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A Rigid Block Model with Cracking Units for Limit Analysis of Masonry Walls subject to In-Plane Loads

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Abstract

In this paper, a rigid block model with cracking units for the limit analysis of masonry structures is presented. A simplified micro-modelling approach is adopted, based on the discretization of the single masonry unit into multiple blocks separated by cohesive contacts. Failure modes, which may occur at mortar and mortar-unit interfaces as well as in the masonry unit itself, involve crushing, cracking and sliding. The modelling approach is validated against a literature case study of unconfined masonry panel, whose collapse mechanism involves cracking of units. To show the accuracy of the proposed modelling approach, the obtained results of the analysis are compared with experimental tests and with the outcomes of other modelling approaches used in the literature.

Keywords: rigid block model; cracking units; masonry; limit analysis; linear programming.

1 Introduction

In standard rigid block limit analysis based on micro-modelling approaches, masonry units are modelled with single blocks expanded by half of the mortar thickness while the behaviour of mortar and mortar-unit interfaces is lumped into a single zero-thickness contact. As a consequence, failure of the units is usually neglected [1-4].

In this paper a micro-modelling approach with rigid blocks is developed for in-plane loaded masonry panels which are able to take into account the failure of the units.

The model is formulated according to different simplified assumptions, concerning block failure modes and cracking behaviour .

For the discretization of masonry texture and for the definition of yield functions, the modelling strategy is similar to the approach used in finite element analysis for simplified micro-modelling [5].

In the proposed model each masonry units schematized with two rigid blocks separated by a contact interface (Figure 1). As for standard rigid block modelling approach, failure is defined at interfaces and includes cracking, crushing and sliding with nonassociate flow rule of masonry units as well as mortar joints. Different yield surfaces are defined for the behaviour of uncracked and cracked contact interfaces. Plastic tensile strength and cohesion are included to consider the cohesive behaviour of the joints. An iterative solution procedure has been developed to take into account the brittle behaviour of joints in cracking whose response is governed by yield functions with a null value of cohesion and tensile strength.

In the next sections the details of the implemented modelling procedure are presented. Afterwards, with the aim of validating the proposed model, the method is used to analyse the response of unconfined shear walls involving cracking of the units. The results of the numerical simulations on the wall samples are presented and compared with the outcomes of other modelling approaches used in limit analysis as well as with experimental tests.

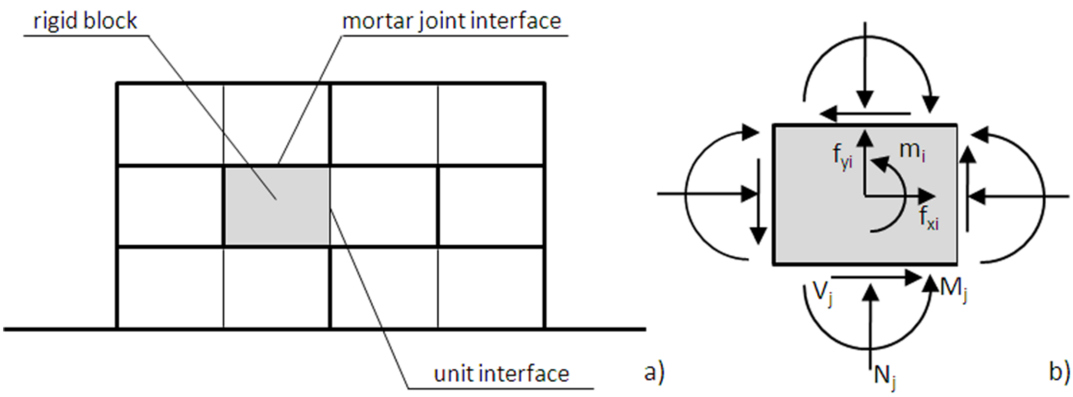

Figure 1: a) Proposed rigid block model with cracking units (masonry unit in thick line); b) Block and contact static variables.

2 The rigid block limit analysis model

In this section, the modelling approach adopted for the formulation of the limit analysis problem as a mathematical program is presented and discussed in detail. Static and kinematic variables as well as equilibrium equations, constitutive laws and flow rules are expressed in matrix form.

The formulation of governing equations, which mainly refers to the work by Ferris & Tin Loi [2], has been developed for two-dimensional block assemblages.
Details of the solution procedure based on iterative linear programming problems with variable contact properties are also provided.

