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Symmetry of Solutions in Discrete and Continuous Structural Topology Optimization

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Abstract

In this paper symmetry and asymmetry of optimal solutions in symmetric structural topology optimization problems are investigated, based on the choice of variables. A group theory approach is used to formally define the symmetry of the structural problems. This approach allows the set of symmetric structures to be described and related to the entire search space. It is shown that, given a symmetric problem with continuous variables, an optimal symmetric solution (if any) necessarily exists. However, it is shown that this does not hold for the discrete case. Finally a number of examples are investigated to demonstrate the findings of the research.

Keywords: structural topology optimization, symmetric topology, asymmetric topology, group theory, symmetry operation, truss topology optimization.

1 Introduction

In structural engineering discrete variable optimization is of great interest, given the discrete nature of building components. Symmetry reduction of structural problems is a well established technique for structural analysis. In the past several decades the mathematical rigour of group theory has been applied to symmetric structural problems [1], leading to the development of structural analysis techniques for discrete structures. Bifurcation problems [2] of framework and latticed domes [3, 4] are amongst the most widely studied of these problems. In static analysis, symmetric frame structures [5, 6] have been studied using group theory symmetry reduction techniques. Group theoretic methods combined with graph products have been developed Kaveh et al. [7] and Kaveh and Nikbakht [8]. Kangwai et al. [9] and Zingoni [1] provide more complete reviews of the applications of symmetry tools and group theory in structural mechanics problems.

In numerous studies of symmetric discrete topology optimization the approach has been to enforce symmetry artificially [10, 11]. These measures lead to significant simplification of the problem by reducing the number of design variables. The consequent reduction in the problem size and hence the computational costs, are strong motivating factors for this approach. However, as will be seen, in discrete topology optimization this simplification often leads to strongly suboptimal solutions. The authors were motivated to undertake this research on the basis of the results of symmetric discrete topology optimization problems. However, a recent paper by Stolpe [12] illustrates the attention this subject is now receiving. Asymmetry in discrete topology optimization was also noted by Achtziger and Stolpe [13]. In topology optimization with continuous design variables Rozvany [14] shows the existence of a symmetric solution, and possible non-uniqueness of the optimal solution, while Kosaka and Swan [15] attributed asymmetric material layout in continuum problems to numerical round-off and local asymmetric solutions. Cheng & Lui [16], inspired by the previously mentioned publications, demonstrated asymmetric solutions in frame topology optimization with frequency objective functions or constraints. However, it is shown that the validity of symmetry reduction in symmetric optimization problems is primarily dependent on the nature of the design variables. Starting from these observations the remainder of the paper is ordered as follows: After an explanation of the scope and several important definitions (section 2), the relation between the search space and its symmetric subset is discussed (section 3). Next objective functions and constraints (section 4) as well as the existence and uniqueness of optimal solutions, with continuous and discrete variables is investigated (section 5). Finally several examples are presented (section 6).

2 Scope and definitions

The investigation focuses on truss-like structures with either continuous or discrete variable bar cross section areas. It is conceivable that the principles developed in this paper apply to topology optimization problems other than the structural kind. Such problems may include thermal, optical and other optimization problems. Furthermore the principles may also apply to discrete sizing and shape variables. However, the authors are primarily interested in structural topology optimization problems. Only single objective optimization problems are considered.

2.1 General definitions

The structural topology optimization problem consists of: (i) A set of nodes with fixed spatial coordinates; (ii) a set of boundary conditions corresponding to selected nodes; (iii) a set of loads applied to selected nodes; and (iv) a set of allowed structural connectivities between the nodes, called a *ground structure*. The nodal connectivity of the ground structure can be represented by an *adjacency matrix* $\mathbf{A}_{GS}$. The structure

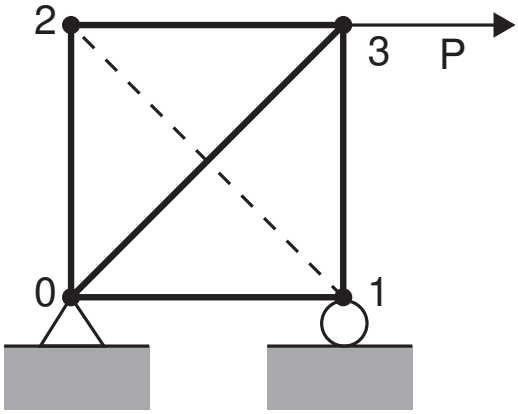

Figure 1: A 6 bar 2D truss design. The dashed line indicates a possible connection between nodes 1 and 2. This connection forms part of the problem ground structure, but is absent in this specific design.

in figure 1 has a ground structure with

$$\mathbf{A}_{GS} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ \text{sym.} & & & 0 \end{vmatrix}.$$

A *design* is a particular structure represented by the problem definition and a specific set of values for the design variables $\mathbf{x}$. For example, in topology optimization, the values of $\mathbf{x}$ may represent the binary existence or non-existence of elements, and as such correspond directly to $\mathbf{A}_{GS}$. The particular design in figure 1 is described by adjacency matrix

$$\mathbf{A} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ & 0 & 0 & 1 \\ & & 0 & 1 \\ \text{sym.} & & & 0 \end{vmatrix}$$

and variable vector

$$\mathbf{x} = [1, 1, 1, 0, 1, 1]^\top .$$

Each design corresponds to a value of the *objective function* $f(\mathbf{x})$, used to evaluate the structural performance. The optimization problem is subject to inequality and equality constraints, respectively denoted $\mathbf{g}(\mathbf{x}) \leq 0$ and $\mathbf{h}(\mathbf{x}) = 0$. Each design is a *member* of a set of all possible designs, the problem *search space* $\mathcal{S}$. The search space is defined by the bounds of the design variables. The *feasible subset* $\Omega_f \subseteq \mathcal{S}$ is the set of values of the variables for which no constraint on the problem is violated, i.e. Ω_f is bounded by the constraints on the problem. The *symmetric subset* $\Omega_{sym} \subseteq \mathcal{S}$ is the set of all possible geometrically and mechanically symmetric structures in $\mathcal{S}$. The *feasible symmetric subset* $\Omega_{f,sym} \subseteq \mathcal{S}$ of the search space is the intersection of the symmetric

