Paper 297


Proceedings of the Eleventh International Conference
on Computational Structures Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

# A Practical Tool for Estimating Errors of Stresses in Assemblies

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## Abstract

In this work, we focus on error estimation for the assembly problem. We develop an estimator dedicated to industrial structures where the cost is of primary importance. The method is based on the constitutive relation error and associated admissible field construction.

**Keywords:** assemblies, error estimation, quantity of interest.

## 1 Introduction

In the framework of industrial design, a common practice is to use finite element codes to model assemblies. We focus in this work on the quality of such simulations [2]. Some works have been developed in the past for controlling finite element simulations with contact [10, 4], but the measure introduced was global. As far as mechanical criteria are often local, a global measure is not sufficient. We propose here to focus on a quantity of interest which is interesting for dimensioning : the stress. The path we follow to obtain an error in a quantity of interest is to extend previous works developed for linear analysis [7]. The final objective is to control the meshes in the framework of assemblies, starting from the local error measure defined in this work.

For more than thirty years the global discretization error (see [1, 9] for an overview) has been extensively developed. More recently, research has focused on *goal-oriented error estimation*, *i.e.* the estimation of the error on specific outputs of interest which may be relevant for design purposes. Several techniques have been proposed for goal-oriented error estimation, and particularly for linear problems [11, 3, 12, 13, 5] based on the resolution of a dual problem. Extensions of this method have also been proposed for different non-linear problems. We propose here an alternative approach that conduce to a lower cost. Results are not proved mathematically but remains very sharp, which is interesting from a practical point of view.

We use here the constitutive error concept to estimate the finite element errror [9]. This method relies on construction of the so-called admissible fields. The first point of this work is to revisit the construction of the admissible stresses. A particular attention is paid to the construction at the boundary of the domain ; especially for boundaries that makes part of the contact zone. From a practical point of view, some information can be missing due to files' transfer between codes and are not available for *a posteriori* error estimation. The second point is to derivate a practical tool for estimating error on the stress.

# 2 Reference model and constitutive relation error

We focus on the quality of such 3D finite element simulations. The error on the results obtained by simulation can be split in two parts. The first part is due to the discretization of the finite element method. The second part is due to the numerical method used for solving the contact problem.

After introducing the reference problem in section 2.1, we introduce an error estimator section 2.2. The estimator introduced allows one to estimate both part of error (classical F.E. discretisation and contact problem).

## 2.1 Reference problem

We consider the problem of an elastic domain $\Omega$. The structure has been divided into sub-domains $\Omega_E$ and interfaces $\Phi_{EE'}$ (fig 1). The structure is submited to body forces $\underline{f}_d$ defined on $\Omega$. Prescribed displacements $\underline{u}_d$ and forces $\underline{F}_d$ on the dirichlet $\partial_1\Omega$ and Neumann $\partial_2\Omega = \partial\Omega - \partial_1\Omega$ boundary limits. The unknowns of the equations are $\underline{u}$ the displacement in the subdomains, $\underline{W}$ the displacement on the interfaces, $\sigma$ the stress field in the subdomains and $\underline{F}$ the tractions on the interfaces.

We designate the material's Hooke's operator by $\mathbf{K}$,and $b$ the bi-potential introducted by De Saxc on [4].

The three sets of equations are then:

- The kinematic constraints

$$\underline{u} = \underline{u}_d \quad on \quad (\partial_1\Omega) \tag{1}$$

$$\underline{u} = \underline{W} \quad on \quad (\Phi_{EE'}) \tag{2}$$

- The equilibrium equations

$$\sigma \in \mathcal{S}, \quad \sigma.\underline{n} = \underline{F}_d \quad on \quad (\partial_2\Omega) \tag{3}$$

$$\sigma \in \mathcal{S}, \quad \underline{div}\,(\sigma) + \underline{f}_d = \underline{0} \quad on \quad (\Omega_E) \tag{4}$$

