=10 m along directions x and y, thickness h=0.238 m, while the material properties are Poisson's ratio ν=0.3, density ρ_p=7850 kg/m^3 and Young's modulus E=206000 GPa. The density of the water is ρ_w=1000 kg/m^3. The natural frequencies of the dry plate presented in Table 1 were obtained using the finite element code FAESOR using H8 elements. Results obtained using the Rayleigh-Ritz method [17], the commercial FEM code COMSOL [18] and experimental results presented by Lindholm *et al* [19] are used to compare the results obtained using the present approach implemented in the free code FAESOR. Results using FAESOR converge from below as the number of elements of the model of the structure increases. Two sets of results using FAESOR are presented in Table 1. The first model was built using a coarse mesh with 4 elements along the sides aligned in the x and y directions and 7 elements across the thickness, while in the second model has a finer mesh with 40 elements in the x and y directions and 7 elements across the thickness. The fine mesh results using FAESOR compare well with the results using the Rayleigh-Ritz method and COMSOL. Results using the Rayleigh-Ritz method were obtained from the non-dimensional frequency parameters presented in [17].

Mode	[19] Test rad/s	[17] Ritz rad/s	COMSOL rad/s	4,4,7 FAESOR rad/s	40,40,7 FAESOR rad/s
1	12.30	12.70	12.84	12.78	12.74
2	30.78	31.13	31.27	29.53	30.87
3	75.46	77.88	78.53	74.08	77.77
4	99.84	99.50	99.96	88.14	98.98
5	110.57	113.28	113.65	102.31	112.18

Table 1. Dry natural frequencies of a clamped plate.

The results of the natural frequencies of the plate in cantilever submerged in water are presented in Table 2. Results obtained in FAESOR are compared to the experimental results presented in Lindholm [19], the numerical results published by Fu and Price [11] and results obtained using the commercial code COMSOL. The first two sets of results using the procedure described in this work correspond to the models presented in Table 1 and the pressure modes were computed directly. In the last column of Table 2, results of a third model are presented interpolating the pressure modes of the coarse model using 4 elements per side in a finer mesh built by 40 elements per side.

Mode	[19] Test rad/s	[11] Numerical rad/s	COMSOL rad/s	4,4,7 FAESOR rad/s	40,40,7 FAESOR rad/s	4,4,7 40,40,7 FAESOR rad/s
1	6.56	7.35	7.34	6.21	7.18	6.16
2	19.66	20.2	20.74	17.64	21.06	18.51
3	45.32	50.45	49.27	48.08	51.75	51.05
4	68.18	70.41	69.67	60.36	72.01	73.87
5	74.69	78.85	78.96	72.61	81.01	80.04

Table 2. Wet natural frequencies of a clamped plate.

Results show that refining the mesh from 4 to 40 elements per side and interpolating the pressure modes did not improve the approximation of the fundamental frequency, but a better approximation was obtained for the second mode, while results for the 3rd, 4th and 5th modes were very similar to those obtained with a finer mesh using 40 elements per side also included in Table 2.

It should be clear that the results of the first five modes of vibration of the cantilever plate obtained in COMSOL shown in Table 2 correspond to the same modes of vibration obtained with the present approach implemented in FAESOR. The modes of vibration obtained with the present approach for the submerged plate are presented in Figure 1.

b) Free square plate

In this section, results of the natural frequencies and modes of vibration of the plate presented above, but in completely free condition are presented using the H20 element in the finite element code FAESOR. Table 3 shows the results of the dry frequencies obtained using the Rayleigh-Ritz method [17], the commercial FEM code COMSOL [18] and two sets of results using FAESOR. The results using the present approach correspond to models with 4 and 20 elements along the sides aligned in the x and y directions and 2 elements across the thickness. Results using the finer mesh obtained using FAESOR compare well with the results obtained by the Rayleigh-Ritz method and COMSOL. Results of the six rigid body modes (zero frequencies) are omitted in all cases.