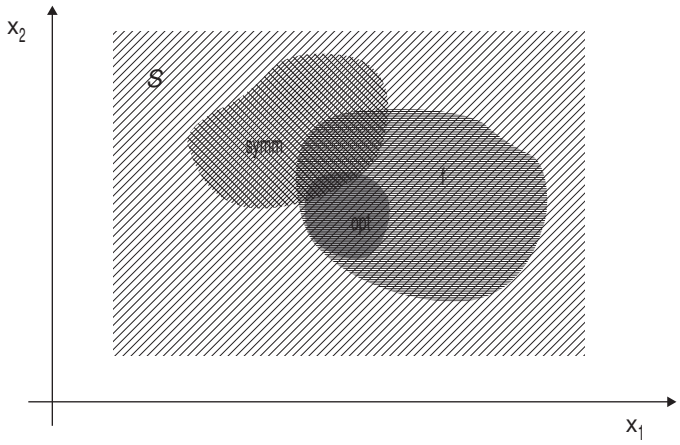

Figure 2: Sets and subsets of the variable space

and feasible subsets $\Omega_{f,sym} = \Omega_f \cap \Omega_{sym}$. The optimal subset $\Omega_{opt} \subseteq \Omega_f \subseteq \mathcal{S}$ is the set of designs corresponding to the values of $\mathbf{x}$ such that the objective function $f(\mathbf{x})$ is minimized (or maximized) in the global sense. A schematic representation of the various sets is summarized in figure 2.

2.2 Group theory and group representation

Structures are symmetric with respect to an operation if they are left unaltered under that operation. The geometric symmetry of structures can be studied through algebraic structures called *symmetry groups*. Simply stated: a symmetry group $\mathcal{G}$ is formed by the set of possible symmetry operations that do not result in a change in the structure. The number of symmetry groups is finite [9] and can be reduced depending on the type of structures under consideration. In this study we consider only finite structures which are not symmetric under dilatation or translation, the so-called *point symmetry groups*. Point groups leave at least one point fixed under all operations in the group [17]. Thirty two unique point groups exist. In the topology optimization problems considered here, the symmetry group of the structure is defined by the non-variable aspects of the problem: the boundary conditions (supports) and non-variable structural components (such as truss elements, etc.). The loading need not necessarily be symmetric [18], however for the sake of simplicity symmetric loading is considered in the examples. Any operation in the point group $\mathcal{G}$, acting on a vector $\mathbf{x} \in \Omega_{sym}$, can be represented by the *permutation matrix* $\mathbf{P}$ such that $\mathbf{P}^\top \mathbf{x} = \mathbf{x}$ [19]. We denote the permutation of $\mathbf{x}$, under symmetry operation $\iota \in \mathcal{G}$, $\iota(\mathbf{x}) = \mathbf{P}_\iota^\top \mathbf{x}$. The set of permutation matrices $\mathbf{P}$ is a *representation* of $\mathcal{G}$. The reader is referred to Hamermesh's excellent book [20] for an overview of group theoretic concepts.

3 Search space and symmetric subset

Using these definitions, Ω_{sym} can be constructed and related to $\mathcal{S}$ in a symmetric topology optimization problem. If the structure represented by a vector $\mathbf{x}$ is symmetric with group $\mathcal{G}$, it can be reduced to a vector $\mathbf{x}'$ such that:

$$\mathbf{x} = \bigcup_{\iota \in \mathcal{G}} \iota(\mathbf{x}').$$

An equivalent algebraic form can be stated as follows:

$$\mathbf{x} = \frac{\sum_{\iota=1}^{m} \sum_{\kappa=\iota}^{m} \left(\iota(\mathbf{x}_{\iota\kappa}) + \kappa(\mathbf{x}_{\iota\kappa})\right)}{2} \tag{1}$$

where $\mathbf{x}_{\iota\kappa}$ is such that:

$$\iota(\mathbf{x}_{\iota\kappa}) = \kappa(\mathbf{x}_{\iota\kappa}), \qquad \iota \neq \kappa$$

and

$$\iota(\mathbf{x}_{\iota\iota}) \bigcap_{\iota,\kappa \in \mathcal{G}} \kappa(\mathbf{x}_{\kappa\kappa}) = \mathbf{0}_n$$

where $\mathbf{0}_n$ is the null vector of length n. Analogously, the symmetric subset can be mapped to the search space by means of the reduced topology permutation matrix $\mathbf{P}''$, constructed as follows:

$$\mathbf{P}' = \bigcup_{\iota \in \mathcal{G}} \mathbf{P}_\iota.$$

All identical rows of $\mathbf{P}'$ can be collapsed, rendering $\mathbf{P}''$ with dimension $m \times n$, where $m \leq n$. In this way all symmetric members of $\mathcal{S}$, the m-dimensional $\mathbf{x}'' \in \Omega_{sym}$ can be constructed as n-dimensional $\mathbf{x} \in \mathcal{S}$:

$$\mathbf{x} = \mathbf{P}''^{\top} \mathbf{x}'' \tag{2}$$

Noting that both $\mathbf{x}$ and $\mathbf{x}''$ can be represented by binary strings (by concatenating the entries), both $\mathcal{S}$ and Ω_{sym} are countable sets which can be mapped to real positive integers, a useful way of ordering the sets.

4 Objective functions and constraints

In this paper we consider only mass minimization and compliance minimization. No explicit constraints (such as maximum stress in elements, buckling of elements, deflection, ...) are considered, with the exception of the kinematic stability of the structures and a constraint on the volume of the minimum compliance problem.

