Paper 97

©Civil-Comp Press, 2012
Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Sets of Admissible Functions for the Rayleigh-Ritz Method

L.E. Monterrubio[1] and S. Ilanko[2]
[1]Department of Structural Engineering
University of California, San Diego, United States of America
[2]School of Engineering
The University of Waikato, Hamilton, New Zealand

Abstract

This paper presents a discussion on the characteristics of sets of admissible functions to be used in the Rayleigh-Ritz method (RRM). Of particular interest are sets that can lead to converged results when penalty terms are added to model constraints and interconnection of elements in vibration and buckling problems of beams, as well as plates and shells of rectangular planform. The discussion includes the use of polynomials, trigonometric functions and a combination of both. In the past, several sets of admissible functions that have a limit on the number of terms that can be included in the solution without producing ill-conditioning were used. On the other hand, a combination of trigonometric and low order polynomials have been found to produce accurate results without ill-conditioning for any number of terms and any number of penalty parameters that can be accommodated by the computer memory.

1 Introduction

1.1 Comparison functions and admissible functions of the Rayleigh-Ritz method

In [1] Meirovitch states that the classical Rayleigh-Ritz method consists of selecting N comparison functions u_i to be included in the Rayleigh quotient. These functions must satisfy natural and geometric boundary conditions and be differentiable $2p$ times (p being the order of the highest differential operator in the functional used) to construct the linear combination

$$w_n = \sum_{i=1}^{N} a_i u_i ,$$

where a_i are unknown coefficients. However, it was noted that a set of admissible functions ϕ_i, which have to satisfy only geometric boundary conditions and be only

p times differentiable can be used instead. Furthermore, Meirovitch [1] also states that orthogonal and normalized functions such as Bessel functions, Legendre polynomials, etc., as well as the Gram-Schmidt orthogonalization process have been often used aiming to reduce computational work, although these operations add computational cost. It is also worth noting that comparison functions are a subset of the admissible functions.

1.2 Building sets of admissible functions with simple polynomials

Simple polynomials have a severe limitation on the number of terms that can be included in the solution before an ill-conditioning problem arises. Other sets of admissible functions built by orthogonal polynomials using the Gram-Schmidt process presented by Bhat in [2] have been proven to give excellent results for plates involving free edges, as shown in a publication by Yuan and Dickinson [3]. This procedure has been used to build sets of admissible functions by many researchers, even though Brown and Stone raised some criticism of this work in [4], where it is stated that the convergence of a vibration problem is independent of the selection of the set of admissible functions (no need for orthogonal polynomials) and that it depends only on the degree of the polynomial represented in the set. In the same work Brown and Stone stated that for plate problems, orthogonality of the functions should be targeted only on the second derivative of the functions, although they also recognized that special polynomials are only needed if higher order polynomials are included in the set of admissible functions. This is to make the set of functions more stable with respect to inversion and the extraction of eigenvalues of the resulting stiffness and mass matrices, although in [5] Li reported that even when orthogonal polynomials are used in the RRM, the higher order polynomials become numerically unstable due to round-off errors.

1.3 Building sets of admissible functions with transcendental functions

Transcendental functions also have some disadvantages. For instance, Li and Daniels [6] show that certain sets of admissible functions built by trigonometric functions have limitations converging when penalty parameters are included in the solution. Sets of functions using trigonometric and hyperbolic functions are very complex and are likely to become numerically unstable when several terms are used in the solution. This was noticed by Blevins [6] who recommends using a high degree of precision when higher modes are included, as well as Jaworski and Dowell [7] who used trigonometric and hyperbolic functions to solve vibration problems of beams with multiple steps using a set of functions for clamped-free beams. Jaworski and Dowell reported that numerical problems arise due to the difference between the values of the hyperbolic functions. In [7] the set of admissible functions built by trigonometric and hyperbolic functions was substituted by an approximation in higher modes with a combination of sine, cosine and exponential functions previously used by Dowell [8]. More recently Dozio [9] published a comprehensive study on the use of a set of trigonometric functions, originally proposed by Beslin and Nicolas [10], used to solve vibration problems of rectangular orthotropic plates. The method by Dozio offers the same advantages as the proposed set of functions of the present work, but the matrices of the system are built with more complex terms, and even though many terms of the matrices using the set of admissible functions by Beslin and and Nicolas [10] become zero, the matrices of the present method are even less sparse.

1.4 Building sets of admissible functions from polynomial and trigonometric functions

In contrast with all the previous options to build sets of admissible functions, several publications including the works by Li [5,11] and Zhou [12] have shown that when polynomials and trigonometric functions are used to build sets of admissible functions, the solutions have a fast convergence rate and results are also accurate for higher modes. Although it is now known that only the sum of the series of the functions should satisfy the boundary conditions, many researchers have proposed to build sets of functions starting with a series containing trigonometric and polynomial functions, but enforcing boundary conditions for each term. This approach was used in [5,11,12]. Li [5,11] built a series of admissible functions by mixing polynomials and trigonometric (cosine) functions. Li stated that the polynomials are introduced to take all the relevant discontinuities with the original displacement and its derivatives at the boundaries. More recently Dal and Morgul [13] presented a similar approach to those presented by Li [5,11] and Zhou [12]. Dal and Morgul used sine functions as in the work by Zhou [12] and also enforced boundary conditions for each term. Polynomials in the publications by Li [5,11] were of order 4, while in the approaches of Zhou [12] and Dal and Morgul [13] the maximum order of a polynomial was 3.
It is important to remember that high order polynomials are the cause of numerical instabilities and ill-conditioning. Thus to keep the solution as simple as possible and free of numerical problems the minimum number of polynomial functions with the lowest order possible are included in the proposed set of admissible functions presented in this work.