2.1 Static and kinematic variables

The structural model consists of an assemblage of rigid blocks *i* connected by interfaces *j*. The masonry unit is represented by two rigid blocks separated by a vertical contact passing through the centroid of the unit itself (Figure 1a).

The static variables are represented by the internal forces (V_j, N_j, M_j) acting at the interface j and referred to the centre of the contact (Figure 1b). These variables are collected in the vector $\mathbf{x_j}$.

$$\mathbf{x_j} = \begin{bmatrix} V_j & N_j & M_j \end{bmatrix}^T . \tag{1}$$

The corresponding kinematic variables are the relative displacement rates at the interfaces (Figure 2). These variables are collected in the vector $\mathbf{q_j}$:

$$\mathbf{q_j} = \begin{bmatrix} \gamma_j & \varepsilon_j & \omega_j \end{bmatrix}^T . \tag{2}$$

The loads are applied to the centroid of the rigid block i and are indicated with the vector $\mathbf{f_i}$:

$$\mathbf{f_i} = \begin{bmatrix} f_{xi} & f_{yi} & m_i \end{bmatrix}^T . \tag{3}$$

The loads $\mathbf{f_i}$ are expressed as the sum of the known dead loads $\mathbf{f_{Di}}$ and live loads $\mathbf{f_{Li}}$ amplified by an unknown scalar multiplier α**.**

$$\mathbf{f_i} = \mathbf{f_{Di}} + \alpha \mathbf{f_{Li}} \tag{4}$$

The displacement rates at the centroid of the block i, that are work conjugated to the nodal loads $\mathbf{f_i}$, are collected in the vector $\mathbf{u_i}$:

$$\mathbf{u_i} = \begin{bmatrix} u_{xi} & u_{yi} & u_{\theta i} \end{bmatrix}^T \tag{5}$$

2.2 Equilibrium equations

For block i and contact j, the equilibrium equations are expressed in the matrix form:

$$\mathbf{A_{i/j}} \mathbf{x_j} = \mathbf{f_i} \tag{6}$$

where $\mathbf{A_{i/j}}$ is a (3 × 3) equilibrium matrix.

For the entire structure the equilibrium in the matrix form gives:

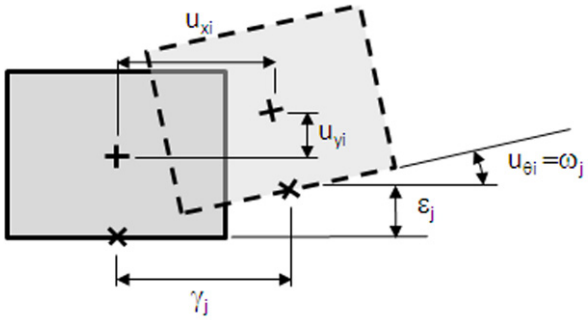

Figure 2: Block and contact kinematic variables.

$$\mathbf{A}_{3b\times 3c}\mathbf{x}_{3c} = \mathbf{f}_{3b} \tag{7}$$

being b the number of blocks and c the number of contacts, and it is obtained through assembly of matrices for each block.

2.3 Yield conditions

Failure is concentrated at mortar joint and unit interfaces and includes different types of failure modes, that are cracking, crushing and sliding.

A different behaviour of the uncracked and cracked interfaces was considered to take into account the brittle behaviour of joints failing in cracking.

For uncracked contact interfaces, the possibility to include plastic tensile strength and cohesion for the definition of yield functions has been considered.

The plastic stress values are determined according to Eq. (8-11), following the approach used by Orduña [4, 6] and originally proposed for concrete limit analysis by Nielsen [7]:

$$f_{cef} = \nu_c \cdot f_c \tag{8}$$

$$\nu_c = 0.7 - \frac{f_c}{200} \tag{9}$$

and

$$f_{tef} = \nu_t \cdot f_t \tag{10}$$

$$\nu_t = 0.6 \tag{11}$$

where f_c and f_t are the compressive and tensile strength of material, expressed in MPa.

The effectiveness factors ν are used to take into account the differences between rigid plastic strength assumed in limit analysis and the softening behaviour of material.

