Paper 233

©Civil-Comp Press, 2012
Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Shape and Topology Optimization of Elastic Contact Problems using the Piecewise Constant Level Set Method

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Abstract

This paper considers the shape and topology optimization of the elastic contact problems using the level set approach. A piecewise constant level set method is used to represent interfaces rather than the standard method. The piecewise constant level set function takes distinct constant values in each subdomain of a whole design domain. Using a two-phase approximation the original structural optimization problem is reformulated as an equivalent constrained optimization problem in terms of the piecewise level set function. The necessary optimality condition is formulated. The finite difference method is applied as the approximation method. Numerical examples are provided and discussed.

Keywords: shape and topology optimization, unilateral problems, piecewise constant level set method, Uzawa method.

1 Introduction

The paper deals with the solution of a structural optimization problem for an elliptic variational inequality. This inequality governs unilateral contact between an elastic body and a rigid foundation. The structural optimization problem for the elastic body in unilateral contact consists in finding such topology of the domain occupied by the body and the shape of its boundary that the normal contact stress along the boundary of the body is minimized. The volume of the body is bounded.

In structural optimization the standard level set method [1, 2] is employed in the numerical algorithms for tracking the evolution of the domain boundary on a fixed mesh and finding an optimal domain. This method is based on an implicit representation of the boundaries of the optimized structure, i.e., the position of the boundary of the body is described as an isocountour of a scalar function of a higher dimensionality.

While the shape of the structure may undergo major changes the level set function remains to be simple in its topology. The evolution of the domain boundary is governed by Hamilton - Jacobi equation. The speed vector field driving the propagation of the level set function is given by the Eulerian derivative of the cost functional with respect to the variations of the free boundary. The solution of this equation requires reinitialization procedure to ensure that it is as close as possible to the signed distance function to the interface. Moreover this approach requires regularization of non-differentiable Heaviside and Dirac functions. Applications of the level set methods in structural optimization can be found, among others, in [1, 3, 4, 5, 6, 7, 8, 9].

Recently, a piecewise constant level set method as a variant of traditional level set method has been proposed for the image segmentation [10], shape recovery [11] or elliptic inverse problems. For a domain divided into 2^N subdomains in standard level set approach is required 2^N level set functions to represent them. Piecewise constant level set method can identify an arbitrary number of subdomains using only one discontinuous piecewise constant level set function. This function takes distinct constant values on each subdomain. The interfaces between subdomains are represented implicitly by the discontinuity of a set of characteristic functions of the subdomains [10]. Comparing to the classical level set method, this method is free of the Hamilton-Jacobi equation and do not require the use of the signed distance function as the initial one. Piecewise constant level set method has been used in [12] to solve numerically topological optimization problem in plane elasticity and in [13] to solve structural optimization problem for the Laplace equation in 2D domain. Moreover in [14] this method is used to solve topology optimization problem for plane elasticity with unilateral boundary condition.

In the paper the original structural optimization problem is approximated by a two-phase optimization problem. Using the piecewise constant level set method this approximated problem is reformulated as an equivalent constrained optimization problem in terms of the piecewise constant level set function only. Therefore neither shape nor topological sensitivity analysis is required. During the evolution of the piecewise constant level set function small holes can be created without use of the topological derivatives. The paper extends results contained in [14]. Necessary optimality condition is formulated. The finite difference method is used as the approximation method. This discretized optimization problem is solved numerically using the augmented Lagrangian method. Numerical examples are provided and discussed.

2 Problem Formulation

Consider deformations of an elastic body occupying two-dimensional domain Ω with the smooth boundary Γ (see Figure 1). Assume $\Omega \subset D$ where D is a bounded smooth hold-all subset of R^2. Let $E \subset R^2$ and $D \subset R^2$ denote given bounded domains. So-called hold-all domain D is assumed to possess a piecewise smooth boundary.