2 Building a set of admissible functions

As mentioned earlier, this work presents a set of functions that can be used to model beams, plates and shells; converges fast and allows the use of a large number of functions without causing ill-conditioning. In addition, the selected set of admissible functions models a structure in a completely free condition and complex boundary conditions can be modelled adding as many constraints as necessary using penalty functions.
In the past some researchers gave guidelines to develop sets of admissible functions such as the ones given in [14] as follows:

a) the set of functions must be complete in energy form (all modes of vibration must be represented and no modes must be missing),

b) the set of functions must be linearly independent,
c) the functions must satisfy boundary conditions and
d) the functions must have derivatives at least up to half of the order of the partial differential equation.

Here a more intuitive method was used to build a set of admissible functions keeping in mind that in vibration problems the stiffness matrix includes derivatives up to the second order with respect to the same variable. The first step in the procedure to build a set of admissible functions combining trigonometric and simple polynomials is to select the trigonometric function. Sine functions are used as a set of functions to exactly model simple supported structures as they constrain the displacement at both ends of the structure, while rotation is allowed. On the other hand, cosine functions constrain rotation and allow translation, modelling sliding structures also in an exact way. Sliding condition is very useful when symmetry is used to model symmetrical modes using half of the structure.

In [15] Budiansky and Hu implemented Lagrangian multipliers in the RRM to constrain edges of a plate. Budiansky and Hu showed that the rate of convergence of the RRM together with the Lagrangian Multiplier Method is faster when a cosine series is used to build the set of admissible functions together with translational constraints to model clamped conditions than the combination of a set of admissible functions built by sine series and rotational constraints.

Similarly, in [16] Li presented a comparison of the convergence of the RRM with admissible functions built by either a sine or a cosine series plus a polynomial. In [16] convergence rates for most boundary conditions were also found to be faster using cosine series than sine series. As expected, in this work by Li, sine series have their fastest rate of convergence for simply supported conditions, while cosine series have their fastest rate of convergence for sliding conditions. In these cases they represent exact modes if the beams are uniform and have no discontinuities. For this reason, cosine series were selected in this work to build the set of admissible functions of a free-free beam. The cosine series used in this work is defined as

$$\cos\left(\frac{i\pi x}{L}\right), \qquad \text{for} \qquad i = 0,1,2\ldots n \tag{1}$$

where x is the axial coordinate of the beam, L is the beam length and n is the number of terms included in the set of admissible functions. Now, it is only necessary to define the simple polynomials in terms of the coordinate system that should be used in the set of admissible functions together with the cosine series. This can be started knowing that the series must include the rigid-body modes of the beam and as mentioned earlier that the set of admissible functions should satisfy the boundary conditions as a whole series and not individually. Thus to keep the solution as simple as possible and free of numerical problems the minimum number of polynomial functions with the lowest order possible (to minimise the chances of ill-conditioning) are included in the set of admissible functions presented in this work. Then, the two rigid-body modes of a beam should be represented by a unit function and a linear function as in the work by Bassily and Dickinson [17], Warburton [18] and Zhou [12]. The unit term releases the translational rigid-body mode and the linear term releases a rotational rigid-body mode. It is important to note that the unit function is already included in the cosine series for $i = 0$, although for simplicity the unit function will be used in the notation. Next, by inspection it is observed that to satisfy all possible combinations of boundary conditions it is only necessary to add one more function that allows a second non-zero slope at one of the ends of the beam. Thus, a square term is added to the set of functions, because it is the lowest order polynomial that can be added to the series.

This completes the set of admissible functions $\phi_i(x)$ used in this work that it is defined as

$$\phi_i(x) = 1, \qquad \text{for} \quad i = 1 \tag{2a}$$

$$\phi_i(x) = \left(\frac{x}{L}\right), \quad \text{for} \quad i = 2 \tag{2b}$$

$$\phi_i(x) = \left(\frac{x}{L}\right)^2, \quad \text{for} \quad i = 3 \tag{2c}$$

$$\phi_i(x) = \cos\frac{(i-3)\pi x}{L}, \quad \text{for} \quad i = 4,5,\ldots n \tag{2d}$$

The set of admissible functions given in Equation (2) are used in the RRM to model the transverse deflection of the beam as

$$w(x,t) = W(x)\sin(\omega t), \tag{3a}$$

where $W(x)$ is the amplitude of the deflection of the neutral axis of the beam defined as

$$W(x) = \sum_{i=1}^{n} c_i \phi_i(x), \tag{3b}$$

where c_i are arbitrary coefficients.

A very important property of the Fourier series is that they are nominally orthogonal functions with respect to each other when integrated to the full span [0 to L]. This property can be defined for cosine functions with the following relationship [19]:

$$\int_0^L \cos\frac{i\pi x}{L}\cos\frac{j\pi x}{L}dx = \begin{cases} 0 & \text{for} \quad i \neq j \\ L/2 & \text{for} \quad i = j \end{cases}, \tag{4}$$