Sliding is governed by a Coulomb type criterion and is expressed by the following relationships:

$$y_j^{s\pm} = \pm\cos\phi V_j - \sin\phi N_j \leq r_{j1} \tag{12}$$

where $y_j^{s\pm}$ are the yield functions for positive and negative sliding, ϕ is the friction angle and $r_{j1} = c_{j1} \cdot \cos\phi$, being c_{j1} the cohesion at joint j expressed as a function of f_{tef}. The friction coefficient is defined as $\mu = \tan\phi$.

For crushing and cracking, a perfectly plastic behaviour in compression and tension was considered.

As it is well known, in this case a parabolic yield function is obtained in the M, N domain. To reduce the resulting optimization problem to a linear program, the corresponding non-linear yield condition was approximated by eight hyperplanes. In this case, the generic expression of the admissibility condition for the internal static variables M and N at joint j is:

$$y_j^{ck\pm} = \pm\cos\psi_{jk} M_j - \sin\psi_{jk} N_j \leq r_{jk} \quad \text{for } k=2,\ldots,5 \tag{13}$$

For cracked interfaces, the yield functions present a null value of tensile strength and cohesion to take into account the brittle behaviour of material.

In matrix notation, the previous limit conditions at a contact interface j can be written as:

$$\mathbf{N}_j^T \cdot \mathbf{x}_j \leq \mathbf{r}_j \tag{14}$$

being $\mathbf{N}_j^T$ the yield function matrix in case of sliding and crushing and $\mathbf{r_j}$ a constant vector.

It is worth to note that yield conditions must be specialized for the mortar joint interface and the unit interfaces, according to the different mechanical properties of the two components.

The extension of previous expression to the whole structure is straightforward.

2.3 Flow rule

The modelling of flow rule has important implications in the formulation of the mathematical program of the limit analysis problem. As mentioned above, this problem results into a linear program in case of associate flow rule and into a mathematical problem with equilibrium constraint in case of non-associate flow rule.

Flow rule provides relationships between generalized strains and resultant strain rates, that is plastic multipliers in limit analysis, and is governed by the following equations:

$$\mathbf{q} = \mathbf{V}\mathbf{z} \tag{15}$$

where $\mathbf{V}$ is the flow rule matrix and $\mathbf{z}$ is the vector of plastic multipliers, being $\mathbf{z} \geq \mathbf{0}$ to ensure energy dissipation of the structural system under applied loads.

In case of associate flow rule, the vector of generalized strains is normal to the yield function **y.** In this case, it is easy to show that:

$$\mathbf{V} = \mathbf{N} \tag{16}$$

The definition of the plastic behaviour of joints is completed by the additional condition:

$$\mathbf{y}^{\mathrm{T}}\mathbf{z} = 0 \tag{17}$$

known in plasticity as complementarity relation, which allows to have positive components of plastic multipliers **z** only when the stress state is on a yield plane.

3 Formulation of the limit analysis problem and solution procedure

It is well known that, under the hypothesis of classical plastic theory, including fully associate plastic flow rules and convex yield functions, the lower and upper bound formulations of the limit analysis problem lead to two dual linear programming problems, static and kinematic, whose unique solution is the load factor α.

On the basis of previous assumptions, the formulation by linear programming of the static theorem of limit analysis, stating that the collapse load corresponds to the maximum load factor associated to a static admissible distribution of internal forces satisfying yield conditions, is then:

$$\begin{aligned} &\max && \alpha \\ &\text{subject to:} && \mathbf{A}\cdot\mathbf{x}=\mathbf{f}_{\mathrm{D}}+\alpha\mathbf{f}_{\mathrm{L}} \\ & && \mathbf{N}^{\mathrm{T}}\cdot\mathbf{x}\le\mathbf{r} \end{aligned} \tag{18}$$

The dual linear programming problem, whose main variables are the displacement rates, is the mathematical formulation of the kinematic theorem and can be expressed as:

$$\begin{aligned} &\min && \mathbf{r}^{\mathrm{T}}\mathbf{z}\text{-}\mathbf{f}_{\mathrm{D}}^{\mathrm{T}}\mathbf{u} \\ &\text{subject to:} && \mathbf{f}_{\mathrm{L}}^{\mathrm{T}}\mathbf{u}=1 \\ & && \text{-}\mathbf{A}^{\mathrm{T}}\mathbf{u}+\mathbf{N}\mathbf{z}=\mathbf{0} \\ & && \mathbf{z}\ge\mathbf{0} \end{aligned} \tag{19}$$

In the above formulation, the first constraint is a governing relation involving positive energy dissipation under live loads. The second constraint is a compatibility condition, relating resultant strain rates as a function of displacement rates and flow rule.