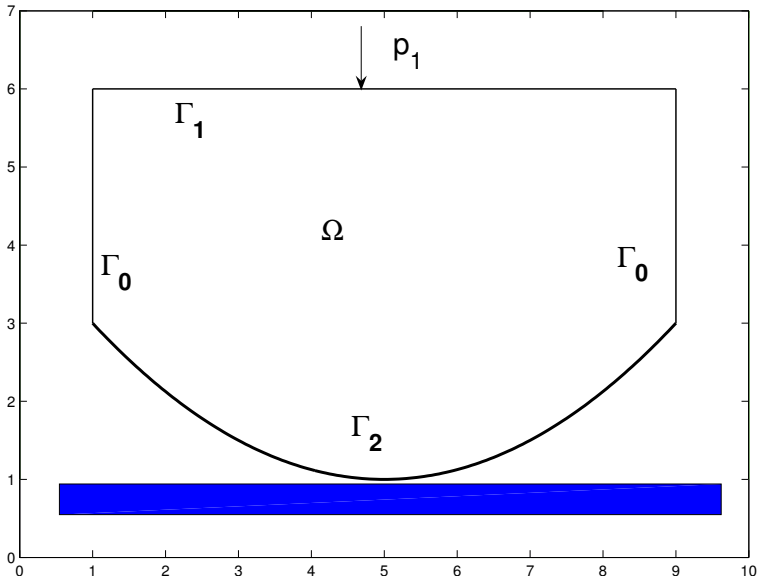

Figure 1: Initial Domain Ω.

Domain Ω is assumed to belong to the set O_l defined as follows:

$$O_l = \{\Omega \subset R^2 : \Omega \text{ is open}, \quad E \subset \Omega \subset D, \quad {}^{\#}\Omega^c \leq l\}, \tag{1}$$

where ${}^{\#}\Omega^c$ denotes the number of connected components of the complement Ω^c of Ω with respect to D and $l \geq 1$ is a given integer. Moreover all perturbations $\delta\Omega$ of Ω are assumed to satisfy $\delta\Omega \in O_l$. The body is subject to body forces $f(x) = (f_1(x), f_2(x))$, $x \in \Omega$. Moreover, surface tractions $p(x) = (p_1(x), p_2(x))$, $x \in \Gamma$, are applied to a portion Γ_1 of the boundary Γ. We assume, that the body is clamped along the portion Γ_0 of the boundary Γ, and that the contact conditions are prescribed on the portion Γ_2, where $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j$, $i, j = 0, 1, 2$, $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1 \cup \bar{\Gamma}_2$.

We denote by $u = (u_1, u_2)$, $u = u(x)$, $x \in \Omega$, the displacement of the body and by $\sigma(x) = \{\sigma_{ij}(u(x))\}$, $i, j = 1, 2$, the stress field in the body. Consider elastic bodies obeying Hooke's law, i.e., for $x \in \Omega$ and $i, j, k, l = 1, 2$,

$$\sigma_{ij}(u(x)) = a_{ijkl}(x) e_{kl}(u(x)). \tag{2}$$

We use here and throughout the paper the summation convention over repeated indices [15]. The strain $e_{kl}(u(x))$, $k, l = 1, 2$, is defined by:

$$e_{kl}(u(x)) = \frac{1}{2}(u_{k,l}(x) + u_{l,k}(x)), \tag{3}$$

where $u_{k,l}(x) = \frac{\partial u_k(x)}{\partial x_l}$. The stress field σ satisfies the system of equations [15]

$$-\sigma_{ij}(x)_{,j} = f_i(x) \quad x \in \Omega, i, j = 1, 2, \tag{4}$$

where $\sigma_{ij}(x)_{,j} = \frac{\partial \sigma_{ij}(x)}{\partial x_j}$, $i, j = 1, 2$. The following boundary conditions are imposed

$$u_i(x) = 0 \quad \text{on} \quad \Gamma_0, \quad i = 1, 2, \tag{5}$$

$$\sigma_{ij}(x) n_j = p_i \quad \text{on} \quad \Gamma_1, \quad i, j = 1, 2, \tag{6}$$

$$u_N + v \leq 0, \quad \sigma_N \leq 0, \quad (u_N + v)\sigma_N = 0 \quad \text{on } \Gamma_2, \tag{7}$$

$$\mid \sigma_T \mid \leq 1, \quad u_T \sigma_T + \mid u_T \mid = 0 \ \text{ on } \Gamma_2, \tag{8}$$

where $n = (n_1, n_2)$ is the unit outward versor to the boundary Γ and $v = v(x)$ is a given profile of the boundary Γ_2. Here $u_N = u_i n_i$ and $\sigma_N = \sigma_{ij} n_i n_j$, $i, j = 1, 2$, represent the normal components of the displacement u and the stress σ, respectively. The tangential components of displacement u and stress σ are given by $(u_T)_i = u_i - u_N n_i$ and $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, $i, j = 1, 2$, respectively. $\mid u_T \mid$ denotes the Euclidean norm in R^2 of the tangent vector u_T.