4.1 Mass objective function

In the context of truss structures, the mass objective function is expressed as follows:

$$f_m(\mathbf{x}) = \sum_{k=1}^{n_e} m_k = \sum_{k=1}^{n_e} \rho_k A_k l_k x_k$$

where n_e is the total number of elements; $x_k \in \mathbf{x}$; m_k, A_k, l_k and ρ_k are respectively the mass, cross-section area, length and density of element k. The $n \times n$ mass matrix $\mathbf{M}$ has entries

$$m_{ij} = \rho_{ij} A_{ij} l_{ij} a_{ij}, \qquad m_{ij} \in \mathbf{M}$$

and so

$$f_m(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} = \frac{1}{2} \mathbf{e}^\top \mathbf{M} \mathbf{e}$$

where $\mathbf{e}$ is a vector containing all ones. The mass matrix is therefore a weighted adjacency matrix, with the same structure as the adjacency matrix $\mathbf{A}$. Since the function $f_m(\mathbf{x})$ is linear in $\mathbf{x} \in \mathbb{R}^n$, it is a convex function, bounded only by the kinematic stability of the structure.

4.2 Compliance objective function

The compliance energy

$$f_c(\mathbf{x}) = \mathbf{f}^\top \mathbf{u} = \mathbf{f}^\top \mathbf{K}^{-1} \mathbf{f}$$

is one of the most widely studied objective functions in structural topology optimization [21]. Here $\mathbf{f}$ is a vector of external loads, $\mathbf{u}$ is the vector of nodal displacements and $\mathbf{K}$ is the stiffness matrix of the structure. It has been shown that, if a structural optimization problem is expressed in terms of variable bar areas, the compliance objective function is convex [22]. As mentioned an inequality constraint is placed on the volume of the structure, usually a maximum volume fraction V_f, the quotient of the volume of the specific design to the volume of the ground structure.

4.3 Constraints

In section 5, the notion of convex combination of the topology of structures is laid out. The following can be shown, given a symmetric problem: the convex combination of a kinematically stable topology $\mathbf{x}$ with other topologies $\iota(\mathbf{x})$, is also kinematically stable. This is because the problem formulation dictates that $\mathbf{x}$ and $\iota(\mathbf{x})$ are equivalent for all $\iota \in \mathcal{G}$. Therefore the kinematic stability is also equivalent. Convex combination never leads to a "reduction" in the total topology, therefore the kinematic stability cannot be violated in the combination of stable structures.

5 Convex combination: existence and uniqueness of solutions

The convex combination of designs i represented by variable vectors $\mathbf{x}_i \in \Omega$ is a design $\mathbf{x}$, such that:

$$\mathbf{x} = \sum_i \lambda_i \mathbf{x}_i. \tag{3}$$

where $\mathbf{x} \equiv \mathbf{A} \in \Omega$, $\lambda_i \in [0, 1]$, $\sum_i \lambda_i = 1$, Ω is some set of designs and

$$\mathbf{x}_i = [x_{i,1},\ x_{i,2},\ \ldots,\ x_{i,n}]^\top$$

In what follows we investigate the case where the design variable vectors have either continuous ($\mathbf{x}_{i,j} \in [0, 1]$) or binary discrete ($\mathbf{x}_{i,j} \in \{0, 1\}$) entries.

5.1 Convex combination and variable mapping

Given a function $f : \mathcal{S} \to \mathbb{R}$, any convex combination of the variables will have a corresponding value to which the function f can map it. If, in addition, the function is an affine mapping, the map of the convex combination of the variables will be the convex combinations of the mappings

$$f\left(\sum_{i=1}^{m} \lambda_i \mathbf{x}_i\right) = \sum_{i=1}^{m} \lambda_i f(\mathbf{x}_i) \tag{4}$$

where $\sum_{i=1}^{m} \lambda_i = 1$. Therefore any convex combination of variables will also be a convex combination in the function space. The following important result follows: if both $\mathbf{x}$ and $f(\mathbf{x})$ are convex, any convex combination of optimal solutions is also optimal, and the convex combination of the variables maps to this optimal function value. Jensen's inequality states that, for a convex function

$$f\left(\sum_{i=1}^{n} \lambda_i \mathbf{x}_i\right) \leq \sum_{i=1}^{n} \lambda_i f\left(\mathbf{x}_i\right). \tag{5}$$

In one variable this can be interpreted as saying that the secant line of a convex function lies "above" the graph where $\sum_{i=1}^{m} \lambda_i = 1$. However, since $f(\mathbf{x}_i) = f(\mathbf{x}_j)$ for all $\mathbf{x}_i, \mathbf{x}_j \in \Omega_{opt}$, on the optimal set, equation (5) becomes equation (4). If an asymmetric optimal solution can be found, the symmetry group permutations of this design are also optimal, since the structures are equivalent according to the problem statement. The complex combination of the variables can be mapped to the complex combination of the objective functions via an affine mapping. The complex combination is therefore also optimal. An intuitive interpretation of this, on a non-convex function, is shown in figure 3.

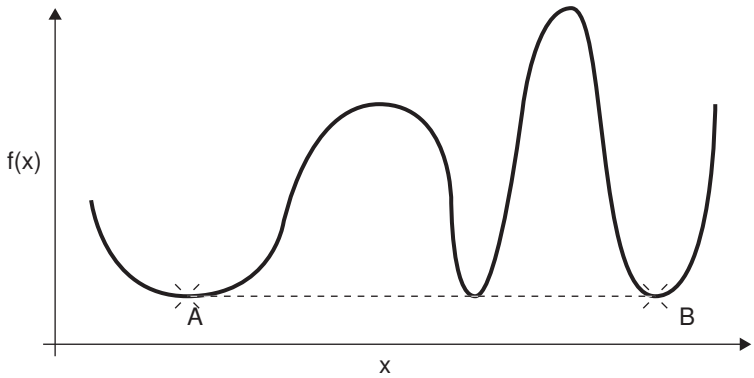

Figure 3: The convex combination of two optima for a non-convex function

5.2 Convex combination with continuous variables

In this section we consider symmetric problems with continuous design variables. These problems are similar to those discussed in classical topology optimization, where the bar cross section areas represented by continuous variables [23, 21]. Here the structure is represented by a weighted adjacency matrix corresponding to a continuous design variable vector

$$\mathbf{x} = [x_1, x_2, \ldots, x_m]^\top .$$

Consider the convex combination of designs

$$\mathbf{x}_i = [x_{i,1},\ x_{i,2},\ \ldots,\ x_{i,m}]^\top .$$

It follows directly from equation (3) that in the maximum case, where $x_{i,j} = 1$ for all i and j

$$\sum_{i=1}^{m} \lambda_i x_{i,j} = \sum_{i=1}^{m} \lambda_i = 1$$

All intermediate values are permitted since all are in $\mathbb{R}$. Since all entries of a weighted adjacency matrix $\mathbf{B}_i$ are either 0 or $x_{i,j}$, all $b_{i,jk} \in \mathbb{R}$. The following theorem is a generalization of the principles stated by Rozvany [14].