A similar relationship applies for sine series. A property of orthogonal functions is that their first and second derivatives are also orthogonal [19]. This property is very useful to obtain the terms of the elastic stiffness, geometrical stiffness and mass matrices of beams, plates and shells. Sets of orthogonal functions used in the RRM produce diagonal mass and stiffness. However, as stated in [20] by Mukhopadhyay a good set of admissible functions may be chosen, so that off-diagonal terms will be relatively small. In the present work, the mass matrix of a beam has off-diagonal terms only in the first three rows and columns, corresponding to the terms that involve the linear and square functions. This is because the linear and square functions are not orthogonal with respect to any other function of the set. The absolute value of the off-diagonal terms of the mass matrix decreases as the number of terms in the set of admissible function increases, starting with the values of the fourth admissible function. However the non-zero second derivative of the series is orthogonal as suggested by Brown and Stone [4]. Thus, the stiffness matrix of a

beam derived with the present set of admissible functions results in a diagonal matrix, although the values of the first two terms in the main diagonal are zero.
In the cases of a completely free plate or a completely free shallow-shell modelled by the set of admissible functions given in Equation (2), the stiffness and mass matrices are sparse. Furthermore, even though neither orthogonalization nor orthonormalization is carried out to define the set of admissible functions presented here, the set of admissible functions does not produce ill-conditioning due to the number of terms used in the series as shown by Monterrubio in [21,22,23]. This was demonstrated for several vibration and buckling problems involving beams, plates and shells and certain connected structures. The aim of the present work in contrast to the previous work by Monterrubio is to show how the traditional procedure to build sets of admissible functions was used to model free structures combining trigonometric functions and simple polynomials. The procedure presented here is extremely simple and the functions were still carefully selected to obtain the simplest set of functions that converges fast and does not have a limitation in the number of functions due to numerical instabilities.
The set of functions given in Equation (2) as was done by Li in [5,11] use a cosine series and a polynomial. The main difference between the present approach and those in [5,11] is that even though the structures could be defined to be completely free in the work by Li, admissible functions were still obtained solving for boundary conditions of a structure with elastic boundary supports.
Next, a comparison between the present set of admissible functions with the Legendre polynomials is carried out to show that these two sets of admissible functions model a free-free beam. The Legendre polynomials are obtained starting from a simple polynomial with the lowest degree that satisfies the boundary conditions of the problem and obtaining the rest of the polynomials using the Gram-Schmidt orthogonalization process [14]. The Legendre polynomials have been used to solve vibration problems of completely free plates [14].
The Legendre polynomials are defined by the following Rodrigue's formula [24]:

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n \left(x^2 - 1\right)^n, \tag{5}$$

Then the first six polynomials are

$$P_0(x) = 1, \tag{6a}$$

$$P_1(x) = x, \tag{6b}$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right), \tag{6c}$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right), \tag{6d}$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right) \quad \text{and} \tag{6e}$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right), \tag{6f}$$

Comparing Figures 1 and 2, it is obvious that the first two functions of both the Legendre polynomials and the set of admissible functions developed in this work are identical. Furthermore the third function in both sets is a square function (the function in the Legendre polynomials is a linear combination of a square term and a constant, both of which appear in the proposed set); while the following functions of both series add a nodal point to the previous function. This makes clear that not all functions satisfy the free boundary conditions at both ends, but as stated by Budiansky and Hu 1946 in [15], the boundary conditions do not have to be satisfied individually by the functions in the set of admissible functions, but by the expansion of the whole set of admissible functions.

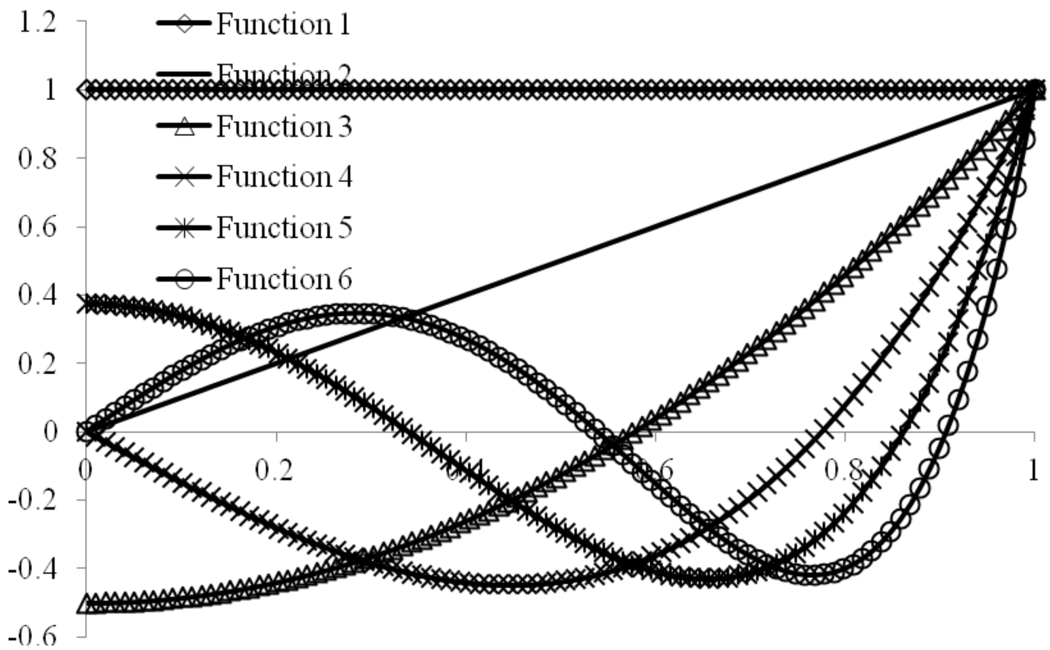

Figure 1. First six Legendre polynomials.