2.1 Variational Formulation of Contact Problem

Let us formulate contact problem (4)-(8) in variational form. Denote by V_{sp} and K the space and set of kinematically admissible displacements:

$$V_{sp} = \{z \in [H^1(\Omega)]^2 = H^1(\Omega) \times H^1(\Omega) \ : \ z_i = 0 \text{ on } \Gamma_0, \ i = 1, 2\}, \tag{9}$$

$$K = \{z \in V_{sp} \ : \ z_N \leq 0 \ \text{ on } \ \Gamma_2\}. \tag{10}$$

Denote also by Λ the set of Lagrange multipliers corresponding to term $\mid u_T \mid$ in equality constraint in Equation (8) [15, 16]:

$$\Lambda = \{\zeta \in L^2(\Gamma_2) \ : \mid \zeta \mid \leq 1\}. \tag{11}$$

The spaces $L^2(\Omega)$ and $H^1(\Omega)$ denote the space of square integrable functions as well as the space of square integrable functions having also square integrable first derivatives on the domain Ω, respectively.

Variational formulation of problem (4)-(8) has the form: *find a pair* $(u, \lambda) \in K \times \Lambda$ *satisfying*

$$\int_\Omega a_{ijkl} e_{ij}(u) e_{kl}(\varphi - u) dx - \int_\Omega f_i(\varphi_i - u_i) dx - \int_{\Gamma_1} p_i(\varphi_i - u_i) ds + \int_{\Gamma_2} \lambda(\varphi_T - u_T) ds \geq 0 \ \ \forall \varphi \in K, \tag{12}$$

$$\int_{\Gamma_2} (\zeta - \lambda) u_T ds \leq 0 \quad \forall \zeta \in \Lambda, \tag{13}$$

$i, j, k, l = 1, 2$. The results concerning the existence of solutions to system (12)-(13) can be found, among others, in [15].

2.2 Structural Optimization Problem

Before formulating a structural optimization problem for (12)-(13) let us introduce first the set U_{ad} of admissible domains. Domain Ω is assumed to satisfy the volume constraint of the form

$$Vol(\Omega) - Vol^{giv} \leq 0, \;\; Vol(\Omega) \stackrel{def}{=} \int_{\Omega} dx, \tag{14}$$

where the constant $Vol^{giv} = const_0 > 0$ is given. Moreover this domain is assumed to satisfy the perimeter constraint [3], [16, p. 126]

$$Per(\Omega) \leq const_1, \;\; Per(\Omega) \stackrel{def}{=} \int_{\Gamma} dx, \tag{15}$$

The constant $const_1 > 0$ is given. The set U_{ad} has the following form

$$\begin{aligned} U_{ad} = \;\; & \{\Omega \in O_l \; : \; \Omega \text{ is Lipschitz continuous,} \\ & \Omega \text{ satisfies conditions (14) and (15) } \}, \end{aligned} \tag{16}$$

The set U_{ad} is assumed to be nonempty. In order to define a cost functional we shall also need the following set M^{st} of auxiliary functions

$$\begin{aligned} M^{st} = \;\; & \{\eta = (\eta_1, \eta_2) \in [H^1(D)]^2 : \eta_i \leq 0 \;\text{ on } D, \\ & i = 1, 2, \quad \| \eta \|_{[H^1(D)]^2} \leq \; 1\}, \end{aligned} \tag{17}$$

where the norm $\| \; \eta \; \|_{[H^1(D)]^2} = \; (\sum_{i=1}^{2} \; \| \; \eta_i \; \|^2_{H^1(D)})^{1/2}$. Recall from [7] the cost functional approximating the normal contact stress on the contact boundary

$$J_\eta(u(\Omega)) = \int_{\Gamma_2} \sigma_N(u) \eta_N(x) ds, \tag{18}$$

depending on the auxiliary given bounded function $\eta(x) \in M^{st}$. σ_N and ϕ_N are the normal components of the stress field σ corresponding to a solution u satisfying system (12)-(13) and the function η, respectively.

