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# The Elastic-Viscoelastic Correspondence Principle and Parameter Identification

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## Abstract

This paper presents an identification method for viscoelastic materials employing the wavelet analysis. The proposed inverse analysis is based on elastic-viscoelastic correspondence principle for linear viscoelastic materials, and two-dimensional discrete wavelet analysis is applied to the system matrix of the iteration equation for identifying viscoelastic materials. The elastic-viscoelastic correspondence principle [1] is that the Laplace time-transformed viscoelastic field equations and boundary conditions are formally identical with the equations for an elastic body of the same geometry. Therefore, identification analysis of viscoelastic materials can be converted to an 'associated' elastic problem in the Laplace domain [2]. By applying the discrete wavelet transform to the system matrix, we estimate unknown material parameters for not only overdetermined systems and determined systems, but also underdetermined systems. Numerical example is calculated to investigate the validity of the method.

**Keywords:** viscoelasticity, correspondence principle, parameter identification, discrete wavelet transform, finite element analysis.

# 1 Introduction

We have experienced an increased demand for identification or back analysis in various engineering fields. Back analysis has been mainly applied for predicting unknown parameters such as the initial stress or material properties. It is assumed that in most of these works the material is elastic. However, soil, rock mass, concrete or most materials, more or less, show time dependency.

The elastic-viscoelastic correspondence principle [1] is that the Laplace time-transformed viscoelastic field equations and boundary conditions are formally identical with the equations for an elastic body of the same geometry. Thus, transformed solutions can be calculated by standard elastic analysis, and then inverted to obtain the time dependent response. Therefore, identification analysis of viscoelastic materials can be converted to an 'associated' elastic problem in the Laplace domain [2].

Most of the inverse problems generally require the solution of an ill-posed system of equations. Especially for a problem in which the number of unknown parameters exceeds the measured data, it is difficult to identify the unknown parameters. The wavelet transform is a mathematical tool and has been widely used for image compression and signal processing. Various FEM or BEM techniques together with wavelet transforms have been studied for solving systems of linear equations [3]–[8]. Doi et al. presented a new inverse method using wavelet analysis in magnetic fields [9], which utilizes the data compression ability and the spectrum resolution ability of the waveforms.

An identification scheme for elastic materials applying the wavelet transform was proposed [10]. In this paper, we adapt this scheme to the identification of viscoelastic materials. Two-dimensional discrete wavelet transform is applied to the rectangular system matrix for identifying viscoelastic materials in the Laplace domain, and then we identify the viscoelastic parameters in the real domain by applying the least-squares estimation. Finite element discretization is used for the linear viscoelastic analysis and viscoelastic parameters for volumetric and deviatoric components are sought using displacement values calculated by the ordinary finite element method as 'observed data'. Numerical example is calculated to investigate the validity of the method.

# 2 System Equation Based on the Correspondence Principle

Applying the elastic-viscoelastic correspondence principle, the system of equibrium equation in the Laplace domain is written as

(Equation of equilibrium)

$$\nabla \cdot \boldsymbol{\sigma}_s = \mathbf{0} \qquad \text{in } \Omega \tag{1}$$

(Boundary conditions)

$$\boldsymbol{u}_s = \hat{\boldsymbol{u}}_s \qquad \text{on } \partial\Omega_u \text{ (displacement boundary)} \tag{2}$$

$$\boldsymbol{\sigma}_s \boldsymbol{n} = \hat{\boldsymbol{t}}_s \qquad \text{on } \partial\Omega_t \text{ (traction boundary)} \tag{3}$$

where $\Omega$ is a given region, $\partial\Omega$ its boundary, and $\boldsymbol{n}$ the outward unit normal on $\partial\Omega$. The subscript $s$ represents the value in the Laplace domain.

For the parameter identification problem, observed data are displacement $\bar{\boldsymbol{u}}_s$ on (a part of) $\partial\Omega_t$, where the traction $\hat{\boldsymbol{t}}_s$ is given, or conversely traction $\bar{\boldsymbol{t}}_s$ on (a part of) $\partial\Omega_u$ where the displacement $\hat{\boldsymbol{u}}_s$ is given. Considering the problem of measuring traction, the boundary $\partial\Omega_u$ is not used in this formulation.