To further clarify the role of the functions on the boundary conditions at the ends of the beam, the following inequalities show that the selected set of functions permit non-zero displacement and translation at both ends

$\phi_i(0) \neq 0$ This condition is satisfied by Equations (2a,2d),

$\phi_i(L) \neq 0$ All functions included in the set defined in Equations (2a,2d) satisfy this condition,

$\left.\frac{\partial \phi_i}{\partial x}\right|_{x=0} \neq 0$ This condition is only satisfied by the linear term defined in Equation (2b) and

$\left.\frac{\partial \phi_i}{\partial x}\right|_{x=L} \neq 0$ This condition is satisfied by the linear and square terms defined in Equations (2b,2c).

The argument above shows that the proposed set of admissible functions is a complete set, which models the deflection of a free-free beam.

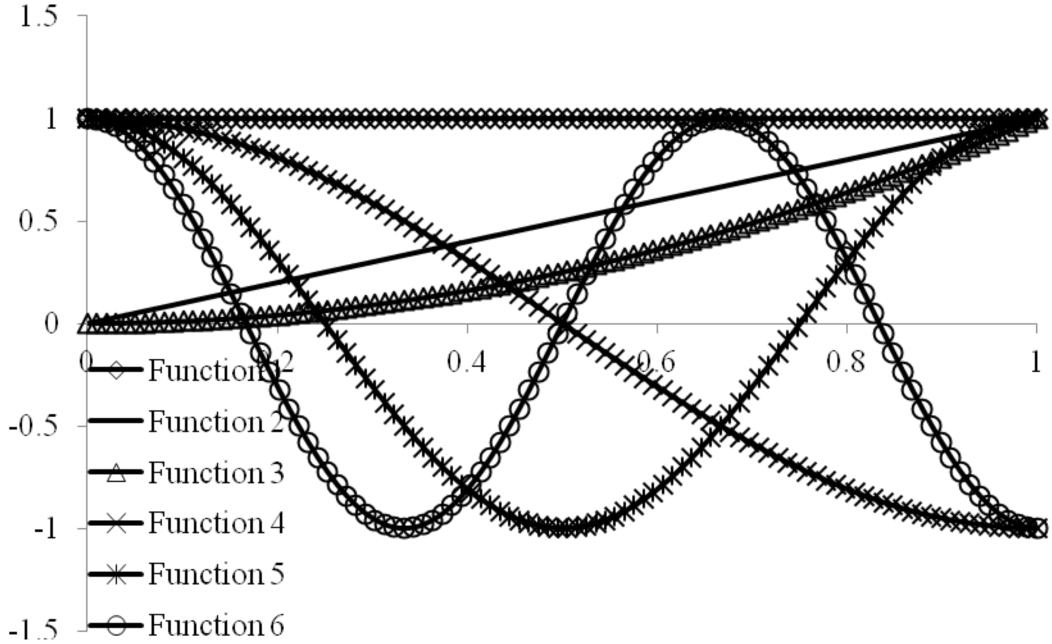

Figure 2. First six admissible functions of the present work.

3 Additional comments on the Rayleigh-Ritz method

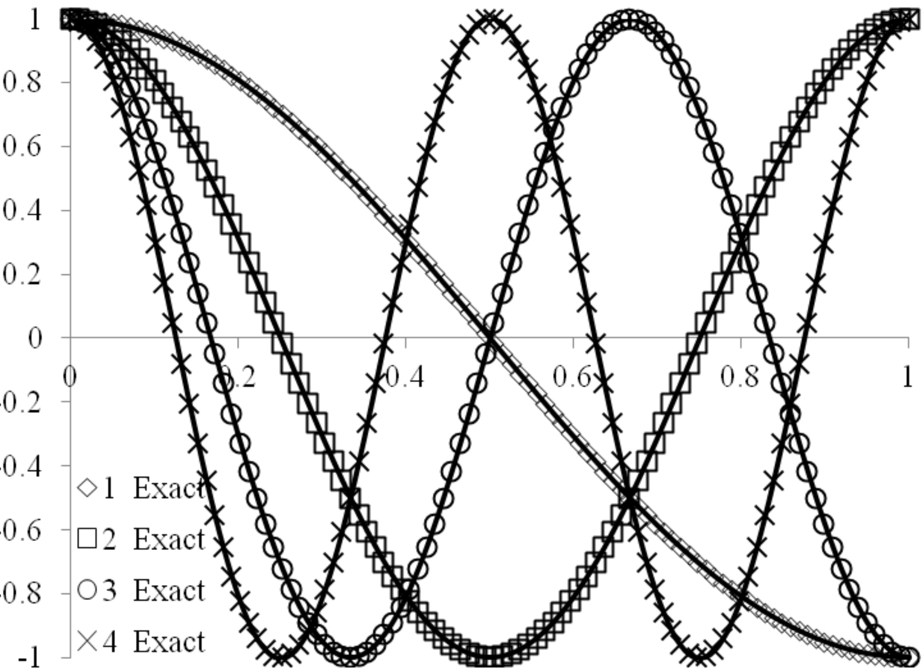

Fig. 3 Modes of vibration of a guide-guide beam.

In Kohn [25] it is stated that the Rayleigh-Ritz method generally gives better approximations of the eigenvalues than of the eigenfunctions and that the magnitude of the errors of the eigenvalues and eigenvectors depends on the smoothness of the set of admissible functions. The set of admissible functions presented here is built by a cosine series and two simple polynomials (a linear term and a square term). Cosine functions are the exact modes of a guide-guide beam and with the addition of a unit function results in modes very similar to the clamped-clamped and free-free modes of a beam (except for the first two modes). Figure 3 shows that the results of the first four non-zero modes of vibration of a G-G beam (line) match the exact results (markers - no line) using the first 7 functions of the set presented in Equation (2) in the RRM as to be expected. Similarly, good approximations to the first four modes of a S-S beam can be obtained using 14 terms in the set of admissible functions. The S-S is the case that has the slowest convergence with respect to the number of functions.
In a publication by Williams [26] two important characteristics of the classical RRM (without penalty parameters) are mentioned. The first characteristic is that when the RRM is used to solve vibration problems, the natural frequencies converge monotonically from above as the number of terms in the set of admissible functions is increased and the second characteristic is that the lower modes converge first.