Theorem 5.1. *For any convex, symmetric topology optimization problem using continuous variables, if a solution exists a symmetric solution exists.*

Proof. Suppose an optimal solution $f(\mathbf{y})$ exists. If the problem is symmetric with respect to operations $\iota \in \mathcal{G}$, $\iota(\mathbf{y})$ also exists and is optimal. A convex combination of vectors $\iota(\mathbf{y})$ such that:

$$\mathbf{x} = \sum_{\iota=1}^{m} \lambda_\iota \iota(\mathbf{y}).$$

Due to the distributivity of ι, a decomposition of $\mathbf{y}$, as described in section 3 can be made

$$\begin{aligned}\mathbf{x} = \sum_{\iota=1}^{m} \lambda_\iota \sum_{\kappa=1}^{m} \iota(\mathbf{y}_{\iota\kappa}) \\ &= \frac{\sum_{\iota=1}^{m} \lambda_\iota \sum_{\kappa=1}^{m} \iota(\mathbf{y}_{\iota\kappa}) + \sum_{\kappa=1}^{m} \lambda_\kappa \sum_{\iota=1}^{m} \kappa(\mathbf{y}_{\kappa\iota})}{2} \\ &= \frac{\sum_{\iota=1}^{m} \sum_{\kappa=1}^{m} (\lambda_\iota \iota(\mathbf{y}_{\iota\kappa}) + \lambda_\kappa \kappa(\mathbf{y}_{\kappa\iota}))}{2}.\end{aligned}$$

Since $\mathbf{y}_{\iota\kappa} = \mathbf{y}_{\kappa\iota}$, terms can be eliminated, and taken up in the expression of λ_ι. Furthermore if we choose all λ_ι equal

$$\mathbf{x} = \lambda_\iota \frac{\sum_{\iota=1}^{m} \sum_{\kappa=\iota}^{m} (\iota(\mathbf{y}_{\iota\kappa}) + \kappa(\mathbf{y}_{\kappa\iota}))}{2}$$

Equation (1) is satisfied and $\mathbf{x} \in \Omega_{f,sym} \cap \Omega_{opt}$. □

5.3 Convex combination with discrete variables

In this section, the bar sizing variables are considered to be discrete, binary variables $x_j \in \{0, 1\}$. The linear combinations of n vectors lead to a system of m equations, where the following holds:

$$(x_j \in \{0, 1\}) \iff \left(\left(\sum_{i=1}^{n} \lambda_i x_{i,j} \right) \in \{0, 1\} \right)$$

Two scenarios are possible for any x_j:

1. $x_{i,j} = x_{l,j}, \ \forall i, l$
2. $\exists i, l : x_{i,j} \neq x_{l,j}$

In the former case the convex combination can be constructed, but this is a trivial case where the vectors are identical. In the latter the linear combination of the vectors, having elements in $\{0, 1\}$, can only be constructed if

$$\sum_{i=1}^{n} x_{i,j} = \begin{cases} \sum_{l=1}^{n} x_{l,j} \\ 0 \end{cases} \qquad \forall j$$

However this construction would violate the condition that $\sum_{i=1}^{n} \lambda_i = 1$.

Theorem 5.2. *For any set of (non-trivial) binary adjacency matrices, no convex combination of these matrices is possible.*

Proof. Assuming $\sum_{i=1}^{m} x_{i,j} = r$ where $r \in \{1 \dots (m-1)\}$. If at least n entries in $\mathbf{x}$ are non-zero, a system of n equations $\sum_{i=1}^{m} \lambda_i x_{i,j} = 1$ can be set up. Furthermore, we assume that none of the vectors $\mathbf{x}_i$ is the trivial null vector, so that $0 < \lambda_i \leq 1$. Assume now that there are some i in a set A of size $m - r$ for which $\mathbf{x}_{i,j} = 0$. Then $\sum_{i=1}^{m} \lambda_i x_{i,j} = \sum_{i \notin A} \lambda_i = 1$. Therefore: $\sum_{i \notin A} \lambda_i + \sum_{j \in A} \lambda_j > 1$ and the linear combination is not convex. □

This automatically leads to the following result:

Corollary 5.3. *In symmetric binary topology optimization problems, if an asymmetric optimal solution exists, no corresponding symmetric solution can necessarily be constructed.*

Proof. As in section 5.2 the symmetric solution can be expressed as the convex combination the permuted asymmetric solution. However it has been shown that the convex combination of any binary adjacency matrices cannot be constructed. Therefore, no corresponding symmetric solution necessarily exists, and if a symmetric solution does exist it is unique. □

Since no convex mapping is possible between symmetric and asymmetric solutions, and at most one optimal symmetric solution exists, the relative sizes of the symmetric and asymmetric feasible solution sets plays a role in the probability of the existence of a symmetric solution. This is now demonstrated by means of several examples.

6 Examples

The first is a 2D truss, with point group S_2, consisting only of the identity operation and a line of 'mirror symmetry' through the middle of the structure. The second is a 3D pylon structure, with three-fold rotational symmetry about a central, vertical axis, and three planes of 'mirror symmetry', as well as the trivial identity operation. This structure therefore has dihedral symmetry group D_3. Both structures are investigated considering two possible objective functions, namely the mass of the structure and the compliance energy under a given loading. All possible topologies (the entire search space) are evaluated and the symmetric subset of the search space constructed as described above. The optimal solutions for the objective functions can then be compared with the best performing symmetric solutions.