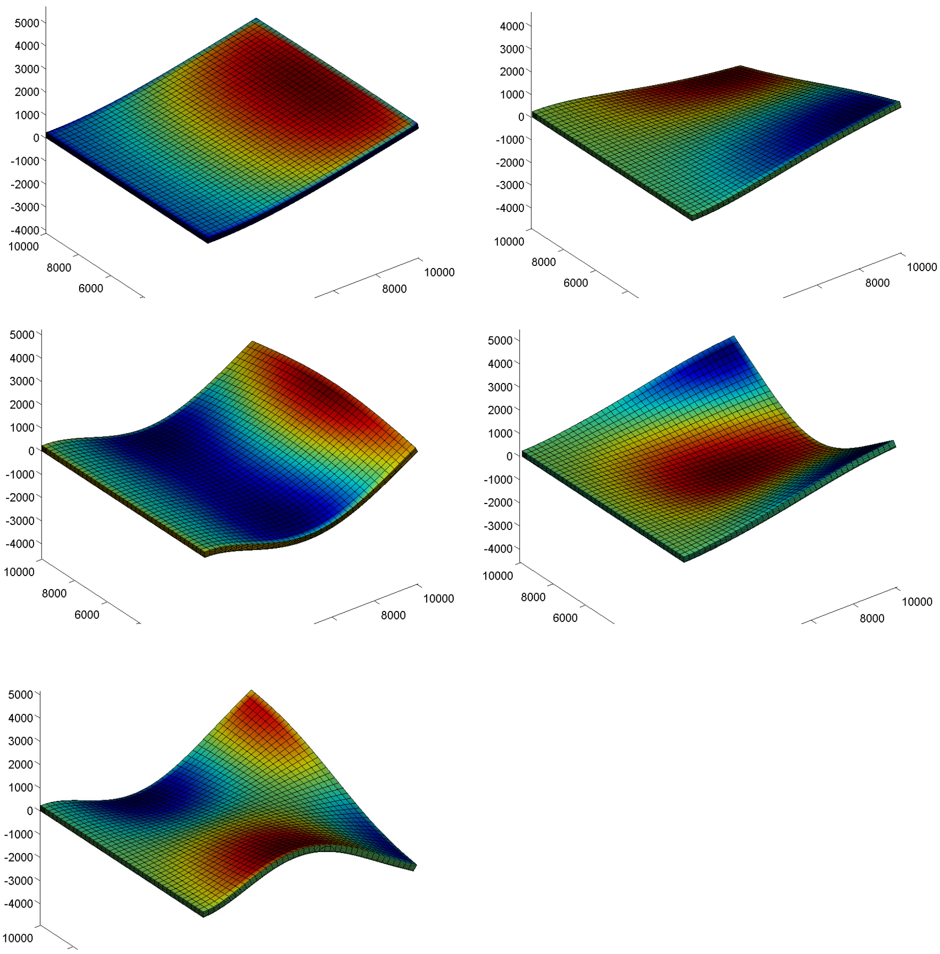

Fig. 1. Modes of vibration of a submerged cantilever plate. The figures correspond to mode 1 (top-left), mode 2 (top-right), mode 3 (center-left), mode 4 (center-right) and mode 5 (bottom-left). Signed pressure is color-coded (red is positive, blue is negative).

Mode	[17] Ritz rad/s	COMSOL rad/s	4,4,2 FAESOR rad/s	20,20,2 FAESOR rad/s
1	49.69	49.33	47.09	49.16
2	72.30	72.18	65.31	71.76
3	89.54	89.38	83.65	88.94
4	128.40	127.36	114.66	126.43
5	128.40	127.37	114.66	126.43

Table 3. Dry natural frequencies of a free plate.