4 Examples

To show the versatility and stability of the set of admissible functions presented in Equation (2) the first six natural frequencies of thin plates with free (F), simply supported (S), guided (G) and clamped edges (C) are presented using 40 terms in each direction. Consider a rectangular plate as shown in Figure 4, with dimensions a and b along directions x and y, thickness h and flexural rigidity D defined as

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}, \tag{7}$$

where ν is Poisson's ratio and E is Young's modulus.

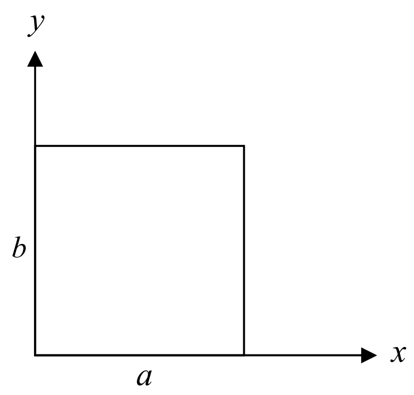

Figure 4. Completely free rectangular plate.

The amplitude of the deflection of the plate defined in terms of the set of admissible functions is

$$W(x,y)=\sum_{j}^{n}\sum_{i}^{n}c_{ij}\phi_i(x)\chi_j(y), \tag{8}$$

where c_{ij} are arbitrary coefficients.

The maximum potential and kinetic energy terms [14] for thin rectangular plates are as follows. The maximum potential energy of the plate V_{plate} due to the strain energy of bending and twisting of the plate is

$$V_{plate}=\frac{D}{2}\int_0^a\int_0^b\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2+2\nu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}+2(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]dxdy \tag{9}$$

The maximum kinetic energy T_{max} of the plate is

$$T_{plate}=\frac{\rho h\omega^2}{2}\int_0^a\int_0^b W^2 dxdy, \tag{10}$$

The maximum kinetic energy function Ψ_{max} is given by

$$\Psi_{max}=T_{max}/\omega^2 \tag{11}$$

The selected set of admissible functions are used to model the deflection of completely free structures. For this reason, all constraint conditions are incorporated through the use of the penalty method. Then, the strain energy of translational and rotational springs along all four edges of the plate ($x=0$, $x=a$, $y=0$ and $y=b$) is defined as

$$\begin{aligned}V_{edge}=&\frac{1}{2}\int_0^b\left(k_{x0}W^2\Big|_{x=0}+k_{xa}W^2\Big|_{x=a}\right)dy+\frac{1}{2}\int_0^a\left(k_{y0}W^2\Big|_{y=0}+k_{yb}W^2\Big|_{y=b}\right)dx\\&+\frac{1}{2}\int_0^b\left(k_{rx0}\left(\frac{\partial W}{\partial x}\right)^2\Bigg|_{x=0}+k_{rxa}\left(\frac{\partial W}{\partial x}\right)^2\Bigg|_{x=a}\right)dy\\&+\frac{1}{2}\int_0^a\left(k_{ry0}\left(\frac{\partial W}{\partial y}\right)^2\Bigg|_{y=0}+k_{ryb}\left(\frac{\partial W}{\partial y}\right)^2\Bigg|_{y=b}\right)dx,\end{aligned} \tag{12}$$

where k_{x0}, k_{xa}, k_{y0} and k_{yb} are the stiffness per unit length of the translational spring supports, while k_{rx0}, k_{rxa}, k_{ry0} and k_{ryb} are the stiffness per unit length of the rotational spring supports located along the edges at $x=0$, $x=a$, $y=0$ and $y=b$, respectively. To model each of the 54 cases of plates with constrained boundary conditions along the edges only the appropriate stiffness coefficients should have a non-zero value. Then the set of linear homogeneous equations of the system are found by minimizing the potential and kinetic energy of the plate including the energy of the artificial springs

$$\left(\frac{V_{plate}}{\partial c_{ij}}+\frac{V_{edge}}{\partial c_{ij}}\right)-\omega^2\frac{\Psi_{max}}{\partial c_{ij}}=0 \tag{13}$$

To obtain results in non-dimensional form non-dimensional coordinates of the plate are introduced and the stiffness and mass matrices are non-dimensionalized by dividing them by D/ab and ρhab, respectively. Furthermore, the penalty matrices are also non-dimensionalized introducing non-dimensional penalty parameters (the same non-dimensional penalty value is used in all cases). Examples for distributed penalty parameters along the edge at $x=0$ are presented, while all other non-dimensional penalty parameters can be obtained in a similar way