In the present study, to take into account the nonassociative behaviour in sliding, the iterative solution procedure proposed by Gilbert et al. [8] was used to find the minimum collapse load, instead of solving the underlying mixed complementarity program.

Moreover, a specific iterative procedure was used to take into account, in a simple way, the brittle behaviour of mortar joints and unit interfaces undergoing cracking failure. The procedure is based on a different plastic behaviour of uncracked and cracked interfaces and it is organized in the following steps: 1. Limit analysis of the rigid block model with effective values of plastic strength based on yield functions related to uncracked behaviour; 2. Detection of contacts undergoing cracking failure; 3. Redefinition of mechanical properties at the selected interfaces according to cracked behaviour; 4. Repetition of steps from 1 to 3 until convergence.

4 Application to a shear wall sample

On the basis of the previous formulation, a computer program has been developed for the limit analysis by iterative linear programming of masonry walls under in-plane loads. For the output, the program computes the failure loads and provides a plot of the corresponding collapse mechanism.

The program was first verified in the original rigid block version and with classic rocking and sliding yield functions against the set of wall panels investigated by Ferris and Tin-Loi [2] and Gilbert et al. [8]. The obtained results were in perfect agreement with the outcomes of other modelling approaches and solution procedures.

Afterwards, to check the accuracy of the implemented code in the 'cracking unit' version, the program was validated against the unconfined shear wall examined by Orduña and Lourenco [6].

The walls under investigation are the specimen labelled as SW30 and SW200 and were tested by Oliveira [9].

The specimens were made of dry stone masonry with dimensions of 1000×1000×200mm. The size of the units is 100×200×200mm.

The specimens were tested under a vertical load of 30kN and 200kN respectively and a monotonically increasing lateral load. The first course at the base of the panel was horizontally restrained in the tests.

Two different collapse mechanisms were observed for the investigated wall panels in experimental tests. For the vertical load of 30kN the panel failed with a rocking mechanism. For higher values of the vertical load cracking failure of blocks was observed.

The rigid block model with cracking units generated for limit analysis of the selected panels is shown in Figure 3 and consists of 91 blocks and 181 interfaces. Considering the test set-up, the first course of blocks was not included in the model developed for the present study. Detail of material model parameters are shown in Table 1. For the sake of clarity, the effective values of plastic strength have been expressed as the product of the material strength by the reduction factor ν. The selected values of the friction angle and the effective compressive strength are determined from experiment tests and are equal to the values used in [4]. The tensile strength is 3.7MPa, according to Ramos [10]. A specific weight of 25 kN/m^3 was considered.

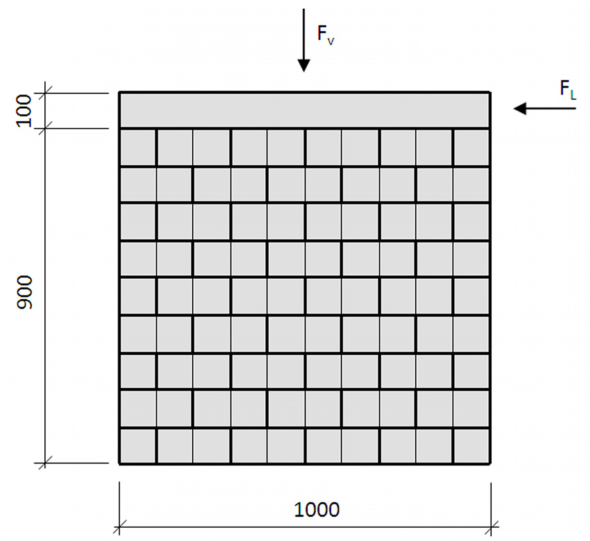

Figure 3: Rigid block model with cracking units of unconfined shear walls SW30 and SW200.

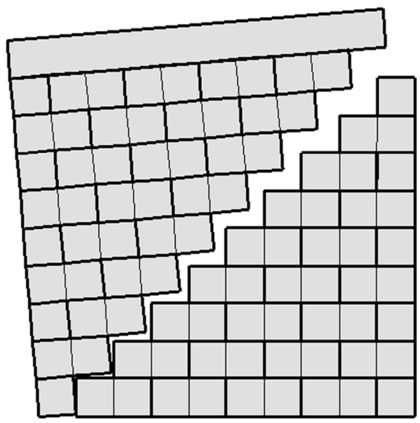

Figure 4: SW30 wall panel: Plot of the failure mode from computer program.