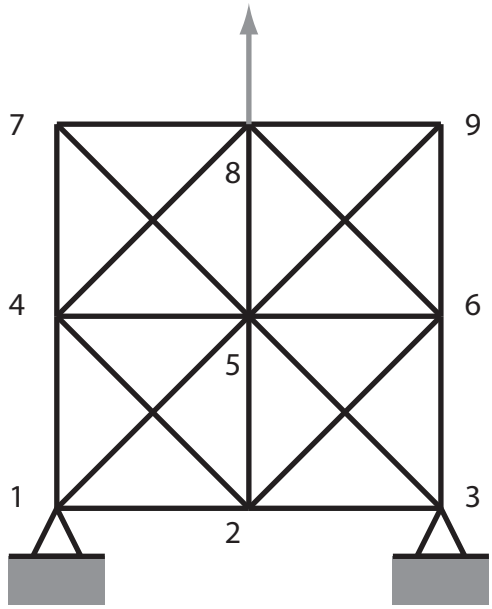

Figure 4: 20 bar 2D problem ground structure

6.1 20-bar 2D truss

6.1.1 Problem

The ground structure of the 20 bar topology optimization problem is shown in figure 4. For this problem the nodes are distributed regularly at the vertices of a grid with spacing one unit in the two (Euclidean) dimensions of the problem. The nodal coordinates, supports and loading are as follows:

$$\mathbf{c} = \begin{bmatrix} 0,0 \\ 1,0 \\ 2,0 \\ 0,1 \\ 1,1 \\ 2,1 \\ 0,2 \\ 1,2 \\ 2,2 \end{bmatrix} \qquad \mathbf{b}_c = \begin{bmatrix} 1,1 \\ 0,0 \\ 0,0 \\ 1,1 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,0 \\ 0,1 \\ 0,0 \end{bmatrix}.$$

The symmetry of the structure is expressed by the amorphism group $\mathcal{G}$ of the graph with adjacency matrix $\mathbf{A}$, representing the topology of the structure. The structural conditions have symmetry point group $S_2 = \{E, \sigma\}$, where E is the identity operation and σ a reflection about a line. The symmetry operation σ corresponds to the

permutation matrix $\mathbf{P}_\sigma$ and adjacency matrix for the ground structure are as follows:

$$\mathbf{A}_{GS} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ & & & 0 & 1 & 0 & 1 & 1 & 0 \\ & & & & 0 & 1 & 1 & 1 & 1 \\ & & & & & 0 & 0 & 1 & 1 \\ & \text{sym.} & & & & & 0 & 1 & 0 \\ & & & & & & & 0 & 1 \\ & & & & & & & & 0 \end{bmatrix} \equiv \mathbf{x}_{GS} = \mathbf{e}_{20} \tag{6}$$

where $\mathbf{e}_{20}$ is a 20 dimensional vector of ones. Keeping the nodal connectivity in mind, it is relatively simple to construct the topological permutation matrix

$$\mathbf{P}_\sigma = \begin{bmatrix}
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
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0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0
\end{bmatrix}.$$

Furthermore, a decomposition of the vector $\mathbf{x}_{GS}$ can be made

$$\mathbf{x}_{\sigma\sigma} = \mathbf{x}_{EE} = [1,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 1,\ 1,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0]^\top$$

and

$$\mathbf{x}_{E\sigma} = \mathbf{x}_{\sigma E} = [0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0]^\top.$$

It can be easily confirmed that, as in equation (1)

$$\frac{2E(\mathbf{x}_{EE}) + E(\mathbf{x}_{E\sigma}) + \sigma(\mathbf{x}_{E\sigma}) + 2\sigma(\mathbf{x}_{\sigma\sigma})}{2} = \mathbf{x}.$$

Ω_{sym} has $2^{11} = 2048$ members, while the topological search space $\mathcal{S}$ has 2^{20} members.

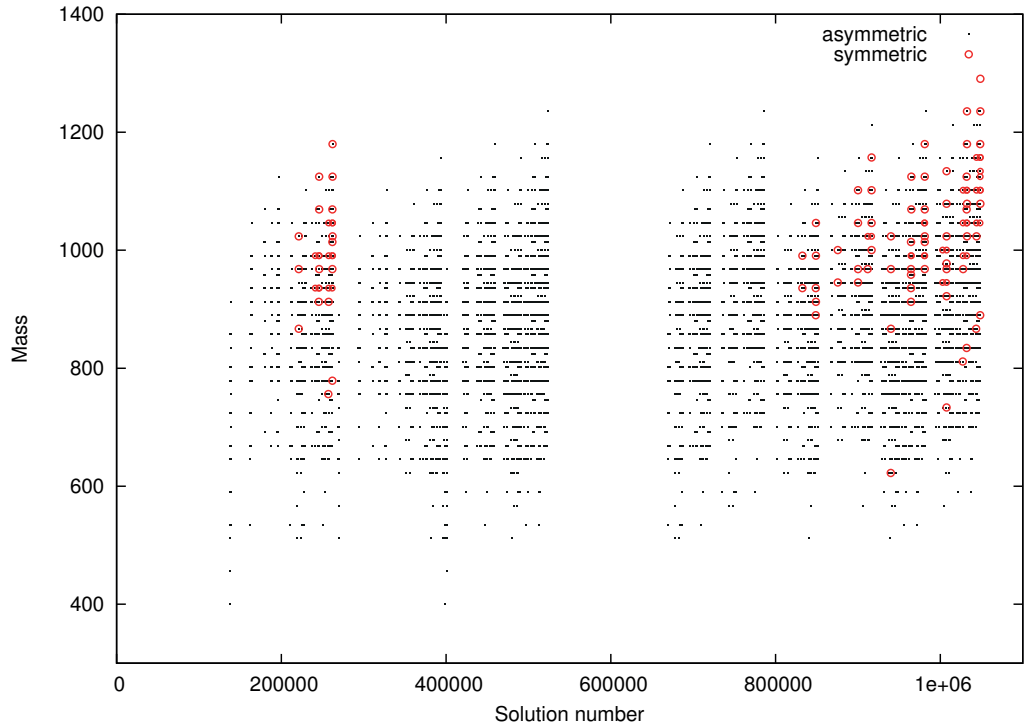

Figure 5: Symmetric and asymmetric structure mass objective functions

6.1.2 Results

For simplicity sake the variables of each design have been mapped to natural numbers $\mathbf{x} \to \mathbb{N}$, since a one-to-one mapping exists and this allows for a meaningful plotting of results. To do this the topology variables are taken to be the digits of a binary number. All feasible masses of the structures in $\mathcal{S}$ and in Ω_{sym} are shown in figure 5. Note that the minimum feasible mass is not symmetric. The two (feasible) minimum mass structure is shown in figure 7. As expected the two solutions are asymmetric, mirror images of one another about a vertical line. For the compliance objective function, taking the volume fraction $V \leq 0.4$, the same topology is found as in figure 7. The minimum symmetric mass solution is 1.55 times greater than the asymmetric minimum mass solution.