- non-dimensional coordinates of the plate

$$\xi = x/a \text{ and } \eta = y/b \tag{14a, 14b}$$

- non-dimensional distributed translational and rotational stiffness parameter

$$\hat{k} = \frac{k_{x0}a^3}{D} = \frac{k_{rx0}a}{D} \tag{15}$$

The non-dimensional eigen-problem obtained after the Rayleigh-Ritz minimization is

$$\left[\mathbf{K}+\mathbf{P}_{edge}\right]\{\mathbf{c}\} - \lambda^2[\mathbf{M}]\{\mathbf{c}\} = \{\mathbf{0}\}, \tag{16}$$

where $\mathbf{P}_{edge}$ is the penalty matrix and λ is the non-dimensional frequency parameter defined as [27]

$$\lambda = \sqrt{\frac{\rho h a^2 b^2 \omega^2}{D}} \tag{17}$$

The terms of the non-dimensional mass $\mathbf{M}$ and stiffness matrices of a plate $\mathbf{K}$ are

$$M_{klij} = E_{ki}^{(0,0)} F_{lj}^{(0,0)} \quad \text{and} \tag{18}$$

$$K_{klij} = a^2 b^2 \left[\frac{1}{a^4} E_{ki}^{(2,2)} F_{lj}^{(0,0)} + \frac{1}{b^4} E_{ki}^{(0,0)} F_{lj}^{(2,2)} + \frac{\nu_p}{a^2 b^2}\left[E_{ki}^{(0,2)} F_{lj}^{(2,0)} + E_{ki}^{(2,0)} F_{lj}^{(0,2)}\right] + \frac{2(1-\nu)}{a^2 b^2} E_{ki}^{(1,1)} F_{lj}^{(1,1)} \right], \tag{19}$$

where $E_{ki}^{(r,s)} = \int_0^1 \left(\frac{d^r \phi_k}{d\xi^r}\right)\left(\frac{d^s \phi_i}{d\xi^s}\right) d\xi$, $F_{lj}^{(r,s)} = \int_0^1 \left(\frac{d^r \chi_l}{d\eta^r}\right)\left(\frac{d^s \chi_j}{d\eta^s}\right) d\eta$,

$k,i,l,j = 1,2,3\ldots n$ and $r,s = 0,1,2$

The terms of the non-dimensional penalty matrices due to the artificial stiffness are

$$\begin{aligned} P_{edge,klij} = \hat{k}\Big[& \varphi_k(0)\varphi_i(0)F_{lj}^{(0,0)} + \varphi_k(1)\varphi_i(1)F_{lj}^{(0,0)} \\ & + E_{ki}^{(0,0)}\chi_l(0)\chi_j(0) + E_{ki}^{(0,0)}\chi_l(1)\chi_j(1) \\ & + \frac{\partial\varphi_k(0)}{\partial\xi}\frac{\partial\varphi_i(0)}{\partial\xi}F_{lj}^{(0,0)} + \frac{\partial\varphi_k(1)}{\partial\xi}\frac{\partial\varphi_i(1)}{\partial\xi}F_{lj}^{(0,0)} \\ & + E_{ki}^{(0,0)}\frac{\partial\chi_l(0)}{\partial\eta}\frac{\partial\chi_j(0)}{\partial\eta} + E_{ki}^{(0,0)}\frac{\partial\chi_l(1)}{\partial\eta}\frac{\partial\chi_j(1)}{\partial\eta}\Big] \end{aligned} \tag{20}$$

An alternative procedure to obtain the frequency parameters is to first solve the eigenproblem of the unconstrained structure

$$[\mathbf{K}]\{\mathbf{c}\} - \lambda^2[\mathbf{M}]\{\mathbf{c}\} = \{\mathbf{0}\}, \tag{21}$$

to obtain the frequency parameters λ_i and the matrix $\mathbf{X}$ whose columns contain the eigenvectors of Equation (21). Then use matrix $\mathbf{X}$ and its transpose to perform a transformation on the stiffness and penalty matrices. Then the frequency parameters of the constrained structure can be obtained solving for the eigenvalues of a matrix resulting from the addition of the transformed stiffness and penalty matrices

$$\left[\mathbf{X}^{\mathrm{T}}\mathbf{K}\mathbf{X} + \mathbf{X}^{\mathrm{T}}\mathbf{P}_{edge}\mathbf{X}\right]\{\mathbf{c}\} = \lambda^2\{\mathbf{c}\} \tag{22}$$

This procedure saves space in the memory of the computer.

5 Results

Results of the frequency parameters of the 55 cases of rectangular plates with simply supported, clamped, guided and free conditions were obtained by assigning appropriate penalty parameters $\hat{k}$. Forty terms of admissible functions were used in each direction. Results in Table 1 correspond to those obtained with the higher penalty value in the series 10^p where $p = 1,2,3,\ldots$ that still converges monotonically from below. This means that the Rayleigh-Ritz method converges from above with respect to the number of terms used in the set of admissible functions, but from below when artificial springs are used to model constraints. The rigid body modes are not included in Table 1. Cases 20 SFFF, 21 FFFF, 38 SGFG, 40 GGFG, 41 FGFG, 42 GGGG, 52 SGFF, GGFF, GFFF have 1, 3, 1, 1, 2, 1, 1, 1, 2 rigid body modes.

Case		Mode					
	$\hat{k}$	1	2	3	4	5	6
1 SSSS	1E+10	19.739	49.348	49.348	78.957	98.698	98.698
2 SCSC	1.E+09	28.951	54.744	69.329	94.589	102.220	129.105
3 SCSS	1.E+09	23.646	51.675	58.647	86.136	100.272	113.233
4 SCSF	1.E+09	12.687	33.065	41.702	63.016	72.399	90.614
5 SSSF	1.E+10	11.685	27.756	41.197	59.066	61.861	90.297
6 SFSF	1.E+09	9.631	16.135	36.726	38.945	46.739	70.741
7 CCCC	1.E+09	35.986	73.397	73.397	108.225	131.592	132.215
8 CCCS	1.E+09	31.827	63.333	71.079	100.798	116.363	130.361
9 CCCF	1.E+09	23.921	39.999	63.224	76.713	80.576	116.665
10 CCSS	1.E+09	27.054	60.540	60.787	92.840	114.562	114.709
11 CCSF	1.E+09	17.537	36.024	51.813	71.078	74.328	105.791
12 CCFF	1.E+09	6.920	23.905	26.585	47.653	62.708	65.535
13 CSCF	1.E+09	23.371	35.572	62.878	66.764	77.378	108.874
14 CSSF	1.E+09	16.792	31.114	51.397	64.022	67.541	101.117
15 CSFF	1.E+09	5.351	19.075	24.671	43.088	52.708	63.760
16 CFCF	1.E+09	22.168	26.407	43.596	61.177	67.179	79.818
17 CFSF	1.E+09	15.192	20.584	39.736	49.449	56.280	77.325
18 CFFF	1.E+09	3.471	8.507	21.285	27.199	30.957	54.188