The results of the natural frequencies of the free plate submerged in water are presented in Table 4. Results obtained in FAESOR are compared to the results obtained using the commercial code COMSOL. The first three sets of results using the procedure described in this work correspond to the models presented in Table 1 computing the pressure modes at the nodes directly. The last column of Table 4 presents results using interpolation to obtain the pressure modes of a fine mesh from a coarser mesh. Results show that refining the mesh from 4 to 20 elements per side and interpolating the pressure modes improved the approximation of the first four non-zero frequencies. Results for higher frequencies were difficult to obtain, possibly due to the symmetry of the modes with respect to the neutral plane of the plate. The modes of vibration obtained with the present approach for the submerged plate are presented in Figure 2.

Mode	COMSOL rad/s	4,4,2 FAESOR rad/s	20,20,2 FAESOR rad/s	4,4,2 20,20,2 FAESOR rad/s
1	35.08	30.30	33.74	32.29
2	51.21	45.38	50.14	52.16
3	59.55	58.49	61.52	63.46
4	89.25	86.68	90.96	91.73

Table 4. Wet natural frequencies of a free plate.

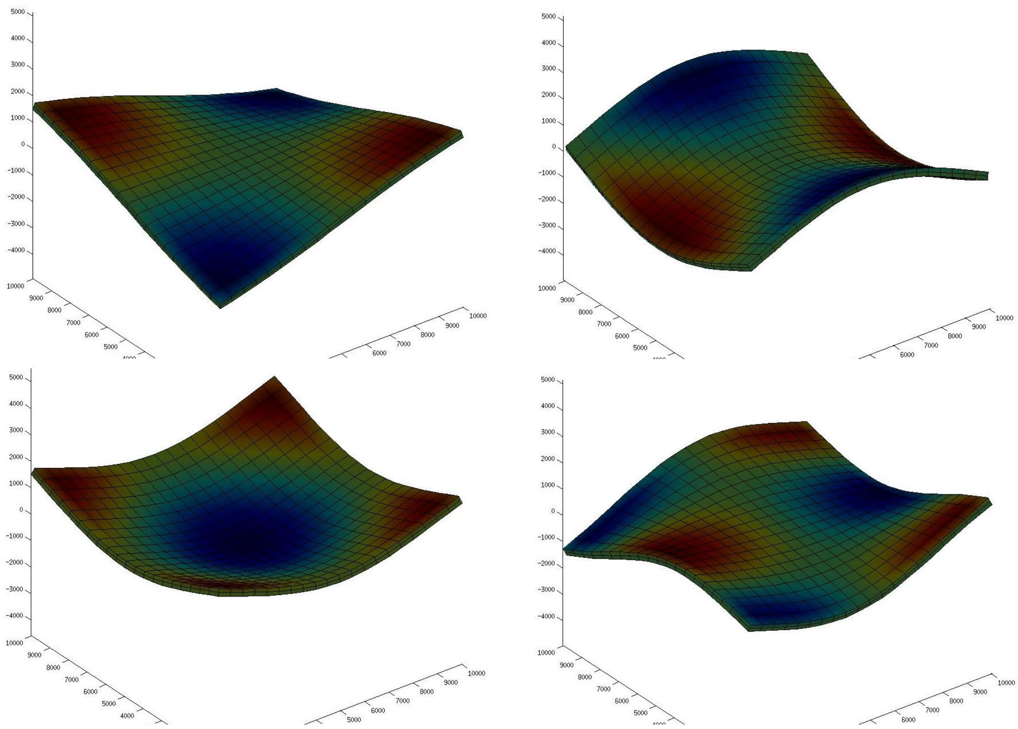

Fig. 2. Modes of vibration of a submerged free plate. The figures correspond to mode 1 (top-left), mode 2 (top-right), mode 3 (bottom-left) and mode 4 (bottom-right). The displacement amplitude is color-coded (red is positive, blue is negative).