$$\sigma.\underline{n} = \underline{F} \quad on \quad (\Phi_{EE'}) \tag{5}$$

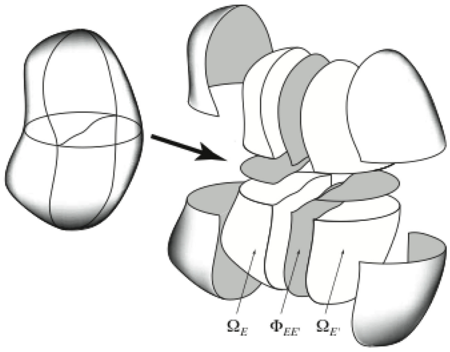


Figure 1: reference model

- The constitutive relation

$$\sigma = \mathbf{K}\epsilon\left(\underline{u}\right) \quad on \quad (\Omega) \tag{6}$$

$$b(\underline{F}, -\underline{W}) = \underline{F}.\underline{W} \quad on \quad (\Phi_{EE'}) \tag{7}$$

## 2.2 Constitutive relation error

The constitutive error concept is a tool for that measure the distance between the unknown solution of the reference problem and its finite element resolution [9]. The approach based on the error in constitutive relation relies on a partitioning of the above equations into two groups:

- the admissibility conditions Eqs (1) to (5)
- the constitutive relation Eqs (6) and (7)

In practice, the constitutive relation is often the least reliable of all the equations of the reference problem. Therefore, it is natural to consider approximate solutions which verify the admissibility conditions exactly. Then we quantify their quality by the degree to which the constitutive relations are checked. This leads to the introduction of the following definition of the constitutive error:

$$E^2_{CRE} = \sum_{\Omega_E} \|\hat{\sigma} - \mathbf{K}\epsilon(\hat{\underline{u}})\|^2_{\Omega_E} + \sum_{\Phi_{EE'}} \|b(\underline{F}, -\underline{W}) - \underline{F}.\underline{W}\|^2_{\Phi_{EE'}} \tag{8}$$

with $\hat{\underline{u}}, \hat{\sigma}$ are admissible field ($\hat{\underline{u}}$ is kinematically admissible and $\hat{\sigma}$ is statically admissible). And where $\| \bullet \|_{\Omega_E}$ (resp. $\| \bullet \|_{\Phi_{EE'}}$) is the energy norm define on $\Omega_E$ (resp. $\Phi_{EE'}$).

We are interested here in estimate $\|\hat{\sigma} - \mathbf{K}\epsilon(\hat{\underline{u}})\|^2_{\Omega_E}$ on one domain $\Omega_E$ (discretization part of the constitutive error) without using the data of other domains $\Omega_E$. The key point of the method is the construction a pair of admissible fields $(\hat{\underline{u}}, \hat{\sigma})$. In fact the finite element solution $\underline{u}_h$ is always kinematically admissible, so we choose usually $\hat{\underline{u}} = \underline{u}_h$.

## 2.3 Admissible field

### 2.3.1 two steps

The construction of a kinematically admissible field is straightforward. The calculation of an admissible stress field $\hat{\sigma}$ is a crucial and technically complicated point. It is based on the finite element solution $\sigma_h$ and the problem's data.
This construction is divided on two step:

- Construction of load densities $\hat{\underline{F}}_h$:
  The load densities $\hat{\underline{F}}_h$ are defined on the edge $\Gamma$ of each element. These densities are calculated such that they are in equilibrium with the given boundaries condition of the reference problem, they check:

$$\hat{\underline{F}}_h = \underline{F}_d \qquad sur \qquad \partial_2\Omega \tag{9}$$

$$\int_E \underline{f}_d . \underline{U}_h^* dE + \int_{\partial E} \eta_E^\Gamma \hat{\underline{F}}_h . \underline{U}_h^* d\Gamma = 0 \quad \forall \underline{U}_h^* \text{ rigid body displacement field over } E. \tag{10}$$

  Where the scalar $\eta_E^\Gamma = \pm 1$ and $\eta_{E_i}^\Gamma . \eta_{E_j}^\Gamma = -1$ for two adjacent elements $E_i$ et $E_j$ on their common edge $\Gamma$ [14, 15]

- Construction of $\hat{\sigma}$ from $\hat{\underline{F}}_h$
  Once the load densities $\hat{\underline{F}}_h$ at hand, we calculate the admissible stress field $\hat{\sigma}$. Analytical method [17, 18] or numerical method can can be used for recovering the stresses. In [19] on shows that using finite element method with higher-order elements provides good results. We chose this solution.