(Observational boundary condition)

$$\boldsymbol{u}_s = \bar{\boldsymbol{u}}_s \qquad \text{on } \partial\Omega_{t'} \subset \partial\Omega_t \tag{4}$$

The finite element discretization for the system of equibrium equations in the Laplace domain is written as

$$\boldsymbol{K}_s(\boldsymbol{P}_s)\boldsymbol{U}_s = \boldsymbol{F}_s \tag{5}$$

$$\boldsymbol{K}_s(\boldsymbol{P}_s) = \int_\Omega \boldsymbol{B}^t\boldsymbol{D}_s(\boldsymbol{P}_s)\boldsymbol{B}dv, \qquad \boldsymbol{F}_s = \int_{\partial\Omega_t} \boldsymbol{N}^t\hat{\boldsymbol{t}}_s ds$$

where $\boldsymbol{N}$ is the matrix of shape functions and $\boldsymbol{B}$ is the strain-displacement matrix. $\boldsymbol{K}_s$ is the total stiffness matrix which involves the unknown material parameters $\boldsymbol{P}_s$, and $\boldsymbol{U}_s$, $\boldsymbol{F}_s$ the matrices of nodal displacement and nodal force in the Laplace domain respectively. $\boldsymbol{D}_s$ is the stress-strain matrix which is composed of the Laplace transformed unknown parameters $\boldsymbol{P}_s$, that is, volumetric and deviatoric modulus of viscoelastic materials $s\bar{K}(s)$ and $s\bar{G}(s)$. $\hat{\boldsymbol{t}}_s$ is the traction vector in the Lapalce domain. The subscript $s$ represents the value in the Laplace domain. In the ordinary forward analysis, by solving the above simultanous equations, we can obtain the values $\boldsymbol{U}_s$, $\boldsymbol{F}_s$ successively for discrete values $s(=\gamma_m)$ in the Laplace domain.

The observed boundary condition (4) is discretized in the Laplace domain as

$$\boldsymbol{S}_u\boldsymbol{U}_s = \bar{\boldsymbol{U}}_s \tag{6}$$

where $\bar{\boldsymbol{U}}_s$ is the observed displacement data at the boundary nodes and $S_u$ is the matrices to select the the suitable nodal displacements corresponding to the observed displacements.

The parameters $\boldsymbol{P}_s$ to be identified are implicitly included in the FEM governing equation (5), combined with the observational boundary condition (6).

Applying the Newton's iteration scheme for Equations (5) and (6), we get the following system equation at the $k$th iteration step:

$$\boldsymbol{A}_s d\boldsymbol{x} = \boldsymbol{R}_s \tag{7}$$

where

$$\boldsymbol{A}_s = \begin{bmatrix} (\frac{\partial\boldsymbol{K}_s}{\partial\boldsymbol{P}_s}\boldsymbol{U}_s) & \boldsymbol{K}_s(\boldsymbol{P}_s) \\ \boldsymbol{0} & \boldsymbol{S}_u \end{bmatrix}^k$$

$$d\boldsymbol{x}_s = \begin{Bmatrix} d\boldsymbol{P}_s \\ d\boldsymbol{U}_s \end{Bmatrix}^k = \begin{Bmatrix} \boldsymbol{P}_s^{k+1} - \boldsymbol{P}_s^k \\ \boldsymbol{U}_s^{k+1} - \boldsymbol{U}_s^k \end{Bmatrix}$$

$$\boldsymbol{R}_s = \left\{ \begin{array}{l} \boldsymbol{F}_s - \boldsymbol{K}_s(\boldsymbol{P}_s)\boldsymbol{U}_s \\ \\ \bar{\boldsymbol{U}}_s - \boldsymbol{S}_u\boldsymbol{U}_s \end{array} \right\}^k$$

Solving Equation (7) with an iterative procedure until the discrepancy between measurements and numerical results is minimized, we can determine the unknown parameters $\boldsymbol{P}_s$ in the Laplace domain.

# 3 Approximate Inverse Matrix Using a Wavelet Transform

Applying the two-dimensional discrete wavelet transform to the system equation, the system matrix $\boldsymbol{A}_s$ is transformed into the wavelet spectrum $\boldsymbol{A}'_s$ as

$$\boldsymbol{A}'_s = \boldsymbol{W}_n \boldsymbol{A}_s \boldsymbol{W}_m^T \tag{8}$$

where $\boldsymbol{W}_i$ denotes the wavelet transform matrix with order $i \times i$ and we assume here $\boldsymbol{A}_s$ is the $n \times m$ matrix. The superscript $T$ refers to the transpose of matrix $\boldsymbol{W}$. Since $\boldsymbol{W}^T = \boldsymbol{W}^{-1}$, the inverse wavelet transform is carried out by the following equation:

$$\boldsymbol{A}_s = \boldsymbol{W}_n^T \boldsymbol{A}'_s \boldsymbol{W}_m \tag{9}$$

The wavelet transform requires that the size of matrix is a power of two. If $n$ and/or $m$ are not power of two, the size of the matrix is fitted for a power of two by embedding the matrix $\boldsymbol{A}_s$ into a zero matrix whose size is a power of two.