Consider the following structural optimization problem: *for a given function $\eta \in M^{st}$, find a domain $\Omega^\star \in U_{ad}$ such that*

$$J_\eta(u(\Omega^\star)) = \min_{\Omega \in U_{ad}} J_\eta(u(\Omega)) \tag{19}$$

From Šverák theorem and [17, Theorem 2] follows the existence of an optimal domain $\Omega^\star \in U_{ad}$ to the problem (19).

3 Level set approach

In [6, 7] the standard level set method [2] is employed to solve numerically problem (19). Consider the evolution of a domain Ω under a velocity field V. Let $t > 0$ denote the artificial time variable. Under the suitable regular mapping $T(t, V)$ we have

$$\Omega_t = T(t, V)(\Omega) = (I + tV)(\Omega), \;\; t > 0.$$

By Ω_t^- we denote the interior of the domain Ω_t and by Ω_t^+ we denote the outside of the domain Ω_t. The domain Ω_t and its boundary $\partial\Omega_t$ are defined by a function $\phi = \phi(x,t) \ : \ R^2 \times [0, t_0) \to R$ satisfying:

$$\begin{aligned} \phi(x,t) &= 0, \quad \text{if } \ x \in \partial\Omega_t, \\ \phi(x,t) &< 0, \quad \text{if } \ x \in \Omega_t^-, \\ \phi(x,t) &> 0, \quad \text{if } \ x \in \Omega_t^+. \end{aligned} \tag{20}$$

Function ϕ satisfying (20) is called the level set function. In the standard level set approach Heaviside function and Dirac function are used to transform integrals from domain Ω into domain D [2].

Assume that velocity field $V = V(x,t)$ is known for every point x lying on the boundary $\partial\Omega_t$, i.e., such that $\phi(x,t) = 0$. Therefore the equation governing the evolution of the interface $\partial\Omega_t$ in $D \times [0, t_0]$, known as Hamilton-Jacobi equation, has the form [2]

$$\frac{\partial\phi(x,t)}{\partial t} + V(x,t) \cdot \nabla_x \phi(x,t) = 0. \tag{21}$$

$$\phi(x,0) = \phi_0(x), \tag{22}$$

where $\phi_0(x)$ is a given signed distance function of the set Ω_t.

3.1 Piecewise constant level set formulation

Recall hold-all domain D is an open bounded domain in R^2. Let us assume D is partitioned into N subdomains $\{\Omega_i\}_{i=1}^N$ such that

$$D = \bigcup_{i=1}^{N} (\Omega_i \cup \partial\Omega_i) \tag{23}$$

where N is a given integer and $\partial\Omega_i$ denotes the boundary of the subdomain Ω_i. Define a level set function $\phi : D \to R$ such that [10, 12, 13]

$$\phi = i \ \text{ in } \Omega_i, \quad i = 1, 2, ..., N. \tag{24}$$

This function is used to identify all the phases in D. In order to guarantee that there is no vacuum or overlap between different subdomains Ω_i assume function ϕ satisfies the following constraint:

$$\tilde{W}(\phi) = 0, \tag{25}$$

$$\tilde{W}(\phi) \stackrel{def}{=} (\phi - 1)(\phi - 2)...(\phi - N) = \prod_{i=1}^{N} (\phi - i). \tag{26}$$

The constraint (26) means that for every $x \in D$ there exists a unique $i \in \{1, 2..., N\}$ such that $\phi(x) = i$. Using this approach the characteristic function χ_i, $i = 1, 2, ..., N$, of the subdomain Ω_i is represented as [10]

$$\chi_i = \frac{1}{\alpha_i} \prod_{j=1, j\neq i}^{N} (\phi - j) \ \text{ where } \ \alpha_i = \prod_{k=1, k\neq i}^{N} (i - k), \tag{27}$$

i.e., it is constructed using one level set function ϕ only. Each characteristic function χ_i is expressed as a product of linear factors of the form $(\phi - j)$ with the ith factor omitted. Therefore as long as (24) holds, $\chi_i(x) = 1$ for $x \in \Omega_i$ and equals zero elsewhere.