### 2.3.2 construction of load densities

In this work we focus on the construction developed in [16, 7, 8]. The principle consists in dividing the load densities into two parts. The first part corresponds to the affine part $\hat{\underline{R}}$ of $\hat{\underline{F}}_h$ and corresponds to the nodes which are vertices of the mesh. And the higher-degree part $\hat{\underline{H}}$ corresponds to the non-vertices nodes of the mesh.

$$\hat{\underline{F}}_h|_\Gamma = \hat{\underline{R}}|_\Gamma + \hat{\underline{H}}|_\Gamma \tag{11}$$

• Part $\hat{\underline{H}}$ is calculated by the method described [17], we recall here the main idea. This method is based on the prolongation condition for each non-vertex nodes. This leads to a local equality. The idea is to construct densities whose associated admissible

stress field has the same energy norm that the finite element stress field.
The local prolongation condition for each node $i$ writes :

$$\forall E \in P_i, \qquad \int_E Tr[(\hat{\sigma}_h - \sigma_h)\,\epsilon\,(\varphi_i)]dE = 0 \tag{12}$$

Where $\varphi_i$ are the finite element basis functions associated to de node $i$, and $P_i$ the set of all element around the node $i$.
Introducing the load densities, this relation can be written:

$$\forall E \in P_i, \qquad \sum_\Gamma \eta_E^\Gamma \int_\Gamma \underline{\hat{F}}_h \varphi_i d\Gamma = \int_E \left(\sigma_h \underline{grad}\,(\varphi_i) - \underline{f}_d \varphi_i\right) dE \tag{13}$$

We obtain $\underline{\hat{H}}|_\Gamma$ by solving the systems constituted of the prolongation condition for each node.
• Part $\underline{\hat{R}}$ is then calculated starting from $\underline{\hat{H}}$ by minimizing the complementary energy of the whole structure. The minimization of the complementary energy is mathematically leads to the minimization of a quadratic functional under linear constrain.

$$\min_{\underline{\hat{R}} \in \mathcal{R}} \left(E_c(\underline{\hat{R}})\right) \tag{14}$$

Where

$$\begin{aligned} \mathcal{R} = \Big\{ \underline{R} / \qquad & \int_{\Gamma \in \partial E} \underline{R} + \underline{\hat{H}}|_\Gamma d\Gamma = \underline{0} \\ & \int_{\Gamma \in \partial E} \left(\underline{R} + \underline{\hat{H}}|_\Gamma\right) \wedge \underline{OM} d\Gamma = \underline{0} \\ & \underline{R}|_{\partial\Omega} = \underline{F}_d - \underline{\hat{H}}|_{\partial\Omega} \text{ on } (\partial_2\Omega) \\ & \underline{R}|_{\partial\Omega} = \underline{F} - \underline{\hat{H}}|_{\partial\Omega} \text{ on } (\Phi_{EE'}) \Big\} \end{aligned} \tag{15}$$

This minimization (14) is solved exactly introducing Lagrange multipliers. When this minimization has been solved we obtain $\underline{\hat{F}}_h = \underline{\hat{R}} + \underline{\hat{H}}$.

# 3 Estimating contribution to global error on a substructure

We are interested in estimating the discretization error part of the constitutive error $e^2_{\Omega_E}$ without using data of the others domains $\Omega_E$.

$$e^2_{\Omega_E} = \|\hat{\sigma} - \mathbf{K}\epsilon(\underline{\hat{u}})\|^2_{\Omega_E} \tag{16}$$

The main problem addressed here is to determine the traction locally, without using the results on the other subdomains. The idea introduced here is to use the finite element on the sub-structure $\Omega_E$ to build traction on the boundary of $\Omega_E$. In fact, the domain $\Omega_E$ is considered as a structure and the boundary conditions are not supposed to be known.

