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Dynamic Stability of a Non-linear Continuous System Subjected to Vertical Seismic Excitation

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Abstract

Easily deformable tall structures exposed to a strong vertical component of an earthquake excitation are endangered by auto-parametric resonance effects. This non-linear dynamic process in a post-critical regime caused heavy damage or collapses of many towers, bridges and other structures in the epicenter area. Vertical and horizontal response components are independent on the linear level. However their interaction takes place as a result of non-linear terms in post-critical regime. Two generally different types of post-critical regimes are presented: (i) post-critical state with possible recovery; (ii) exponentially rising horizontal response leading to a collapse. Special attention is paid to processes of transition from semi-trivial to post-critical state in the case of the time limited excitation period because it concerns the seismic processes. The solution method combining analytical and numerical approaches is developed and used. Its applicability and shortcomings are commented. A few hints for engineering applications are given. Some open problems are indicated.

Keywords: auto-parametric systems, semi-trivial solution, dynamic stability, system recovery, post-critical response.

1 Introduction

Papers devoted to dynamics of slender structures (towers, masts, chimneys, bridges, *etc.*) under earthquake attack are dealing mostly with effects of horizontal excitation component. However a strong vertical component in epicenter area represents very often the most dangerous condition leading to structure collapse due to auto-parametric resonance. This highly non-linear dynamic process caused in the past heavy damages or collapses of towers, bridges and other structures. In sub-critical linear regime vertical and horizontal response components are independent. So if no horizontal excitation is taken into account, no horizontal response component is observed. The semi-trivial solution gives a full image of the structure behavior. If the frequency of a vertical excitation in a structure foundation finds in a certain interval and its amplitude exceeds a certain limit, the vertical response component loses dynamic stability and dominant horizontal response component is generated. This post-critical regime (auto-parametric resonance) follows from a strong non-linear interaction of vertical and horizontal response components which can lead to a failure of the structure. Consequently, very widely used linear approach, usually doesnt provide any interesting knowledge in such a case.

Auto-parametric systems have been intensively studied for the last three or four decades. A few theoretical studies dealing with these effects have been published even sooner namely in the period of 1968-1985, see e.g. [10], [21], [30]. Then followed many papers, monographs and other studies dealing with analytical, numerical as well as experimental aspects of auto-parametric systems and their applications. They have been given mostly by Tondl and co-authors, see for instance papers: [14], [15], [27], [28] or one of numerous monographs [29]. Certainly, many other authors contributed to this topic significantly, see e.g. [3], [9], etc. Some ideas and selected results can be found also in papers by authors of this study, e.g. [17], [19]. Basic definitions and results on a level of the rational mechanics can be found in [7], extension onto the stochastic approach, see e.g. monographs [6], [23], [12] or papers, [16], etc.

Similar auto-parametric systems have been studied during recent years, see e.g. authors papers [18], [20]. The mathematical models used in these studies idealized the vertical structure as one concentrated mass related with the basement by a massless spring. However a following-up research revealed that such approach is not satisfactory in many particular cases. In principle easily deformable tall structures are the most sensitive regarding effects of auto-parametric resonance. Therefore the structure itself should be modeled as a console with continuously distributed stiffness and mass in order to respect the whole eigen-value spectrum. Concerning subsoil conventional model including internal viscosity can be retained.

2 Mathematical model

Let us consider the theoretical model in a vertical plane outlined in the Fig. 1. The system is Hamiltonian, see for instance [2]. To deduce the governing differential system in the form of Lagrange equations the kinetic and potential energies of the moving system are formulated as follows:

$$\begin{aligned}
T(t) =& \tfrac{1}{2}M(\dot{y}^2(t) + r^2\dot{\varphi}^2(t)) + && \text{(a)}\\
&+\tfrac{1}{2}\mu\int_0^l [(\dot{\varphi}(t)x + \dot{u}(x,t))^2 + \dot{y}^2(t) - 2\dot{y}(t)(\dot{\varphi}(t)x + \dot{u}(x,t))\sin\varphi(t)]\mathrm{d}x, &&\\
U(t) =& Mg\cdot y(t) + \tfrac{1}{2}C((y(t) - y_0(t))^2 + r^2\varphi^2(t)) + && \text{(b)}\\
&+\mu g\int_0^l [y(t) - x(1-\cos\varphi(t)) - u(x,t)\sin\varphi(t)]\mathrm{d}x + \tfrac{1}{2}EJ\int_0^l u''^2(x,t)\mathrm{d}x. &&
\end{aligned} \tag{1}$$

In Eqs (1) following notification has been introduced:

$y = y(t)$	-	vertical displacement of the B point;
$y_0 = y_0(t)$	-	kinematic excitation (seismic random process);
$\varphi = \varphi(t)$	-	angular rotation of the system in the B point;
$u = u(x,t)$	-	bending deformation of the vertical console;
M	-	foundation effective mass;
C	-	subsoil effective stiffness;
μ	-	console uniformly distributed mass;
EJ	-	console bending stiffness (constant);
η_c, η_e	-	viscous damping parameters of the C, EJ stiffness following Kelvin definition;
r, l	-	geometric parameters;
x	-	length coordinate along the console.