4 Conclusions and future work

Results obtained using the present approach reduced the computational cost of the simulations. For instance the computation of the pressure modes of the cantilever plate can be reduced from 5 min 24 s using 40 elements along the sides of the plate to 0.054 s using 4 elements along the sides of the plate. The reduction was achieved interpolating the values of the pressure modes. This avoids the computation of a large number of numerical integrations in the BEM, as well as solving systems of coupled algebraic equations with large dense matrices to obtain the results of the Laplacian problems. To improve results (especially of the fundamental frequency) future work will include the investigation of other interpolation methods, such as Laplacian interpolation. Implementation of tetrahedral and triangular elements in the FEM and BEM is also being considered.

References

[1] Z. Lin, S. Liao, "Calculation of added mass coefficients of 3D complicated underwater bodies by FMBEM". Communications in Nonlinear Science and Numerical Simulation. 16, 187–194, 2011.

[2] H. Ghassemi, E. Yari, "The added mass coefficient computation of sphere, ellipsoid and marine propellers using Boundary Element method", Polish Maritime Research. 1(68) Vol. 18, 17-26, 2011, DOI: 10.2478/v10012-011-0003-1.

[3] T.L. Geers, "Doubly asymptotic approximations for transient motions of submerged structures", Journal of the Acoustical Society of America, 64(5), 1500-1508, 1978.

[4] J.A Deruntz, T.L. Geers, "Added mass computation by the boundary integral method", International Journal for Numerical Methods in Engineering, 12, 531-548, 1978.

[5] I. Antoniadis, A. Kanarachos, "A decoupling procedure for the modal analysis of structures in contact with incompressible fluids", Communications in Applied Numerical Methods, 3, 507-517, 1987.

[6] G. Sandberg, "A new strategy for solving fluid-structure problems", International Journal for Numerical Methods in Engineering, 38, 357-370, 1995.

[7] B. Ugurlu, A. Ergin, "A hidroelasticity method for vibrating structures containing and/or submerged in flowing water", Journal of Sound and Vibration, 290, 572-596, 2006.

[8] J. Rajasankar, N.R. Iyer, V.S.R.A. Rao, "A new 3-D finite element model to evaluate added mass for analysis of fluid-structure interaction problems", International Journal for Numerical Methods in Engineering", 36, 997-1012, 1993.

[9] M.K. Kwak, Hydroelastic vibration of rectangular plates. Journal of Applied Mechanics, 63, 110-115, 1996.

[10] G.C. Everstine, "Prediction of low frequency vibrational frequencies of submerged structures", Journal of Vibration and Acoustics, 113, 187-191, 1991.
[11] Y. Fu, W.G. Price, "Interactions between a partially or totally immersed vibrating cantilever plate and the surrounding fluid", Journal of Sound and Vibration 118(3), 495-513, 1987.
[12] M. Wilken, G. Of, C. Cabos, O. Steinbach, "Efficient calculation of the effect of water on ship vibration", in "Analysis and Design of Marine Structures", Guedes Soares and Das, (Editors), Taylor m.& Francis, London, 93-101, 2009.
[13] F.B. Jensen, W.A. Kupperman, M.B. Porter, H. Schmidt, "Computational Ocean Acoustics", Published Woodbury, New York : American Institute of Physics, 1994.
[14] C. Pozrikidis, "A practical guide to boundary element methods with the software library BEMLIB", Chapman & Hall/CRC, 2002.
[15] C. Pozrikidis, "Numerical computation in engineering and science", New York: Oxford University Press, 1998.
[16] iso2mesh: URL: http://iso2mesh.sourceforge.net.
[17] L.E. Monterrubio, S. Ilanko, "Sets of admissible functions for the Rayleigh-Ritz method, Monterrubio, L.E., The Eleventh International Conference on Computational Structures Technology. Dubrovnik, Croatia, September 4-7, 2012, In print.
[18] COMSOL, Acoustics Module User Guide, 2009.
[19] U.S. Lindholm, D.D. Kana, W.H. Chu, H.N. Abramson, "Elastic vibration characteristics of cantilever plates in water", Journal of Ship Research, 9, 11-22, 1965.