6.2 24 bar 3D truss

6.2.1 Problem

In this example a 24 bar 3D truss is investigated. The structure has dihedral symmetry group

$$\mathcal{G} = D_3 = \left\{E, C_3, C_3^2, C_2^a, C_2^b, C_2^c\right\}$$

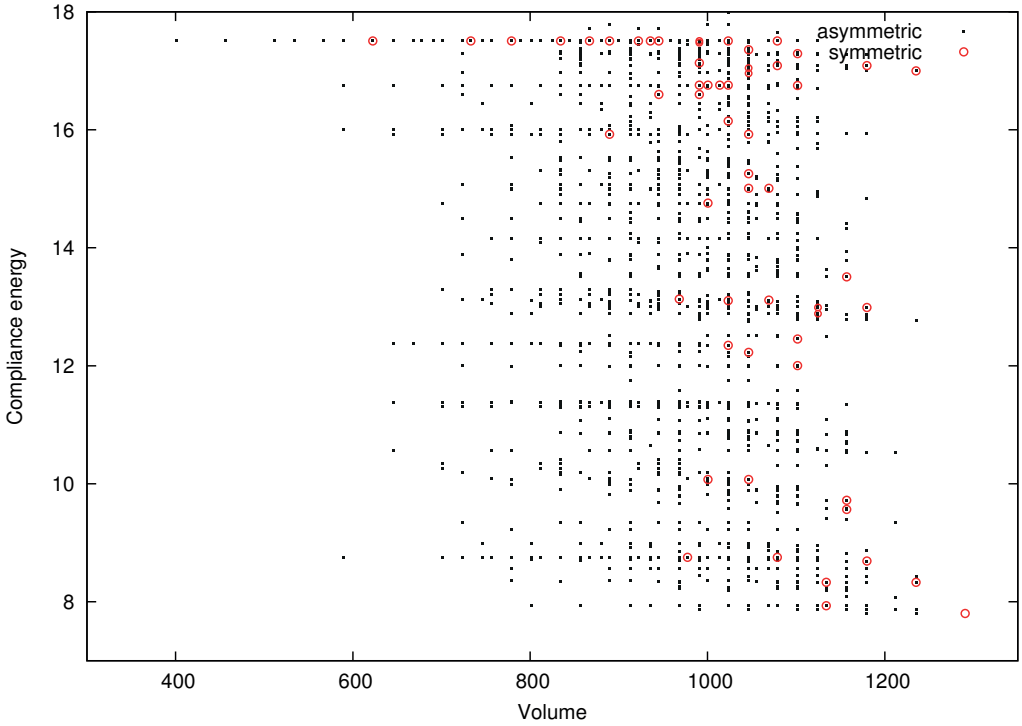

Figure 6: Symmetric and asymmetric structure for variable volume constraint

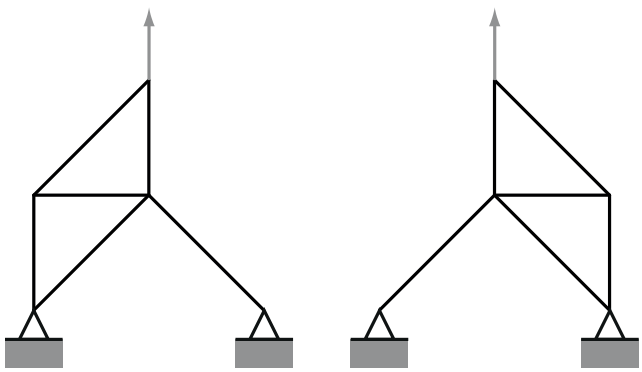

Figure 7: Minimum objective function values: both mass and compliance ($V \leq 0.4$)

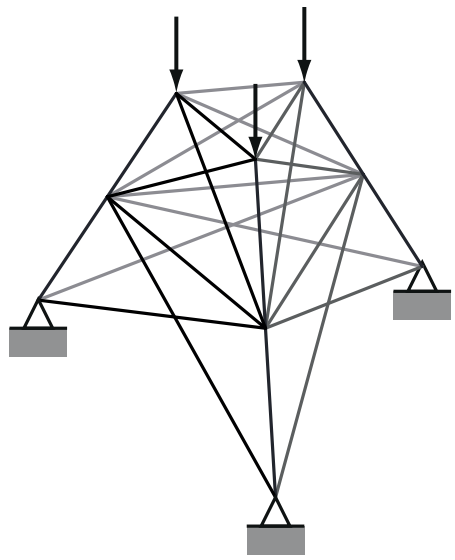

Figure 8: 24 bar 3D structure ground structure

The ground structure can be seen in figure 8, with a plan view (with the three reflection axes) shown in figure 9. Three vertical, unitary point forces act on the three highest nodes on the structure. Only binary topology variables are used and the a unit bar section area taken for simplicity sake. All three supports are pinned, restraining displacements translationally in all three dimensions.