3.1.1 Density function

Consider piecewise constant density function $\rho : D \to R^2$ defined as

$$\rho(x) = \begin{cases} \epsilon & \text{if } x \in D \setminus \bar{\Omega}\,, \\ 1 & \text{if } x \in \Omega. \end{cases} \tag{28}$$

where $\epsilon > 0$ is a small constant. Function (28) can be constructed as a weighted sum of the characteristic functions χ_i. Denoting by $\{\rho_i\}_{i=1}^N$ a set of real scalars, we can represent a piecewise constant function ρ taking these N distinct constant values by

$$\rho(x) = \sum_{i=1}^{N} \rho_i \chi_i(\phi(x)). \tag{29}$$

3.2 Constrained optimization problem

We confine to consider a two-phase problem in the domain D. Therefore we set $N = 2$ and $D = \bar{\Omega}_1 \cup \bar{\Omega}_2$. Using (27) and (29) we have

$$\chi_1(x) = 2 - \phi(x) \;\text{ and }\; \chi_2(x) = \phi(x) - 1, \tag{30}$$

and

$$\rho(x) = \rho_1\chi_1(x) + \rho_2\chi_2(x) = (1-\epsilon)\phi(x) + 2\epsilon - 1. \tag{31}$$

Moreover function (26) takes the form

$$W(\phi) \stackrel{def}{=} (\phi - 1)(\phi - 2). \tag{32}$$

Using (24) as well as (31) the structural optimization problem (19) can be transformed into the following one: *find* $\phi \in U_{ad}^{\phi}$ *such that*

$$\min_{\phi \in U_{ad}^{\phi}} J_\eta(\phi) = \int_{\Gamma_2} \rho(\phi)\sigma_N(u_\epsilon)\eta_N ds \tag{33}$$

where the set U_{ad}^{ϕ} of the admissible functions is given as

$$\begin{aligned} U_{ad}^{\phi} = \;& \{\phi \in H^1(D) \;:\; Vol(\phi) - Vol^{giv} \le 0, \\ & W(\phi) = 0, \;\; Per(\phi) \le const_1\}, \end{aligned} \tag{34}$$

where

$$Vol(\phi) \stackrel{def}{=} \int_\Omega \rho(\phi)dx, \;\; Per(\phi) \stackrel{def}{=} \int_\Omega \mid \nabla\phi \mid dx. \tag{35}$$

The element $(u_\epsilon, \lambda_\epsilon) \in K \times \Lambda$ satisfies the state system (12)-(13) in the domain D rather than Ω:

$$\int_D \rho(\phi) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(\varphi - u_\epsilon) dx - \int_D \rho(\phi) f_i(\varphi_i - u_{\epsilon i}) dx -$$

$$\int_{\Gamma_1} p_i(\varphi_i - u_{\epsilon i}) ds + \int_{\Gamma_2} \lambda_\epsilon(\varphi_T - u_{\epsilon T}) ds \geq 0 \ \ \forall \varphi \in K, \tag{36}$$

$$\int_{\Gamma_2} (\zeta - \lambda_\epsilon) u_{\epsilon T} ds \leq 0 \ \ \ \forall \zeta \in \Lambda. \tag{37}$$

For the existence of the optimal solution $\phi \in H^1(D)$ to the optimization problem (33)-(37) see [11] or [18, Theorem 3.2.1, p. 75].