Starting from the finite element solution, if the mesh is sufficiently fine, we have :

$$\underline{F}_d \simeq \sigma_h.\underline{n} \tag{17}$$

For a coarse mesh (17) is not satisfied. This leads to an error, but we consider this very simple approximation in a first step. The original technique proposed here is then replaced by an approximated construction of the loads densities. It is always based on a minimization but the constraint are approximated :

$$\min_{\underline{\hat{R}} \in \tilde{\mathcal{R}}} \left( E_c(\underline{\hat{R}}) \right) \tag{18}$$

Where

$$\begin{aligned} \tilde{\mathcal{R}} = \Big\{ \underline{R} / \quad & \int_{\Gamma \in \partial E} \underline{R} + \underline{\hat{H}}|_\Gamma d\Gamma = \underline{0} \\ & \int_{\Gamma \in \partial E} \left( \underline{R} + \underline{\hat{H}}|_\Gamma \right) \wedge \underline{OM} d\Gamma = \underline{0} \\ \underline{R}|_{\partial\Omega} = \sigma_h.\underline{n} - & \underline{\hat{H}}|_{\partial\Omega} \text{ on } (\partial_2\Omega) \text{ or } (\Phi_{EE'}) \Big\} \end{aligned} \tag{19}$$

This method leads to the determination of densities in a very simple manner. The same techniques are used for solving this minimization as for (14).

# 4 First results

We present in this part two types of results. The first one concern an academical problem where an overkill solution is available. The second ones shows the very first results on an industrial structure.

## 4.1 2D academic example

First we study the result of the method proposed on an academic 2D problem. This problem is constituted of only one sub-structure, in order to have a very small test case. The simplicity of such a problem allows one to derivate an exact solution. An illustration of error distribution using the above presented method is given on Figure 2. A comparison between actual error and estimated error makes it possible to estimate the efficiency of the method.

The effectivity index $\eta$ is classically defined as :

$$\eta = \frac{\|\hat{\sigma} - \mathbf{K}\epsilon(\underline{\hat{u}})\|^2_{\Omega_E}}{\|\sigma_{ex} - \mathbf{K}\epsilon(\underline{u}_{ex})\|^2_{\Omega_E}} \tag{20}$$

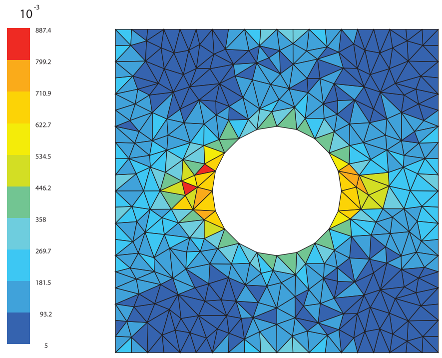


Figure 2: 2D academic example

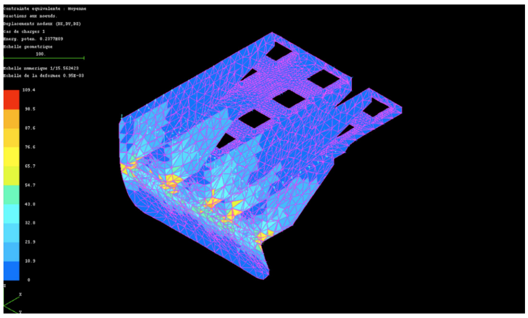

Figure 3: 3D industrial example

where $(\sigma_{ex}, \underline{u}_{ex})$ is the exact solution pair of the problem

We obtain for this test $\eta = 1.27$. This efficiency is close to $1$, which give a estimation a good estimate of the actual error.

### 4.2 3D industrial example

We are interested into the error into the substructure due to the loading that come from the Finite Element simulation where the boundary limits are representative of a assembly. To simplify, we are not interested into the error due to the second term of the constitutive relation error that measure the numerical scheme employed to simulate the contact. Further details will be given in the conference presentation and in a forthcoming paper. An illustration of the studied structure is given in Figure 3.

## 5 Conclusion

The results obtained here correspond to a first step in the verification of assembly problems. The next step will be to estimate the error due to the numerical scheme employed to solve the contact problem. A demonstrator is about to be implemented into SANCEF for contact problems. Introducing an adaptive technique also remains an objective, it is in fact an adaptation of previous work [6] on the contact problem.

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