Table 1. Frequency parameters of a plate with the 55 possible combinations of boundary conditions including free, simply supported, guided and clamped conditions.

Case		Mode					
	$\hat{k}$	1	2	3	4	5	6
19 SSFF	1.E+10	3.367	17.316	19.293	38.211	51.036	53.487
20 SFFF	1.E+09	6.644	14.902	25.376	26.001	48.450	50.579
21 FFFF		13.468	19.596	24.270	34.801	34.801	61.093
22 SSSG	1.E+09	12.337	32.076	41.946	61.685	71.555	91.296
23 SCSG	1.E+09	13.686	38.694	42.587	66.300	83.490	91.706
24 SGSF	1.E+09	9.736	17.685	39.189	42.384	47.967	74.526
25 SGSG	1.E+09	9.870	19.739	39.479	49.348	49.348	78.957
26 CSCG	1.E+09	23.816	39.090	63.537	75.843	79.528	114.785
27 CSSG	1.E+09	17.332	35.051	52.099	69.914	73.440	106.483
28 SSGG	1.E+09	4.935	24.674	24.674	44.413	64.153	64.153
29 CSGG	1.E+08	7.238	25.554	32.274	49.953	64.654	76.830
30 SSGF	1.E+09	4.034	18.821	24.010	41.174	53.026	63.287
31 SCGF	1.E+08	5.704	24.694	24.944	45.755	63.681	64.403
32 SGGF	1.E+09	2.408	9.181	21.997	30.510	33.426	56.190
33 SFGF	1.E+09	2.378	6.881	21.821	26.372	29.208	51.646
34 CGSG	1.E+09	15.418	23.646	49.966	51.675	58.647	86.136
35 CGCG	1.E+09	22.374	28.951	54.744	61.675	69.330	94.589
36 SGGG	1.E+08	2.467	12.337	22.207	32.076	41.946	61.685
37 CGGG	1.E+08	5.593	13.686	30.226	38.694	42.587	66.300
38 SGFG	1.E+08	11.685	15.418	27.756	41.197	49.965	59.066
39 CGFG	1.E+08	3.516	12.687	22.035	33.065	41.702	61.698
40 GGFG	1.E+07	5.593	9.736	17.685	30.226	39.188	42.384
41 FGFG	1.E+07	9.631	16.135	22.373	36.726	38.945	46.738
42 GGGG	1.E+07	9.870	9.870	19.739	39.478	39.478	49.348
43 CCCG	1.E+09	24.578	44.771	63.985	83.277	87.256	123.256
44 CCSG	1.E+09	18.349	41.251	52.632	74.086	85.147	106.843
45 CCGF	1.E+09	7.776	25.850	32.217	51.192	64.917	76.337
46 CCGG	1.E+07	8.996	32.895	33.051	55.008	77.226	77.291
47 CGCF	1.E+09	22.259	27.495	48.533	61.402	68.199	90.289
48 CGSF	1.E+09	15.293	21.897	45.058	49.684	57.400	82.031
49 CSGF	1.E+09	6.601	19.954	31.677	47.034	53.632	76.003
50 CGGF	1.E+08	5.541	10.898	30.024	34.223	37.326	61.183
51 CFGF	1.E+08	5.508	8.986	27.359	29.857	36.177	56.973
52 SGFF	1.E+08	8.700	15.273	26.365	32.867	49.568	53.854
53 CGFF	1.E+08	3.493	10.181	21.838	31.427	34.029	58.071
54 GGFF	1.E+07	4.899	6.068	15.922	29.277	30.611	40.376
55 GFFF	1.E+07	5.366	14.621	22.002	29.681	36.045	40.050

Table 1 (cont.). Frequency parameters of a plate with the 55 possible combinations of boundary conditions including free, simply supported, guided and clamped conditions.

The guided results can be compared with those presented in the work by Bert and Malik [28]. In most cases there was no difference between the results presented here and the results in [28] and the maximum difference between the two sets of results were always found in the third decimal place.
Cases with two opposite edges either simply supported or guided have an analytical solution, while the solution of the remaining cases can be solved using approximate or numerical methods [28]. Furthermore Bert and Malik classified the 55 cases presented in Table 1 according to the boundary conditions and type of solutions:

- Cases 1 to 6 with two opposite edges simply supported have an analytical solution.
- Cases 7 to 21 are possible by approximate or numerical methods only.
- Cases 22 to 25 with two opposite edges simply supported have an analytical solution
- Cases 26 to 33 with one edge simply supported and opposite edge guided have an analytical solution.
- Cases 34 to 42 with two opposite edges guided have an analytical solution.
- Cases 43 to 55 are possible by approximate or numerical methods only.