It is known that the wavelet spectrum has large absolute values around the mother wavelet. Then, we extract a square matrix $\boldsymbol{S}$ which is composed of the dominant elements from the entire wavelet spectrum $\boldsymbol{A}'_s$ and calculate the inverse matrix $\boldsymbol{S}^{-1}$. Combining $\boldsymbol{S}^{-1}$ with a zero matrix with order $m \times n$ yields an approximate inverse matrix $\boldsymbol{A}'^{-1}_{Appro}$ in the wavelet spectrum space. Thus, an approximate inverse matrix $\boldsymbol{A}^{-1}_{Appro}$ of the system matrix is obtained by applying the inverse wavelet transform to the matrix $\boldsymbol{A}'^{-1}_{Appro}$ as

$$\boldsymbol{A}^{-1}_{Appro} = \boldsymbol{W}_m^T \boldsymbol{A}'^{-1}_{Appro} \boldsymbol{W}_n \tag{10}$$

Finally, the unknown vector $d\boldsymbol{x}_s$ can be obtained by

$$d\boldsymbol{x}_s = \boldsymbol{A}^{-1}_{Appro} \boldsymbol{R}_s \tag{11}$$

Then a new vector $\boldsymbol{P}_s$ for the next iteration $(k+1)$ is determined as

$$\boldsymbol{P}_s^{k+1} = \boldsymbol{P}_s^k + \lambda d\boldsymbol{P}_s^k \tag{12}$$

where $\lambda$ is a scalar which is determined by means of the one-dimensional optimization method along the vector $d\boldsymbol{P}_s$.

The convergence is acceptable if either of the following conditions are satisfied:

$$\frac{||\bar{\boldsymbol{U}}_s - \boldsymbol{U}_s^{k+1}||}{||\bar{\boldsymbol{U}}_s||} \ < \ \varepsilon_u \tag{13}$$

$$Max\left\{\left|\frac{P_s^{k+1} - P_s^k}{P_s^{k+1}}\right|_i\right\} \ < \ \varepsilon_p\ , \quad i = 1, 2, \ldots, N^p \tag{14}$$

where $N^p$ is the number of parameters to be identified and $0 < \varepsilon_u,\ \varepsilon_p \ll 1$ are given convergence tolerances for the displacements and material parameters, respectively.

# 4 Identification in the Real Domain

$s\bar{K}(s)$ and $s\bar{G}(s)$, which are the volumetric and deviatoric modulus of viscoelastic materials in the Laplace domain, are identified from the obseved data $\bar{U}(s)$. The scheme is similar to the elastic case by applying the elastic-viscoelastic correspondence principle. We can obtain the values $s\bar{K}(s)$, $s\bar{G}(s)$ for discrete values $s(=\gamma_m)$ as follows ;

$$\begin{aligned} s\bar{K}(s) &= \gamma_1\bar{K}(\gamma_1)\,,\ \gamma_2\bar{K}(\gamma_2)\,,\ \cdots\ ,\ \gamma_n\bar{K}(\gamma_n) \\ s\bar{G}(s) &= \gamma_1\bar{G}(\gamma_1)\,,\ \gamma_2\bar{G}(\gamma_2)\,,\ \cdots\ ,\ \gamma_n\bar{G}(\gamma_n) \end{aligned} \tag{15}$$

where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are different positive real constants.

In case that the volumetric and deviatoric relaxation function in the real domain are assumed as

$$K(t) = K_0 + K_1 e^{-t/\tau_{K_1}} + K_2 e^{-t/\tau_{K_2}}, \qquad \tau_{K_i} = \frac{\eta_i^v}{K_i}$$

$$G(t) = G_0 + G_1 e^{-t/\tau_{G_1}} + G_2 e^{-t/\tau_{G_2}}, \qquad \tau_{G_i} = \frac{\eta_i^d}{G_i} \tag{16}$$

viscoelastic parameters identified are as follows ;

$$\boldsymbol{P} = [K_0, G_0, K_1, G_1, \eta_1^v, \eta_1^d, K_2, G_2, \eta_2^v, \eta_2^d]$$

From the Laplace transform of (16) we can obtain $s\bar{K}(s)$ and $s\bar{G}(s)$ as

$$s\bar{K}(s) = K_0 + \frac{sK_1}{s + 1/\tau_{K_1}} + \frac{sK_2}{s + 1/\tau_{K_2}}$$

$$s\bar{G}(s) = G_0 + \frac{sG_1}{s + 1/\tau_{G_1}} + \frac{sG_2}{s + 1/\tau_{G_2}} \tag{17}$$

Each value of (15) must be equal to a value given by (17) when $s$ is replaced by $\gamma_m(m = 1, 2, \cdots, n)$. We can consequently identify unknown parameters $\boldsymbol{P}$ in the real domain by applying the least-squares estimation in (15) and (17).