6.2.2 Results

All 2^{24} designs in the search space were calculated. Only 64 structures in the search space are D_3 group symmetric. The 8 least mass solutions are shown in figure 10. Two distinct solutions can be seen. Taking structure 1 as one of these, the following mappings are present:

$$\begin{aligned}
\mathbf{x}_2 &= \sigma_a(\mathbf{x}_1) \\
\mathbf{x}_4 &= C_3^2(\mathbf{x}_1) \\
\mathbf{x}_5 &= C_2^b(\mathbf{x}_1) \\
\mathbf{x}_7 &= C_3(\mathbf{x}_1) \\
\mathbf{x}_8 &= C_2^b(\mathbf{x}_1).
\end{aligned}$$

The second type is structures number 3 and 6. For this structure the following holds:

$$\begin{aligned}
\mathbf{x}_7 = C_2^a(\mathbf{x}_3) = C_2^b(\mathbf{x}_3) &= C_2^b(\mathbf{x}_3) \\
\mathbf{x}_3 = C_3(\mathbf{x}_3) &= C_3^2(\mathbf{x}_3) \\
\mathbf{x}_7 = C_3(\mathbf{x}_7) &= C_3^2(\mathbf{x}_7).
\end{aligned}$$

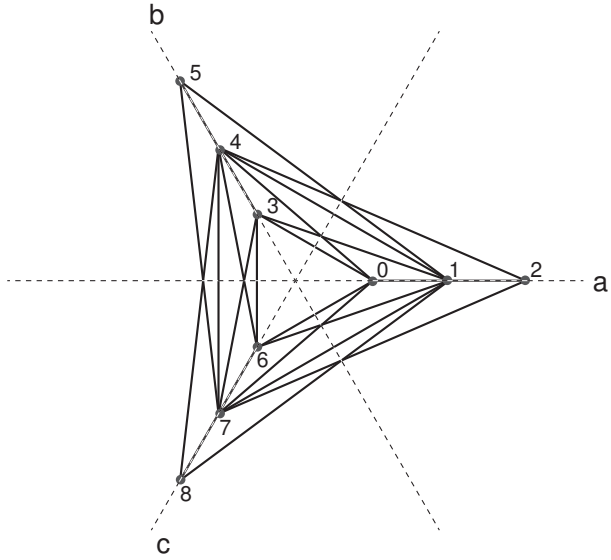

Figure 9: 24 bar 3D structure ground structure plan view

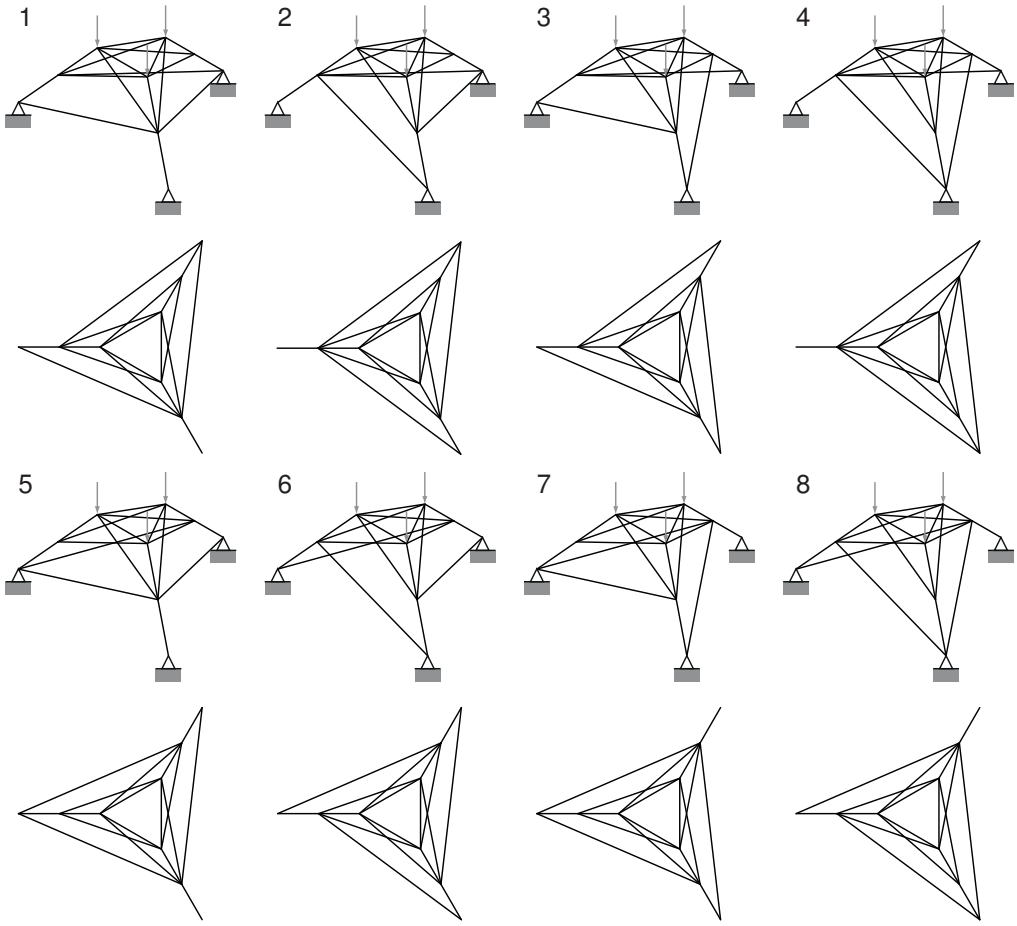

Figure 10: 24 bar 3D structure: least mass solutions

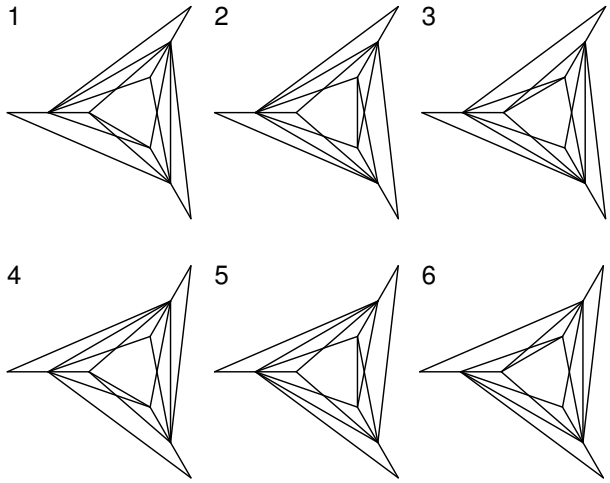

Figure 11: 24 bar 3D structure: lowest compliance energy solutions

In figure 11 the minimum compliance results are shown, with volume fraction constraint $V \leq 0.7$. These results demonstrate one single, asymmetric solution, and the 5 other permutations corresponding to the D_3 group. None of the optimal structures are symmetric with respect to the symmetry group D_3 In the case of the compliance minimization problem, the structures do posses any of the non-trivial symmetries in this group. The symmetric minimum mass solution was approximately 1.2 times the mass of the asymmetric solution, while for the compliance solutions, this ratio was approximately 1:1.14.