6 Conclusions

In this paper a discussion on set of admissible functions to be used in the RRM is presented and it has been shown how a set built by cosine functions and a linear and square terms can be used to model beams, plates and shells in free condition and then constraints can be added using the penalty method. Because the set of functions presented here do not impose a limit in the number of terms of functions that can be used, a large number of constraints can be used to model complex constraints. The availability of large number of terms, limited only by computer memory, also helps to improve the accuracy of the natural frequencies and modes of vibration. Results show that in most cases frequency parameters of rectangular plates with any type of boundary conditions converged to the exact results to at least the fourth significant number. The method used to build the set of functions presented in Equation (2) is intuitive and avoids normalization or orthogonalization of the functions. Furthermore, the integrals that define the mass, stiffness, geometric stiffness and penalty matrices using this method can be easily solved in close form by hand and the set of admissible functions presented in this work seems to be the simplest set that does not causes ill-conditioning when a large number of functions are included in the solution.

References

[1] L. Meirovitch, "Analytical Methods in Vibration", New York, McMillan, 1967.

[2] R.B. Bhat, "Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh-Ritz method", Journal of Sound and Vibration, 102(4), 493-499, 1985.

[3] J. Yuan, S.M. Dickinson, “The flexural vibration of rectangular plate systems approached by using artificial springs in the Rayleigh-Ritz method”, Journal of Sound and Vibration, 159(1), 39-55, 1992.
[4] R.E. Brown, M.A. Stone, “On the use of polynomial series with the Rayleigh-Ritz method”, Composite Structures, 39(3-4), 191-196, 1997.
[5] W.L. Li, “Vibration analysis of rectangular plates with general elastic boundary supports”, Journal of Sound and Vibration, 273(3), 619-635, 2004.
[6] R.D. Blevins, “Formulas for natural frequency and mode shape”, Krieger Pub. Co., Malabar, Fla, 2000.
[7] J. W. Jaworski, E.H. Dowell, “Free vibration of a cantilevered beam with multiple steps: Comparison of several theoretical methods with experiment”, Journal of Sound and Vibration, 312(4-5), 713-725, 2008.
[8] E.H. Dowell, “On asymptotic approximations to beam model shapes”, Journal of Applied Mechanics, Transactions ASME, 51(2), 439, 1984.
[9] L. Dozio, “On the use of the Trigonometric Ritz method for general vibration analysis of rectangular Kirchhoff plates”, Thin-Walled Structures, 49(1), 129–144, 2011.
[10] O. Beslin, J. Nicolas, “A hierarchical functions set for predicting very high order plate bending modes with any boundary conditions”, Journal of Sound and Vibration, 202, 633–55, 1997.
[11] W.L. Li, “Free vibrations of beams with general boundary conditions” Journal of Sound and Vibration, 237, 709-725, 2000.
[12] D. Zhou, “Natural frequencies of rectangular plates using a set of static beam functions in Rayleigh-Ritz method”, Journal of Sound and Vibration, 189(1), 81-87,1996.
[13] H. Dal, Ö.K. Morgül, “Vibrations of elastically restrained rectangular plates”, Scientific Research and Essays, 6(34), 6811-6816, 2011, DOI: 10.5897/SRE11.356.
[14] G.M. Oosterhout, P.J.M. van der Hoogt, R.M.E.J. Spiering, "Accurate calculation methods for natural frequencies of plates with special attention to the higher modes", Journal of Sound and Vibration, 183(1), 33-47, 1995.
[15] B. Budiansky, P.C. Hu, “The Lagrangian multiplier method of finding upper and lower limits to critical stresses of clamped plates”, NACA report 848, 1946.
[16] W.L. Li, “Comparison of Fourier sine and cosine series expansions for beams with arbitrary boundary conditions” Journal of Sound and Vibration, 255, 185-194, 2002.
[17] S.F. Bassily, S.M. Dickinson, “On the use of beam functions for problems of plates involving free edges”, Journal of Applied Mechanics, Transactions ASME, Ser E, 42(4), 858-864, 1975.
[18] G.B. Warburton, “The vibration of rectangular plates”, Proceedings of the Institute of Mechanical Engineers, Ser. A, 168, 371-381, 1954.
[19] R. Szilard, “Theories and applications of plate analysis: classical, numerical, and engineering methods”, John Wiley, Hoboken, NJ, USA, 2004.
[20] M. Mukhopadhyay, “Structural dynamics: vibration and systems”, Ane Books India for CRC Press, New Delhi, India, 2008.

[21] L.E. Monterrubio, “Free vibration of shallow shells using the Rayleigh–Ritz method and penalty parameters”, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 23, 2263–2272, 2009.
[22] L.E. Monterrubio, “The use of eigenpenalty parameters in structural stability analysis”, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, *In Press*, 226, April 2012, DOI: 10.1177/0954406211417367.
[23] L.E. Monterrubio, “Frequency and buckling parameters of box-type structures”, In Press, Computers & Structures.
[24] A.L. Garcia, “Numerical methods for physics”, Prentice Hall, Englewood Cliffs, N.J, 1994.
[25] W. Kohn, “Improvement of Rayleigh-Ritz Eigenfunctions”, SIAM Review, 14(3), 399-419, 1972.
[26] Williams, T. W. C. (1987). "Rounding error effects on computed Rayleigh-Ritz estimates." Journal of Sound and Vibration, 117(3), 588-593.
[27] P.G. Young, S.M. Dickinson, “On the free flexural vibration of rectangular plates with straight or curved internal line supports”, Journal of Sound and Vibration, 162(1), 123-135, 1993.
[28] C.W. Bert, M. Malik, “Frequency equations and modes of free vibrations of rectangular plates with various edge conditions”, Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science, 208, 307-319, 1994, DOI: 10.1243/PIME PROC 1994 208 133 02.