# 5 Numerical Example

To check the validity of the proposed method, we here consider a two layered slope problem shown in Figure 1. Viscoelastic parameters for volumetric and deviatoric components are sought using displacement values calculated by the ordinary finite element method as 'observed data'. Linear triangle elements are used and the numerical inversion of the Laplace transform is evaluated using Schapery's collocation method [11] because this method has a merit that the Laplace parameters $s(=\gamma_m)$ can be chosen arbitrarily. Eight discrete $s$ values are used which are 0.01, 0.03, 0.05, 0.1, 0.25, 0.5, 1.0 and 5.0 from the variation of $s\bar{U}(s)$ (Figure 2).

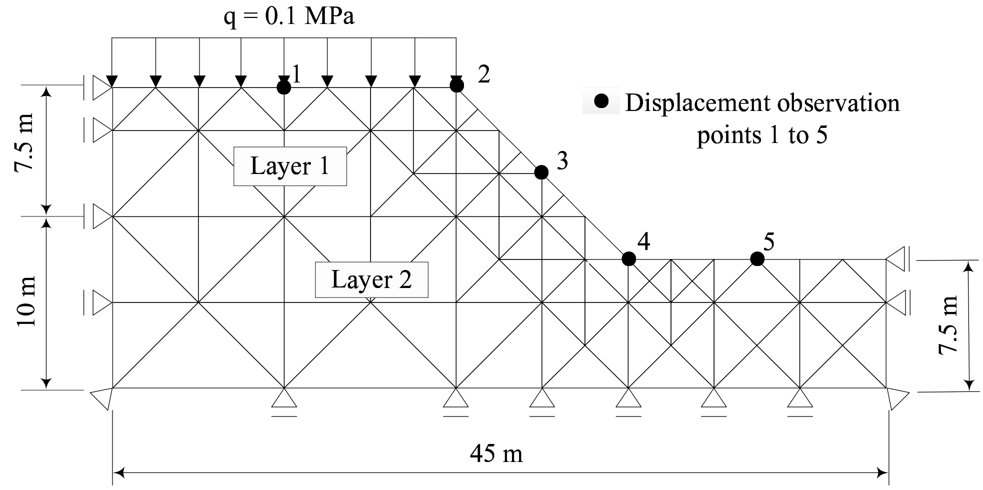


Figure 1: Two layered vertical slope problem.

Maxwell models for volumetric and deviatoric components with 5-elements are used and the material responses are assumed to be given by

$$\begin{aligned} \text{Layer 1:} \quad K(t) &= 2.08 + 0.33e^{-t/3} + 0.08e^{-t/30} \\ G(t) &= 0.96 + 0.15e^{-t/3} + 0.04e^{-t/30} \\ \\ \text{Layer 2:} \quad K(t) &= 4.17 + 1.11e^{-t/5} + 0.28e^{-t/50} \\ G(t) &= 3.13 + 0.83e^{-t/5} + 0.21e^{-t/50} \\ &\qquad \text{elapsed time } t\text{: (min.)} \end{aligned}$$

The number of unknown parameters which should be identified is 20. Basis function derived by Daubechies [12] is used in the wavelet analysis. This wavelet basis is linearly transformed to produce the wavelet transform matrix $\boldsymbol{W}$ in Equation (8). The displacement observation points are 1 to 5 shown in Figure 1.

The size of the system matrix $\boldsymbol{G}$ is $126\times122$. Then the size of the wavelet transform matrix $\boldsymbol{W}$ is set to be $128 \times 128$ and the matrix $\boldsymbol{S}$ which is extracted from the entire