3.3 Necessary optimality conditions

Let us formulate the necessary optimality condition for the optimization problem (33)-(37). In order to do it we introduce the Lagrangian $L(\phi, \tilde{\lambda})$:

$$L(\phi, \tilde{\lambda}) = L(\phi, u_\epsilon, \lambda_\epsilon, p^a, q^a, \tilde{\lambda}) = J_\eta(\phi) +$$

$$\int_D \rho(\phi) a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a) dx - \int_D \rho(\phi) f_i p_i^a dx - \int_{\Gamma_1} p_i p_i^a ds + \tag{38}$$

$$\int_{\Gamma_2} \lambda_\epsilon p_T^a ds + \int_{\Gamma_2} q^a u_{\epsilon T} ds + \tilde{\lambda} c(\phi) + \sum_{i=1}^{3} \frac{1}{2\mu_i} c_i^2(\phi),$$

where $i, j, k, l = 1, 2$, $\tilde{\lambda} = \{\tilde{\lambda}_i\}_{i=1}^3$, $c(\phi) = \{c_i(\phi)\}_{i=1}^3 = [Vol(\phi), W(\phi), Per(\phi)]^T$, $c^T(\phi)$ denotes a transpose of $c(\phi)$, $\mu_m > 0$, $m = 1, 2, 3$, is a given real. Element $(p^a, q^a) \in K_1 \times \Lambda_1$ denotes an adjoint state defined as follows:

$$\int_D \rho(\phi) a_{ijkl} e_{ij}(\eta + p^a) e_{kl}(\varphi) dx + \int_{\Gamma_2} q^a \varphi_T ds = 0 \ \ \ \forall \varphi \in K_1, \tag{39}$$

$$\int_{\Gamma_2} \zeta(p_T^a + \eta_T) ds = 0 \ \ \forall \zeta \in \Lambda_1. \tag{40}$$

The sets K_1 and Λ_1 are given by

$$K_1 = \{\xi \in V_{sp} \ : \ \xi_N = 0 \ \text{ on } A^{st} \}, \tag{41}$$

$$\Lambda_1 = \{\zeta \in \Lambda \ : \ \zeta(x) = 0 \text{ on } B_1 \cup B_2 \cup B_1^+ \cup B_2^+\}, \tag{42}$$

while the coincidence set $A^{st} = \{x \in \Gamma_2 \ : \ u_N + v = 0\}$. Moreover $B_1 = \{x \in \Gamma_2 : \lambda(x) = -1\}$, $B_2 = \{x \in \Gamma_2 : \lambda(x) = +1\}$, $\tilde{B}_i = \{x \in B_i : u_N(x) + v = 0\}$, $i = 1, 2$, $B_i^+ = B_i \setminus \tilde{B}_i$, $i = 1, 2$. The derivative of the Lagrangian L with respect to ϕ has the form:

$$\frac{\partial L}{\partial \phi}(\phi, \tilde{\lambda}) = \int_D \rho'(\phi)[a_{ijkl} e_{ij}(u_\epsilon) e_{kl}(p^a + \eta) - f(p^a + \eta)] dx +$$

$$\tilde{\lambda} c'(\phi) + \sum_{i=1}^{3} \frac{1}{\mu_i} c(\phi) c'(\phi), \tag{43}$$

where the derivatives are equal to

$$\rho'(\phi) = 1 - \epsilon, \quad c'(\phi) = [Vol'(\phi), W'(\phi), Per'(\phi)], \tag{44}$$

$$Vol'(\phi) = 1, \quad W'(\phi) = 2\phi - 3, \tag{45}$$

$$Per'(\phi) = \chi_{\{\partial\Omega = const_0\}} \max\{0, -\nabla \cdot (\frac{\nabla\phi}{\mid \nabla\phi \mid})\} - \chi_{\{\partial\Omega > const_0\}} \nabla \cdot (\frac{\nabla\phi}{\mid \nabla\phi \mid}). \tag{46}$$

The necessary optimality condition takes the form [16, 19]: if $\hat{\phi} \in U_{ad}^{\phi}$ is an optimal solution to the problem (33)-(37) than there exists Lagrange multiplier $\tilde{\lambda} \in R^3$ such that $\tilde{\lambda}_1, \tilde{\lambda}_2 \geq 0$ satisfying

$$L(\hat{\phi}, \tilde{\lambda}) \leq L(\hat{\phi}, \tilde{\lambda}^\star) \leq L(\phi, \tilde{\lambda}^\star) \;\; \forall(\phi, \tilde{\lambda}) \in U_{ad}^{\phi} \times R^3, \tag{47}$$

Condition (47) implies that [15, 16] for all ϕ and $\tilde{\lambda}$

$$\frac{\partial L(\hat{\phi}, \tilde{\lambda})}{\partial \phi} \geq 0 \;\text{ and }\; \frac{\partial L(\hat{\phi}, \tilde{\lambda}^\star)}{\partial \tilde{\lambda}} \leq 0, \tag{48}$$

holds at the point $(\hat{\phi}, \tilde{\lambda}^\star)$.