7 Conclusions and discussion

In this paper we have shown that for point symmetric topology optimization problems of the type discussed above, if a solution exists, at least one symmetric solution exists. Furthermore, we have proven and demonstrated in the examples, that, given a binary topology problem of the type discussed above, no symmetric solution necessarily exists. One interesting future prospect could be the investigation of design variables with discrete domains of a greater size (i.e. variables able to take on more than two discrete values). It is reasonable to predict an increase in the probability of symmetric solutions in these cases, tending towards the 100% probability shown in the continuous case, as the size of the discrete set increases. We have, in research into multiobjective discrete topology optimization, also noted the asymmetric phenomena in results. Extension to the multiobjective case could also be an interesting future prospect for this research. While it is strongly embedded in the engineering tradition to enforce symmetry in symmetric problems, it has been shown that the discrete nature of the optimization variables calls this practice into question. In the examples, the asymmetric solutions performed significantly better compared to the best symmetric

solutions. The authors believe that the evidence suggests that there is some room for discussion on this front. It is the hope that a relaxing of this assumption may lead to more efficient and elegant designs in practice.

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References

[1] A. Zingoni, "Group-theoretic exploitations of symmetry in computational solid and structural mechanics", *International Journal for Numerical Methods in Engineering*, 79(February): 253–289, 2009.

[2] "On the stability analysis of symmetrical frameworks", *Quarterly Journal of Mechanics and Applied Mathematics*, 17: 175–197, 1964.

[3] T. J., Healey, "A group-theoretic approach to computational bifurcation problems with symmetry", *Computer Methods in Applied Mechanics and Engineering*, 67(3): 257 – 295, 1988.

[4] K. Ikeda, K. Murota, "Bifurcation analysis of symmetric structures using block-diagonalization", *Comput. Methods Appl. Mech. Eng.*, 86: 215–243, March 1991.

[5] Đ. Zloković, *Group theory and G-vector spaces in structures: vibrations, stability, and status*, Ellis Horwood series in civil engineering. E. Horwood, 1989.

[6] A. Zingoni, M. Pavlovic, G. Zlokovic, "Application of group theory to the analysis of space frames", in G. Parke, C. Howard (Editors), *Space Structures*, pages 1334–1347. Thomas Telford: London, 1993.

[7] A. Kaveh, M. Nikbakht, H. Rahami, "Improved group theoretic method using graph products for the analysis of symmetric-regular structures", *Acta Mechanica*, 210: 265–289, 2010, ISSN 0001-5970.

[8] A. Kaveh, M. Nikbakht, "Analysis of space truss towers using combined symmetry groups and product graphs", *Acta Mechanica*, 218: 133–160, 2011.

[9] R. Kangwai, S. Guest, S. Pellegrino, "An introduction to the analysis of symmetric structures", *Computers & Structures*, 71(6): 671–688, 1999.

[10] K. Deb, S. Gulati, "Design of truss-structures for minimum weight using genetic algorithms", *Finite Elements in Analysis and Design*, 37(5): 447–465, 2001.

[11] Y. Bai, E. Klerk, D. Pasechnik, R. Sotirov, "Exploiting group symmetry in truss topology optimization", *Optimization and Engineering*, 10(3): 331–349, 2008.

[12] M. Stolpe, "On some fundamental properties of structural topology optimization problems", *Structural and Multidisciplinary Optimization*, 41(5): 661–670, 2010.

[13] W. Achtziger, M. Stolpe, "Truss topology optimization with discrete design variables–guaranteed global optimality and benchmark examples", *Structural and Multidisciplinary Optimization*, 34(1): 1–20, 2007.

[14] G. Rozvany, "On symmetry and non-uniqueness in exact topology optimization", *Structural and Multidisciplinary Optimization*, pages 1–21, 2010.

[15] I. Kosaka, C.C. Swan, "A symmetry reduction method for continuum structural topology optimization", *Computers & Structures*, 70(1): 47 – 61, 1999.

[16] G. Cheng, X. Liu, "Discussion on symmetry of optimum topology design", *Structural and Multidisciplinary Optimization*, pages 1–5, 2011.

[17] J. Leech, "How to use Groups", *American Journal of Physics*, 38: 273, 1970.

[18] A. Zingoni, M.N. Pavlovic, G.M. Zlokovic, "A symmetry-adapted flexibility approach for multi-storey space frames: eneral outline and symmetry-adapted redundants", *Structural Engineering Review*, 7(2): 107–119, 1995.

[19] K. Balasubramanian, "Graph theoretical perception of molecular symmetry", *Chemical Physics Letters*, 232(5âĂŞ6): 415 – 423, 1995.

[20] M. Hamermesh, *Group theory and its application to physical problems*, Dover books on physics and chemistry. Dover Publications, 1989.

[21] M. Bendsøe, O. Sigmund, *Topology optimization: theory, methods, and applications*, Springer, 2003.

[22] K. Svanberg, "On local and global minima in structural optimization", in E. Atrek, R.H. Gallagher, K.M. Ragsdell, O.C. Zienkiewicz (Editors), *New directions in optimal structural design*. John Wiley and Sons, 1984.

[23] M.P. Bendsøe, A. Ben-Tal, J. Zowe, "Optimization methods for truss geometry and topology design", *Structural and Multidisciplinary Optimization*, 7: 141–159, 1994.