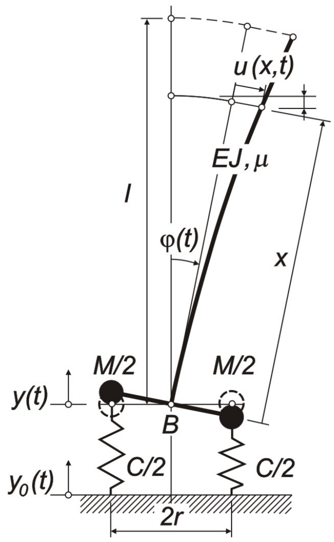

Figure 1: Outline of an autoparametric model with a continuous vertical console

Non-dimensional response and excitation components are useful to be introduced:

$$\begin{aligned}&\zeta_0(t) = y_0(t)/l,\ \zeta(t) = y(t)/l,\ \varphi(t),\ u(x,t)/l = \psi(\xi,t),\\ &\xi = x/l,\ \varrho = r/l,\ m = \mu l\end{aligned} \tag{2}$$

The material damping of the console is proportional. Therefore the deformation of that can be expressed in a form of a convergent series:

$$u(x,t) = \sum_{i=1}^{n} \alpha_i(t) \cdot \psi_i(x) \quad \text{or dimensionless:} \quad \psi(\xi,t) = \sum_{i=1}^{n} \alpha_i(t) \cdot \chi_i(\xi), \\ (\psi_i(x) = l \cdot \chi_i(\xi)), \tag{3}$$

where basis functions $\chi_i(\xi)$ are eigen functions (eigen forms) of the differential equation:

$$\chi_i^{''''}(\xi) + \lambda_i \chi_i(\xi) \ = \ 0, \quad (\lambda_i/l)^4 = \mu\omega_i^2/EJ \tag{4}$$

with boundary conditions valid for a console beam: $\chi_i(0) = 0,\ \chi_i^{'}(0) = 0,\ \chi_i^{''}(1) = 0,\ \chi_i^{'''}(1) = 0$. This approach is useful due to proportional damping which makes time coordinates $\alpha_i(t)$ independent and so the phase shift of each eigen form is constant over the whole definition interval if the damping is sub-critical.

Let us deduce Lagrangian equations for components $\zeta(t), \varphi(t)$ and components $\alpha_i(t)$ arithmetizing coordinates $\chi_i(\xi)$. Let us adopt approximately $(1 - \cos\varphi \approx 0)$

and ($\sin\varphi \approx \varphi$). Hence the system of Lagrangian equations reads:

$$\begin{aligned}
&\ddot{\zeta}(t) - \tfrac{1}{4}\kappa_0(\varphi^2(t))^{\bullet\bullet} - \\
&\qquad -\kappa_0 \sum_{i=1}^{n}[(\varphi(t)\dot{\alpha}_i(t))^{\bullet}\,\Theta_{0,i}] + \omega_0^2[\zeta(t) - \zeta_0(t) + \eta_c(\dot{\zeta}(t) - \dot{\zeta}_0(t))] = 0, \quad \text{(a)} \\
&\ddot{\varphi}(t) - \tfrac{1}{2}\kappa_1\ddot{\zeta}(t)\varphi(t) + \\
&+\kappa_1 \sum_{i=1}^{n}[\ddot{\alpha}_i(t)\Theta_{1,i} + (\dot{\zeta}(t)\dot{\alpha}_i(t) - \omega_2^2\alpha_i(t))\Theta_{0,i}] + \omega_1^2[\varphi(t) + \eta_c\dot{\varphi}(t)] = 0, \quad \text{(b)} \\
&\ddot{\alpha}_i(t)\cdot\Theta_{2,i} + \ddot{\varphi}(t)\cdot\Theta_{1,0} - \\
&\qquad\qquad -[(\dot{\zeta}(t)\varphi(t))^{\bullet} + \omega_2^2\varphi(t)]\cdot\Theta_{0,i} + \omega_3^2[\alpha_i(t) + \eta_e\dot{\alpha}(t)]\Theta_{3,i} = 0, \quad \text{(c)}
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\kappa_0 = \frac{m}{M+m}, \qquad \kappa_1 = \frac{m}{M\varrho^2 + m/3}, \\
&\omega_0^2 = \frac{C}{M+m}, \qquad \omega_1^2 = \frac{C\varrho^2}{M\varrho^2 + m/3}, \qquad \omega_2^2 = \frac{g}{l}, \qquad \omega_3^2 = \frac{EJ}{ml^3} \\
&\Theta_{0,i} = \int_0^1 \chi_i(\xi)\mathrm{d}\xi,\ \Theta_{1,i} = \int_0^1 \xi\chi_i(\xi)\mathrm{d}\xi,\ \Theta_{2,i} = \int_0^1 \chi_i^2(\xi)\mathrm{d}\xi,\ \Theta_{3,i} = \int_0^1 (\chi_i''(\xi))^2\mathrm{d}\xi.
\end{aligned} \tag{6}$$

Regarding parameters $\Theta_{j,i}$, eigen functions of Eq. (4) with respective boundary conditions have a detailed form as follows:

$$\begin{aligned}
&\chi_i(\xi) = (C_1\cdot\cos\lambda_i\xi + C_2\cdot\sin\lambda_i\xi + C_3\cdot\mathrm{ch}\lambda_i\xi + C_4\cdot\mathrm{sh}\lambda_i\xi,) \\
&\qquad C_1 = \sin\lambda_i\mathrm{sh}\lambda_i, \quad C_2 = -\sin\lambda_i\mathrm{ch}\lambda_i - \cos\lambda_i\mathrm{sh}\lambda_i, \\
&\qquad C_3 = -\sin\lambda_i\mathrm{sh}\lambda_i, C_4 = \sin\lambda_i\mathrm{ch}\lambda_i + \cos\lambda_i\mathrm{sh}\lambda_i, \quad \mathrm{ch}\lambda_i\cdot\cos\lambda_i + 1 = 0.
\end{aligned} \tag{7}$$

where $\lambda_i = 1.8751, 4.6941, 7.8548, 10.9955, ...$, etc. are solutions of a transcendent equation: $\mathrm{ch}\lambda_i\cdot\cos\lambda_i + 1 = 0$. In principal analytical form of parameters $\Theta_{j,i}$ can be carried out. However, the results are very complicated and don't provide any information important from physical point of view. Therefore they will be replaced by numerical integration results in particular cases.