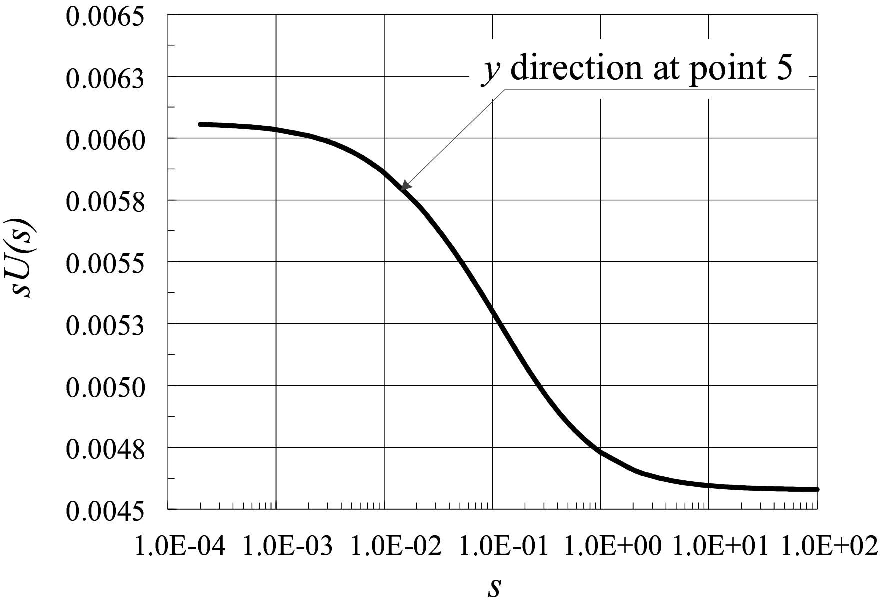


Figure 2: Relation between $s$ and $s\bar{U}(s)$ in the $y$ direction at observation point 5.

wavelet spectrum $\boldsymbol{G}'_{Addzero}$ with order $128 \times 128$ is set to be a $126 \times 126$ matrix. The convergence tolerances are $\varepsilon_u = 0.05$ and $\varepsilon_p = 0.005$.

$K_0$ and $G_0$ can be identified independently at $s = 0$ by using the response data in the Laplace domain as follows :

$$K_0 = \lim_{s \to 0} s\bar{K}(s) = \lim_{s \to 0} \sum_i \left[ K_0 + \frac{K_i s}{s + 1/\tau_{K_i}} \right] = K_0$$

$$G_0 = \lim_{s \to 0} s\bar{G}(s) = \lim_{s \to 0} \sum_i \left[ G_0 + \frac{G_i s}{s + 1/\tau_{G_i}} \right] = G_0$$

The scheme is the same as for the elastic case[10]. The true, initial and identified values are found in Table 1 for volumetric and deviatoric components respectively. In Table 1, symbol 'Daub18' implies Daubechies's 18th-order wavelet basis.

# 6 Conclusion

In this paper, a parameter identification procedure using the discrete wavelet analysis is proposed. The approximate inverse matrix of the rectangular system matrix is obtained by applying the two-dimensional discrete wavelet analysis to the system matrix of the parameter identification equation. The formulation of the method using the correspondence principle is similar to the elastic case in the Laplace domain, and is easier to deal with than the incremental method.

$(K_i$ , $G_i$ : MPa ; $\eta_i^v$ , $\eta_i^d$ : MPa/ min)

| (Volumetric components) | | $K_0$ | $K_1$ | $\eta_1^v$ | $K_2$ | $\eta_2^v$ |
|---|---|---|---|---|---|---|
| Layer 1 | True | 2.080 | 0.330 | 0.990 | 0.080 | 2.500 |
| | Initial | 4.000 | 0.500 | 1.500 | 0.100 | 3.125 |
| | Daub18 | 2.024 | 0.524 | 1.673 | 0.067 | 2.710 |
| Layer 2 | True | 4.170 | 1.110 | 5.550 | 0.280 | 13.888 |
| | Initial | 7.000 | 2.000 | 10.000 | 0.500 | 24.800 |
| | Daub18 | 4.432 | 2.458 | 63.851 | 0.166 | 13.666 |

| (Deviatoric components) | | $G_0$ | $G_1$ | $\eta_1^d$ | $G_2$ | $\eta_2^d$ |
|---|---|---|---|---|---|---|
| Layer 1 | True | 0.960 | 0.150 | 0.461 | 0.040 | 1.150 |
| | Initial | 2.000 | 0.400 | 1.228 | 0.100 | 2.875 |
| | Daub18 | 0.951 | 0.401 | 1.242 | 0.094 | 2.731 |
| Layer 2 | True | 3.130 | 0.830 | 4.150 | 0.210 | 10.416 |
| | Initial | 6.000 | 2.000 | 10.000 | 0.500 | 24.800 |
| | Daub18 | 3.251 | 2.185 | 12.973 | 0.324 | 27.029 |

Table 1: Calculated results of the two layered slope problem

Further detailed studies are required such as the choice of the effective order of the wavelet basis functions, the effect of the data compressibility of the wavelet spectrum and the effective choice of the Laplace transform discrete $s(=\gamma_m)$ values.

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