4 Numerical implementation

Problem (33) - (37) is discretized using the finite difference approximation [2, 13]. Denote by $\phi_{i,j} = \phi(t, x_1^i, x_2^j)$ the approximation of ϕ at the grid point (x_1^i, x_2^j) for a given time t. Recall forward, backward and central differences as

$$\delta_+^{x_1}\phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i,j}}{h_{x_1}}, \quad \delta_+^{x_2}\phi_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j}}{h_{x_2}}, \tag{49}$$

$$\delta_-^{x_1}\phi_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{h_{x_1}}, \quad \delta_-^{x_2}\phi_{i,j} = \frac{\phi_{i,j} - \phi_{i,j-1}}{h_{x_2}}, \tag{50}$$

$$\delta_0^{x_1}\phi_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2h_{x_1}}, \quad \delta_0^{x_2}\phi_{i,j} = \frac{\phi_{i,j+1} - \phi_{i,j-1}}{2h_{x_2}}. \tag{51}$$

The regularization term is approximated as follows

$$\int_D \mid \nabla\phi \mid dx \approx \int_D \sqrt{\phi_{x_1}^2 + \phi_{x_2}^2 + \varepsilon_1} dx, \tag{52}$$

where $\varepsilon_1 > 0$ is small positive number and ϕ_{x_1}, ϕ_{x_2} are partial derivatives of ϕ with respect to x_1 and x_2 respectively discretized using central differences,

$$(\phi_{x_1})_{i,j} = \delta_0^{x_1}\phi_{i,j}, \quad (\phi_{x_2})_{i,j} = \delta_0^{x_2}\phi_{i,j}. \tag{53}$$

The curvature term is approximated using the following formulas:

$$\begin{aligned} \nabla \cdot (\frac{\nabla\phi}{\mid \nabla\phi \mid})_{i,j} = [(\frac{\phi_{x_1}}{\mid \nabla\phi \mid})_{x_1} + (\frac{\phi_{x_2}}{\mid \nabla\phi \mid})_{x_2}]_{i,j} \approx \\ \frac{1}{h_{x_1}}[(\frac{\phi_{x_1}}{\mid \nabla\phi \mid})_{i+1/2,j} - (\frac{\phi_{x_1}}{\mid \nabla\phi \mid})_{i-1/2,j}] + \\ \frac{1}{h_{x_2}}[(\frac{\phi_{x_2}}{\mid \nabla\phi \mid})_{i,j+1/2} - (\frac{\phi_{x_2}}{\mid \nabla\phi \mid})_{i,j-1/2}], \end{aligned} \tag{54}$$

where

$$
\begin{aligned}
(\frac{\phi_{x_1}}{\mid \nabla\phi \mid})_{i+1/2,j} &= \frac{\delta_+^{x_1}\phi_{i,j}}{\sqrt{(\delta_+^{x_1}\phi_{i,j})^2 + \frac{1}{4}(\delta_0^{x_2}\phi_{i,j} + \delta_0^{x_2}\phi_{i+1,j})^2 + \varepsilon_1}},\\
(\frac{\phi_{x_1}}{\mid \nabla\phi \mid})_{i-1/2,j} &= \frac{\delta_-^{x_1}\phi_{i,j}}{\sqrt{(\delta_-^{x_1}\phi_{i,j})^2 + \frac{1}{4}(\delta_0^{x_2}\phi_{i-1,j} + \delta_0^{x_2}\phi_{i,j})^2 + \varepsilon_1}},\\
(\frac{\phi_{x_2}}{\mid \nabla\phi \mid})_{i,j+1/2} &= \frac{\delta_+^{x_2}\phi_{i,j}}{\sqrt{(\delta_+^{x_2}\phi_{i,j})^2 + \frac{1}{4}(\delta_0^{x_1}\phi_{i,j} + \delta_0^{x_1}\phi_{i,j+1})^2 + \varepsilon_1}},\\
(\frac{\phi_{x_2}}{\mid \nabla\phi \mid})_{i,j-1/2} &= \frac{\delta_-^{x_2}\phi_{i,j}}{\sqrt{(\delta_+^{x_2}\phi_{i,j})^2 + \frac{1}{4}(\delta_0^{x_1}\phi_{i,j-1} + \delta_0^{x_1}\phi_{i,j})^2 + \varepsilon_1}}.
\end{aligned}
$$