The system (5) represents a simultaneous differential system for $\zeta(t), \varphi(t)$ and $\alpha_i(t)$ having a size related with a number of eigen-forms (4) taken into account. Although the console bending is considered linear, components $\alpha_i(t)$ are non-linearly related with $\zeta(t), \varphi(t)$. Nevertheless a mutual link of $\alpha_i(t)$ components is not complicated. This fact follows from the linearity of the bending component, proportionality of its damping and so the orthogonality of relevant eigen forms χ_i as well as their second derivatives χ_i'' in the meaning of Eq. (4) and respective boundary conditions. Concerning the excitation process $\zeta_0(t)$, it will be considered as harmonic in the first step in order to investigate limits of stable semi-trivial and post-critical regimes. Later the random non-stationary character of $\zeta_0(t)$ will be respected.

3 Semi-trivial solution and its stability

Let us consider the harmonic excitation transformed into the dimensionless form:

$$y_0 = A_0 \sin \omega t \quad \Rightarrow \quad \zeta_0 = a_0 \cdot \sin \omega t \ , \quad A_0 = a_0 \cdot l \tag{8}$$

and assume that the stationary semi-trivial solution exists. Its general form can be written as follows:

$$\zeta_s = a_c \cdot \cos \omega t + a_s \cdot \sin \omega t \ , \quad \varphi = 0 \ , \quad \alpha_i = 0 \tag{9}$$

Substituting Eqs (9) into the system (5), Eqs (5b) and (5c) are satisfied identically, while Eqn (5a) doing obvious modifications provides the coefficients a_c, a_s:

$$a_c = -\frac{a_0\omega_0^2}{\delta}\omega^3\eta_c \ , \ a_s = \frac{a_0\omega_0^2}{\delta}(\omega_0^2 - \omega^2 + \omega_0^2\omega^2\eta_c^2) \ , \ \delta = (\omega^2 - \omega_0^2)^2 + \omega_0^4\omega^2\eta_c^2 \tag{10}$$

Expression (9) together with coefficients (10) represents an approximate simple linear stationary solution of the single degree of freedom (SDOF) system moving in vertical direction being excited kinematically in the point B. The resonance curve of the response amplitude has the form:

$$R_0^2 = a_c^2 + a_s^2 = \frac{a_0^2\omega_0^4}{\delta}(1 + \omega^2\eta_c^2) \tag{11}$$

which can be seen in the Figure 2. However the solution being characterized by this curve can be unstable beyond a certain value of the excitation amplitude a_0 in some intervals of the excitation frequency ω. For this reason the stability analysis must be carried out. Very well known general monographs dealing with this topic appeared together with their re-editions, i.e. [7]. Nevertheless, dynamic stability of non-linear systems with one or a couple of degrees of freedom has been discussed using various methods by many authors in problem oriented papers, e.g. [3], [5], or in auto-parametric system focused monographs, e.g. [26].

Let us adopt the linear perturbation approach in order to assess the stability limits of the semi-trivial solution (9). Indeed, it can be written approximately in the arbitrarily small neighborhood of the semi-trivial solution:

$$\begin{array}{llll}
\zeta(t) = \zeta_s(t) + q(t) & = \zeta_s(t) + q_c(t)\cos\omega t & + q_s(t)\sin\omega t \ , & (a) \\
\varphi(t) = \ 0 \ + p(t) & = \qquad p_c(t)\cos\frac{1}{2}\omega t & + p_s(t)\sin\frac{1}{2}\omega t \ , & (b) \\
\alpha_i(t) = \ 0 \ + s_i(t) & = \qquad s_{c,i}(t)\cos\frac{1}{2}\omega t & + s_{s,i}(t)\sin\frac{1}{2}\omega t \ . & (c)
\end{array} \tag{12}$$

where absolute value of the perturbation amplitudes $q_c(t), q_s(t), ...$ are small. The argument (t) will be omitted in further text whenever possible $(\zeta, \varphi, \alpha_i, q, q_c, q_s, ...)$ etc. Introducing expression (12a) into Eqn (5a) and taking into account that ζ_s represents its semi-trivial solution, following equation for perturbation q can be extracted:

$$\ddot{q} + \omega_0^2(q + \eta_c\dot{q}) \ = \ 0 \tag{13}$$

Eqn (13) is linear and homogeneous. It is obvious that $\lim_{t\to\infty} q_c, q_s = 0$ if $\eta_c > 0$ and so stationary solution vanishes. For this reason the vertical response component ζ remains independent and stable in the neighborhood of the semi-trivial solution ζ_s (on the level of the linear perturbation approach).