The results of the limit analysis are shown in Figures 4, 5 and the values of obtained collapse load are reported in Table 2. For these panels, the iterative procedure for the cracking behaviour of joints takes two steps only to converge. Time required to solve the problem was 5.8 sec on a CPU Intel Xeon 3.3GHz with 8.0 GB of RAM.

The failure mode obtained by the limit analysis model for F_V=30kN is overturning, according to experimental outcome (Figure 4). The obtained collapse load is 16.4 kN, in a good approximation with test measurement, considering that the values reported in Table 2 are the peak loads.

The results of the limit analysis program developed for this study are in perfect agreement with the outcomes of the rigid block analysis carried out in [4, 6], even though different approach were used for the solution of the mixed complementarity problem and the definition of yield functions for crushing.

For F_V=200kN, the obtained failure loads and collapse modes are shown in Figure 5 and Table 2.

Also in this case, the results are in accordance with experimental tests (see Figure 6) and in a perfect agreement with the solution provided in [4, 6]. The assumption of brittle behaviour for unit interfaces undergoing cracking made in this paper is indeed consistent with the rigid block model presented in [4, 6]. However, it is worth to note that in this study a different limit analysis procedure is adopted and an automatic method is used to model the cracking of the units.

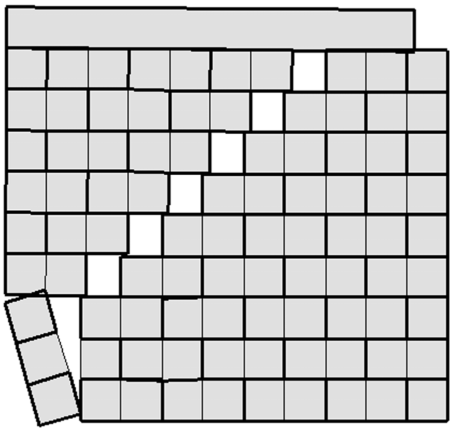

Figure 5: SW.200: Computed failure mode

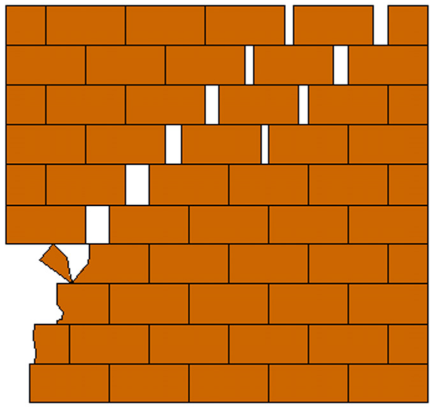

Figure 6: Experimental failure mechanisms for panel SW200 (adapted from Oliveira [9]).

Joints				Units			
f_{tef} [N/mm^2]	f_{cef} [N/mm^2]	μ	c_{ef} [N/mm^2]	f_{tef} [N/mm^2]	f_{cef} [N/mm^2]	μ	c_{ef} [N/mm^2]
0	$\nu_c \cdot 57.1$	0.66	0	$\nu_t \cdot 3.7$	$\nu_c \cdot 57.1$	0.75	$1.2 \cdot f_{tef}$

Table 1: Interface mechanical properties for unconfined shear walls SW30 and SW200.

Specimen	F_v [kN]	F_L [kN]		
		Experimental	Rigid block without cracking units	Rigid block with cracking units
SW.30	30	22.0	16.4	16.4
SW.200	200	72.0	101.1	75.6

Table 2: Comparison of experimental and numerical collapse loads.

5 Conclusions

A rigid block model with cracking units has been developed for limit analysis of masonry shear walls under in-plane loads based on a micro-modelling approach. Masonry panels are discretized with two rigid blocks per masonry unit divided by a vertical interface. Failure is concentrated at mortar and masonry units contact interfaces and includes cracking, crushing and sliding with non associative flow rule. The corresponding mathematical problem is solved using an iterative linear programming.

To evaluate the accuracy of the proposed formulation, a validation study was carried out and an application to a benchmark problem from the literature was presented.

The results of the numerical validation analysis were in perfect accordance with other limit analysis formulation approaches based on the direct definition of the cracking units in the model. Comparison with experimental tests showed a good agreement in terms of collapse load and mechanism.

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