5 Numerical experiments

The discretized structural optimization problem (33)-(37) is solved numerically. We employ Uzawa type algorithm [15] to solve numerically optimization problem (33)-(37). The algorithm is programmed in Matlab environment. As an example a body occupying 2D domain

$$\Omega = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\}, \tag{55}$$

is considered. The boundary Γ of the domain Ω is divided into three pieces

$$
\begin{aligned}
\Gamma_0 = \{(x_1, x_2) \in R^2 : x_1 = 0, 8 \ \wedge \ 0 < v(x_1) \leq x_2 \leq 4\},\\
\Gamma_1 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = 4\},\\
\Gamma_2 = \{(x_1, x_2) \in R^2 : 0 \leq x_1 \leq 8 \ \wedge \ x_2 = v(x_1)\}.
\end{aligned} \tag{56}
$$

The domain Ω and the boundary Γ_2 depend on the function v given as in [6, 14]. Figure 2 presents the obtained optimal domain. The areas with low values of density function appear in the central part of the body and near the fixed edges. The obtained normal contact stress is almost constant along the optimal shape boundary and has been significantly reduced comparing to the initial one.

6 Conclusions

The numerical results obtained seem to be in accordance with physical reasoning. They indicate that the proposed numerical algorithm allows for significant improvements of the structure from one iteration to the next and is more efficient than the algorithms based on standard level set approach. Unlike in the previous papers here the original topology optimization problem is approximated by the two-phase optimization problem. This problem is transformed into the constrained optimization

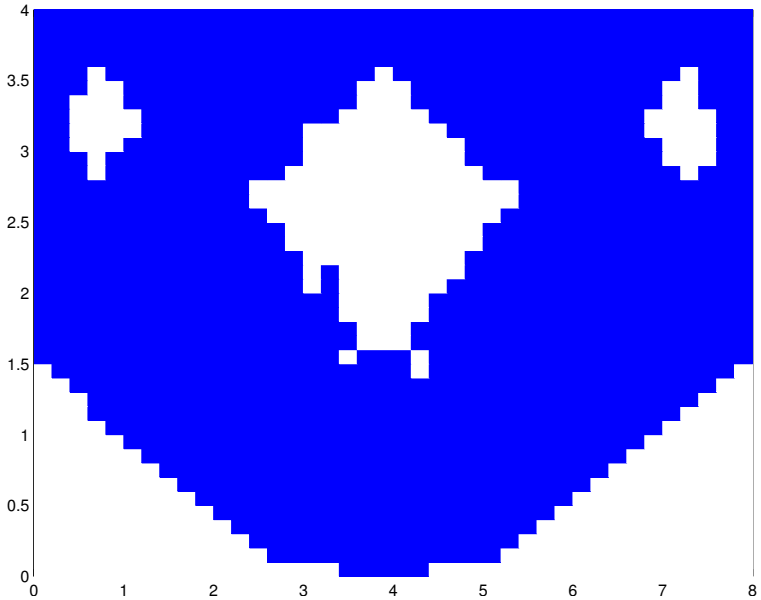

Figure 2: Optimal domain $\Omega^\star$.

problem where the piecewise constant level set function is variable subject to optimization. Compared with the standard level set approach the proposed approach does not require the solution of the Hamilton - Jacobi equation or to perform the reinitialization process of the signed distance function. The proposed method has also a hole nucleation capabilities as with topological gradient based methods.

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