Let us put now the second column of the expressions (12a-c) into Eqs (5b,c). Keeping only the linear terms of perturbations p, s and respecting that $q \equiv 0$, one obtains the following differential system:

$$\begin{aligned}
&\ddot{p}(t) - \tfrac{1}{2}\kappa_1\ddot{\zeta}_s(t)p(t) + \\
&\quad +\kappa_1 \sum_{i=1}^{n}[\ddot{s}_i(t)\theta_{1,i} + \dot{\zeta}_s(t)\dot{s}_i(t)\theta_{0,i} - \omega_2^2 s_i(t)\theta_{0,i}] + \omega_1^2(p(t) + \eta_c\dot{p}(t)) = 0, \\
&\ddot{s}_i(t)\theta_{2,i} + \ddot{p}(t)\theta_{1,i} - \\
&\quad -(\ddot{\zeta}_s(t)p(t) + \dot{\zeta}_s(t)\dot{p}(t))\theta_{0,i} - \omega_2^2 p(t)\theta_{0,i} + \omega_3^2(s_i(t) + \eta_e\dot{s}_i(t))\theta_{3,i} = 0.
\end{aligned} \tag{14}$$

The system (14) as expected is linear similarly like Eq. (13). However three coefficients include harmonic components due to $\ddot{\zeta}_s, \dot{\zeta}_s$ terms being given by Eqn (9). Hence the system (14) is of the Mathieu type (with parametric excitation) and its solution stability should be verified, cf. for instance [1] or [31].

As the next step functions p, s in Eqs (14) should be replaced by means of their first harmonics represented by the third column in Eqs (12a-c). The method of harmonic balance enables to obtain the following homogeneous algebraic system for $p_c, p_s, s_{c,i}, s_{s,i}$ parameters:

$$\begin{array}{ccccccccc}
\mathbf{P}\cdot\mathbf{p} & + & \mathbf{S}_1\cdot\mathbf{s}_1 & + & \mathbf{S}_2\cdot\mathbf{s}_2 & + \ldots + & \mathbf{S}_n\cdot\mathbf{s}_n & = & 0 \\
\mathbf{S}_1\cdot\mathbf{p} & + & \mathbf{D}_1\cdot\mathbf{s}_1 & + & 0 & + \ldots + & 0 & = & 0 \\
\mathbf{S}_2\cdot\mathbf{p} & + & 0 & + & \mathbf{D}_2\cdot\mathbf{s}_2 & + \ldots + & 0 & = & 0 \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \\
\mathbf{S}_n\cdot\mathbf{p} & + & 0 & + & 0 & + \ldots + & \mathbf{D}_n\cdot\mathbf{s}_n & = & 0
\end{array} \quad \Longrightarrow \quad (\mathbf{P} - \sum_{i=1}^{n}\mathbf{S}_i\cdot\mathbf{D}_i^{-1}\cdot\mathbf{S}_i)\cdot\mathbf{p} = 0, \tag{15}$$

where sub-matrices $\mathbf{P}, \mathbf{S}_i, \mathbf{D}_i \in \mathbb{R}^{2\times 2}$ and vectors $\mathbf{p}, \mathbf{s}_i \in \mathbb{R}^2$ have a form as follows:

$$\mathbf{P} = \begin{bmatrix} \frac{1}{4}\omega^2\cdot a_c + (-\frac{1}{4}\omega^2 + \omega_1^2)\kappa_1^{-1}, & \frac{1}{4}\omega^2\cdot a_s + \frac{1}{2}\omega_1^2\omega\eta_c\kappa_1^{-1} \\ \frac{1}{4}\omega^2\cdot a_s - \frac{1}{2}\omega_1^2\omega\eta_c\kappa_1^{-1}, & -\frac{1}{4}\omega^2\cdot a_c + (-\frac{1}{4}\omega^2 + \omega_1^2)\kappa_1^{-1} \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} p_c \\ p_s \end{bmatrix},$$

$$\mathbf{S}_i = \begin{bmatrix} +\frac{1}{4}\theta_{0,i}\omega^2\cdot a_c - \frac{1}{4}\theta_{1,i}\omega^2 - \theta_{0,i}\omega_2^2, & \frac{1}{4}\theta_{0,i}\omega^2\cdot a_s \\ \frac{1}{4}\theta_{0,i}\omega^2\cdot a_s, & -\frac{1}{4}\theta_{0,i}\omega^2\cdot a_c - \frac{1}{4}\theta_{1,i}\omega^2 - \theta_{0,i}\omega_2^2 \end{bmatrix}, \quad \mathbf{s}_i = \begin{bmatrix} s_{c,i} \\ s_{s,i} \end{bmatrix}, \tag{16}$$

$$\mathbf{D}_i = \begin{bmatrix} -\frac{1}{4}\omega^2\theta_{2,i} + \omega_3^2\theta_{3,i}, & \frac{1}{2}\omega\omega_3^2\eta_e\theta_{3,i} \\ -\frac{1}{2}\omega\omega_3^2\eta_e\theta_{3,i}, & -\frac{1}{4}\omega^2\theta_{2,i} + \omega_3^2\theta_{3,i} \end{bmatrix}.$$

The system (15) is presented in two versions: $(2n+2) \times (2n+2)$ (large) and 2×2 (compact). The latter one is enabled due to special form of the large version. In such a case sub-vectors s_i can be easily eliminated and the system in the compact version can be obtained. However matrix elements of the compact version are very complicated indeed and so applicability can be a bit problematic. Anyway each form is suitable for a particular purposes of analytical or numerical treatment. For instance the basic analysis of stability can be done using the compact version of the system (15). Obtaining eigen-vectors $\mathbf{p}_{(j)}$ sub-vectors $s_{i(j)}$ can be subsequently easily derived by back substitution into large version of the system (15). The only sensitive step can be find inversion of matrices $\mathbf{D}_i$. Inspection of Eqs (16) provides that the determinant of $\mathbf{D}_i$ is always positive whenever the damping η_e (console) is positive and matrices $\mathbf{D}_i$ are all regular. If there is $\eta_e = 0$, one of the determinants det(D_i) can vanish for ω coinciding with the eigen-frequency of the console as it corresponds with particular λ_i. This case however is very seldom and should be treated by a special way. It manifests as a turning point on a stability limit.

Let us be aware that the algebraic system Eqs (15) is meaningful only under certain conditions. The system response should be fully or at least nearly stationary in order to be entitled to apply the harmonic balance method. In other words functions $p_c, p_s, s_{c,i}, s_{s,i}$, although being dependent on time, should enable to be approximated by constants within the interval of one period or at least to be considered as functions of the "slow time". Under circumstances of a chaotic or quasi-periodic response with noticeable energy transfer between ζ and φ, α_i components, the harmonic balance method is inapplicable and the system (15) becomes meaningless. Rich references can be addressed to get experiences with early stage of the post-critical processes with dominating chaotic component, see e.g. papers [1], [4], [9], or monographs [11], [22], [24] or even with random character, e.g. [13][19]. For special considerations regarding non-linear dynamic systems, see [25].

If the above general condition is complied with, $p_c, p_s, s_{c,i}, s_{s,i}$ can be taken as parameters. The system (15) being homogeneous cannot provide non-trivial solution unless the determinant of its matrix vanishes. So that the zero determinant of the system matrix will lay out the shape of the stability limit.

4 Numerical experiments - domain with possible system recovery

Let us recall the system at the Fig. 1 and verify its properties regarding the dynamic stability. Zero determinant of the system (15) will be repeatedly evaluated in a certain interval of frequency ω for various combinations of parameter values presented in the table below. Various combinations of values presented throughout the table have been applied in order to obtain typical results concerning the semi-trivial solution stability. The standard code and programming of Wolfram Mathematica package and some in house developed blocks have been used.

To get an overview about influence of system parameters onto the semi-trivial solution stability let us investigate at first Fig. 2. The black graphs represent resonance

M	C	μ	EJ	η_c	η_e	l	$\varrho = r/l$
10,0	11,0	0,125	100	0,05	0,05	8,0	0,05
			250	0,10	0,10		0,10
			500	0,15	0,15		0,15
			1000	0,20	0,20		0,25
			2500	0,25	0,25		0,35
			5000	0,30	0,30		0,45

Table 1: Parameters of the system analyzed

curves following Eq. (11) for various excitation amplitudes a_0. The red curves stand in stability limits under circumstances that the console bending stiffness is employed by one, two or three eigen-forms. Respective pictures (a)-(f) are evaluated for six bending stiffness levels of the console. In principle it is obvious that increasing number of eigen-forms taken into consideration leads always to drop of the stability limit as the system is getting to be weaker. A certain exception represent narrow areas around eigen-frequencies of the system, whatever type they are.

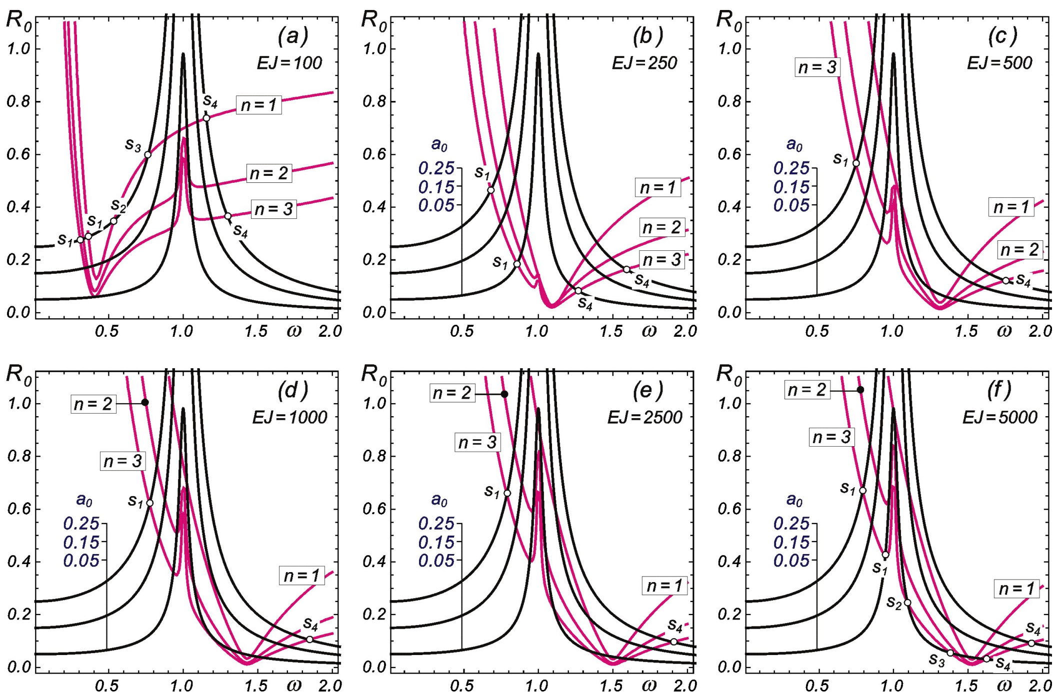

Figure 2: Stability limits of the semi-trivial solution including one, two or three console eigen-forms ($n = 1, 2, 3$) for various bending stiffness of the console; dampings $\eta_c = 0,05, \eta_e = 0,05$, ratio $\varrho = 0,2$.

Picture (a) demonstrates that low bending stiffness leads to the stability loss being concentrated in the area around the 1st eigen-frequency of the console. In this

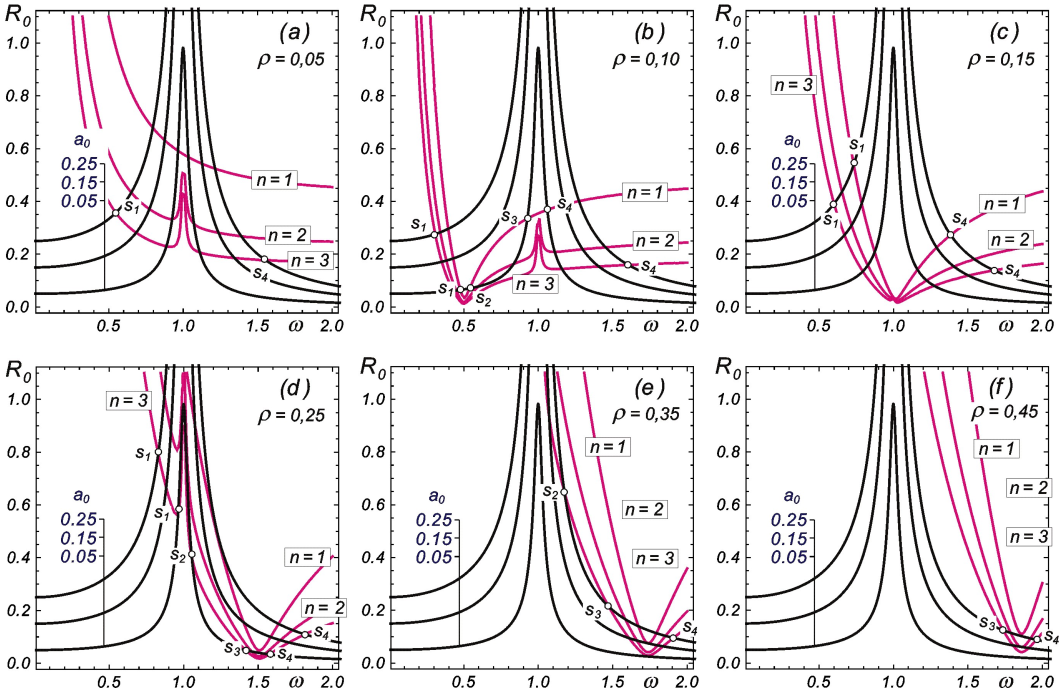

Figure 3: Stability limits of the semi-trivial solution ($n = 1, 2, 3$) for various ratio $\varrho = r/l$ (ground width/height); dampings $\eta_c = 0,05, \eta_e = 0,05$, $EJ = 500$.

frequency domain the number of eigen-forms taken into account is very weak and stability is lost even for small excitation amplitudes a_0. We can see a local maximum in the neighborhood of $\omega = 1,0$ (eigen-frequency of the semi-trivial solution) at the same picture, when two or more eigen-forms ($n = 2$ or more) are taken into account. This interval is however very short and respective positive influence should be neglected. Finally it can be stated that one or two instability interval have been ascertained $\omega \in (s_1 - s_2)$ and $\omega \in (s_3 - s_4)$ or $\omega \in (s_1 - s_4)$ depending on the excitation amplitude a_0 and the number n of eigen-forms respected.

Instability intervals are concentrating mostly in proximity of frequencies $\omega_0, \omega_1, \omega_3$ (sub-soil and system basic properties) and $\omega_{4,5,6,..} = \omega_3 \cdot \lambda_{1,2,3,..}$ (console flexibility). Therefore it is obvious that minimum of stability limits is moving to higher frequencies with increasing bending stiffness of the console. As a special case can be considered picture (b) where nearly ω_0 and ω_5 coincide and twofold eigen-frequency occurs. Thereafter for higher EJ the stability minimum exceeds $\omega = 1$, see pictures (c)-(f). This knowledge can serve as an instruction for engineering practice.

Let us have a look at the Fig. 3 demonstrating an evolution of the stability limits when the ratio $\varrho = r/l$, i.e. ground width/console height is changing. We start with the picture (b). It represents an approximate boundary (exact value is $\varrho_c = 0,086$ keeping other parameters) below which the static stability is violated. In other words for $\varrho < \varrho_c$ the system is instable even in a static state leading to final collapse. Therefore the dynamic problem is worthy to be investigated for $\varrho > \varrho_c$. Of course a position of the static stability boundary in general is a function of all system parameters. The above value ϱ_c corresponds to parameters in use.

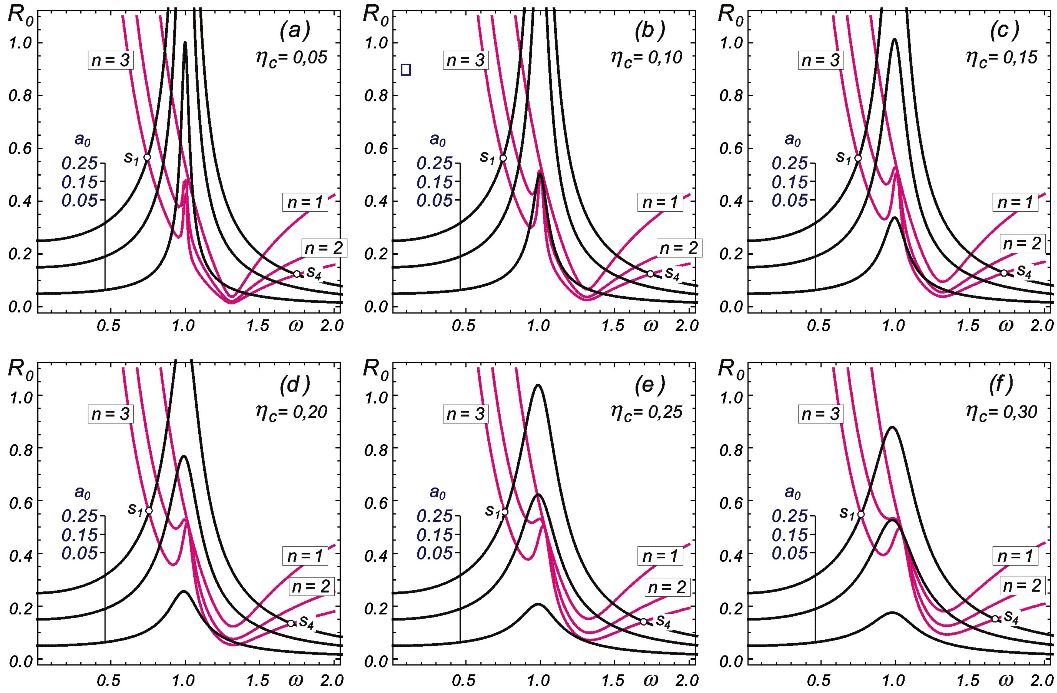

Figure 4: Stability limits of the semi-trivial solution ($n = 1, 2, 3$) for various values of the sub-soil viscous damping; other parameters $\eta_e = 0,05$, $EJ = 500$, $\varrho = 0,2$.

Position of dynamic stability limits minimum is well expressed for each ratio ϱ. The position as well as the value of the minimum are nearly independent from number of eigen-forms n being taken into consideration. The position on the frequency axis is visibly rising with increasing ratio ϱ abandoning resonance area of semi-trivial solution. As it follows from pictures (d)-(f), the stability loss is less and less probable even for higher amplitudes of excitation. Therefore the broad band excitation is also less and less dangerous. This attribute should be taken into account in a practical engineering, despite its technical application is much more complex as adjusting of the console stiffness.

The third parameter significantly influencing the semi-trivial solution (or the system) stability is the sub-soil viscous damping. Although a lot different models of the damping can be discussed, Voigt model is probably able to describe the principle properties of the system response respecting the damping. It follows from Fig. 4, that resonance curves of the semi-trivial system are rapidly dropping with increasing η_c parameter while the shape of stability limits doesn't change considerably. Instability area concentrates around frequency ω_0 and more or less keeps its position and extent. Therefore for design practice is recommended to try as much increase the sub-soil viscosity as possible using some special stuffs for material treating. Internal damping of the console η_e influences the stability limits as well, see Fig. 5. However variation of this parameter didn't lead to considerable changes in shape and character of respective stability limits provided that other system parameters are kept. Indeed the interval $\eta_e \in (0,05; 0,30)$ where the system was tested is large enough to cover usual damping ratio values encountered in civil engineering regarding concrete or steel. Hence

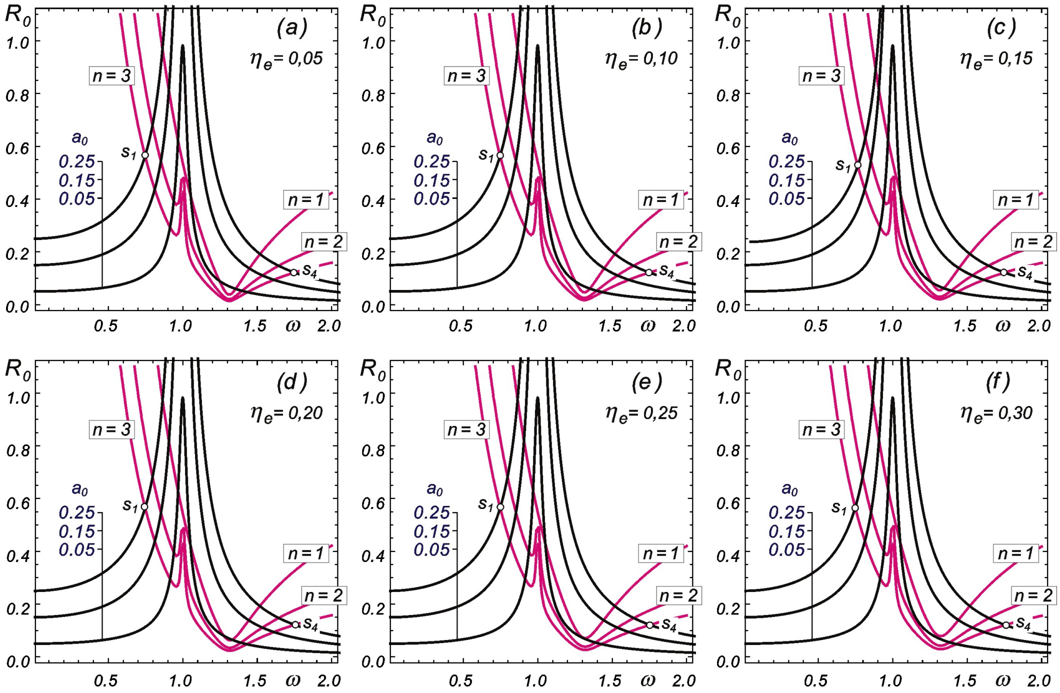

Figure 5: Stability limits of the semi-trivial solution ($n = 1, 2, 3$) for various values of the console material damping; other parameters $\eta_c = 0,05$, $EJ = 500$, $\varrho = 0,2$.

in practice $\eta_e \neq 0$ should be respected, however its value occurring within a domain recommended by the most standards. Slightly different is the case of the sub-soil damping ratio η_c, where the real values can approach the critical damping.

5 Numerical experiments - the limit of irreversibility

As it has been mentioned above, the post-critical regime can be of two types. Both of them are governed by the full differential system (5). The first type means a response process running within a certain limits around the semi-trivial solution. When the excitation is stopped, the system is able to recover and to return to a standstill. Overstepping the limit of irreversibility (or the outer stability limit) the second regime emerges leading to inevitable collapse of the system. The response becomes non-periodic rising exponentially beyond all limits. The mathematical model (5) is not able any more to give a true picture of such terminal states. Its applicability finishes shortly after the limit of the irreversibility.

To trace this limit the analytical investigation of the system (5) doesn't probably provide any understandable results. Therefore simulation processes should be undertaken in order to outline this limit. Numerical solution of the system (5) in full version has been multiply performed as long as the numerical process fails due to numerical stability loss. This collapse occurs in a certain time from the beginning of the integration, because the cumulative errors lose an ability to eliminate themselves. So that the moment when this state occurs indicate that the limit of irreversibility has been

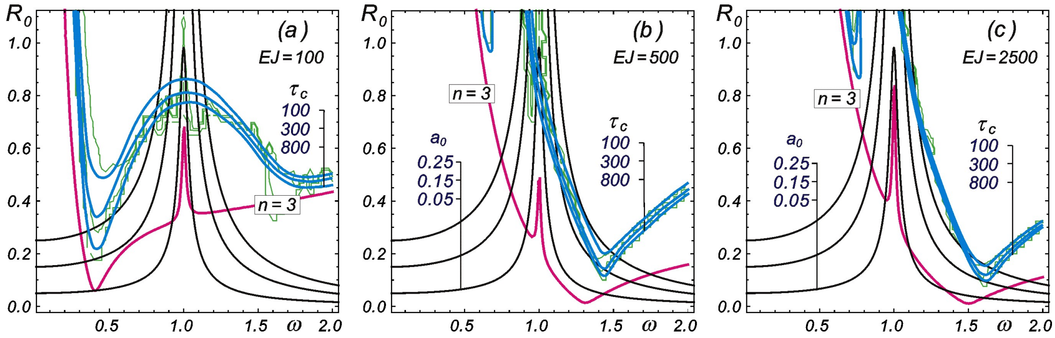

Figure 6: Outer stability limit (limit of irreversibility) of the system - blue curves for the increasing console bending stiffness; other parameters $\eta_e = 0,05$, $\varrho = 0,2$.

reached.

Some result have been plotted in Fig. 6 for three console bending stiffness. The stability limit of the semi-trivial solution has been plotted only for $n = 3$ (three eigen-forms considered). Green curves represent limits of irreversibility. There are plotted three limits in every picture (a)-(c). Each one demonstrates interconnection of points when numerical process collapsed after a certain time τ_c. Three levels of τ_c have been investigated. It is obvious, that increasing τ_c, the result converges to a fixed curve making a lower envelope of all partial results. Therefore there exists a limit curve characterizing the limit of irreversibility independent from τ_c and the solution process itself.

Results demonstrate that the blue curves are approaching stability limits of the semi-trivial solution especially for higher values of the bending stiffness of the console. Special problems emerged for low bending stiffness when the eigen-frequency ω_0 oversteps the first bending eigen-frequency of the console.

6 Conclusion

Authors deal with easily deformable tall structures which are very sensitive to effects of auto-parametric resonance (chimneys, towers, etc.). If the amplitude of a vertical excitation in a structure foundation exceeds a certain limit, a vertical response component loses stability and dominant horizontal response component arises. This post-critical regime (auto-parametric resonance) follows from the non-linear interaction of vertical and horizontal response components and can lead to a failure of the structure.

The Hamiltonian functional is formulated and subsequently a respective Lagrangian governing system is carried out. The differential system shows that horizontal and vertical response components are independent on the linear level. Their interaction takes place arising from the non-linear terms in post-critical regime only. Two generally different types of the post-critical regimes are presented in the paper: (i) Although in the post-critical state, a certain area in the neighbourhood of the stable state exists from where the structure response can return back to the stable state, when the stability conditions are regained; sensitivity of the system parameters concerning auto-parametric stability loss is carefully analysed; (ii) Beyond the primary area of the instability the rocking response component rises rather exponentially leading inevitably to a failure of the structure. Consequently, the presence of the horizontal component in the system response does not automatically mean inevitable collapse of the structure. Such a response can keep the stationary character and can disappear, if the excitation is removed. However, if the limit of the irreversibility is overstepped, horizontal response components rise beyond any limits and the structure collapses.

In principle solution methods combining analytical and numerical approaches have been developed and used. Their applicability and shortcomings are commented upon. A few suggestions for engineering applications in a design practice are given and some open problems are indicated.

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