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On the Convergence of a Nonconforming Plane Quadrilateral Finite Element

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Abstract

The convergence of the nonconvex quadrilateral nonconforming finite element RQ6 is proven using the subdivision of nonunisolvent quadrilaterals and Stummel's generalized patch test. The error estimates are derived with the use of standard geometric conditions. The results of various numerical examples completely confirm the theoretical findings.

Keywords: nonconforming finite element method, convergence, generalized patch test, nonconvex quadrilateral.

1 Introduction

Analytical proofs of convergence of nonconforming finite elements have been performed only occasionally for a specific nonconforming finite elements, see, e.g. Lesaint [1] and Shi [2], or, as in Wang [3], for a particular class of simple nonconforming elements. Because mathematics involved in proving convergence analytically is highly specific and demanding, convergence of the majority of nonconforming finite elements proposed in literature has only been subject to the numerical convergence test introduced by Irons (termed the 'patch test') [4, 5], which is simple to employ. According to [2, 6], however, Irons' patch test as the convergence condition is in general neither sufficient nor necessary for convergence of a nonconforming finite element, and is thus of a limited applicability. An example of such a nonconforming finite element that fully passes the patch test, yet is not unconditionally convergent, is the quadrilateral finite element RQ6, introduced recently by Cheung et al. [7] and discussed in the present paper. Here we analytically derive the sufficient conditions for convergence of the element RQ6 in solving the Dirichlet BVP of a second order elliptic equation on convex domain. Our analytical proof of convergence is based on

Stummel's generalized patch test [6] and the requirement, that the approximability condition is satisfied. In deriving error estimates, we follow the methodology of Shi [2] and Lesaint [1] and employ the inequalities given by Brenner and Scott [8] and Verfürth [9]. The derivations show that the RQ6 finite element is convergent even if its quadrilaterals' domain is nonconvex. This convenient property of the RQ6 element is an advantage when compared to standard isoparametric elements. Due to the use of the Cartesian base functions, the interpolation matrix A of the RQ6 finite element may experience rank deficiency. This inconvenience is here resolved by a further division of such a quadrilateral into more quadrilaterals. This step is here called the *quadriangulation of nonunisolvent quadrilaterals*, and described in detail in Section 3.2. Throughout the text terms *division* and *subdivision* are related to the domain quadriangulation and the nonunisolvent quadrilaterals quadriangulation, respectively. The results of various numerical examples completely confirm the present theoretical findings, in particular the linear rate of convergence in the energy norm, and the quadratic convergence in the L^2 norm. They also show that convergence and robustness of results depend not only on the finite element itself, but also on the meshing employed such that, e.g. the results obtained by dividing schemes using only convex quadrilaterals are somewhat more accurate compared to the results obtained by a dividing schemes using nonconvex quadrilaterals. In what follows we will use the abbreviations for classical as well as for generalized derivatives $\partial_{00}v := v$, $\partial_{10}v := \partial_1 v := \frac{\partial v}{\partial x}$, $\partial_{01}v := \partial_2 v := \frac{\partial v}{\partial y}$, $\partial_{11}v := \frac{\partial^2 v}{\partial x \partial y}$, $\partial_{20}v := \frac{\partial^2 v}{\partial x^2}$, for the set of locally integrable functions $L^1_{\text{loc}}(\Omega)$, where L^p norm is marked with $\|\cdot\|_{0,p,\Omega} : f \mapsto \left(\int_\Omega |f(\boldsymbol{x})|^p \, d\boldsymbol{x}\right)^{\frac{1}{p}}$, for the Sobolev seminorm $|\cdot|_{m,p,\Omega} := \left(\sum_{|i+j|=m} \|\partial_{ij} \cdot\|^p_{0,p,\Omega}\right)^{\frac{1}{p}}$ and norm $\|\cdot\|_{m,p,\Omega} := \left(\sum_{i=0}^m |\cdot|^p_{m,p,\Omega}\right)^{\frac{1}{p}}$ for an integer m and real $1 \le p < \infty$ [10]. In case $p = \infty$, we define the subnorm $|\cdot|_{m,\infty,\Omega} := f \mapsto \max_{|i+j|=m} (\text{ess.sup}_{x\in\Omega} |\partial_{ij} f(x)|)$ and the norm $\|\cdot\|_{m,\infty,\Omega} := \max_{0\le k\le m} |\cdot|_{k,\infty,\Omega}$. If $p = 2$, we will use the abbreviations $|\cdot|_{m,\Omega} \equiv |\cdot|_{m,2,\Omega}$, $\|\cdot\|_{m,\Omega} \equiv \|\cdot\|_{m,2,\Omega}$, $|\cdot|_{m,h} \equiv |\cdot|_{m,2,h} := \left(\sum_{Q\in\mathcal{Q}_h} |\cdot|^2_{m,Q}\right)^{\frac{1}{2}}$ and $\|\cdot\|_{m,h} \equiv \|\cdot\|_{m,2,h} := \left(\sum_{Q\in\mathcal{Q}_h} \|\cdot\|^2_{m,Q}\right)^{\frac{1}{2}}$, where $\mathcal{Q}_h$ denotes the division of the polygonal domain Ω into generally non–convex quadrilaterals Q. The Sobolev spaces used here will be denoted as $H^m(\Omega) := W^m_2(\Omega)$ and defined by $W^m_p(\Omega) := \{f \in L^1_{\text{loc}}(\Omega); \|f\|_{m,p,\Omega} < \infty\}$. $H^m_0(\Omega)$ is a closure of $C^\infty_0(\Omega)$ in $H^m(\Omega)$ and V_h is the finite element space, constructed from finite elements RQ6 in Section 3. In order to simplify the notation, we denote most of the constants with the same name c, i.e. c is a constant, which can take different meanings and different values at different places.

2 Boundary value problem

We consider the weak form of the Dirichlet boundary value problem of a second order elliptic equation with variable coefficients on the convex polygonal domain $\Omega \subset \mathbb{R}^2$, where we sarch for the weak solution $u^* \in H_0^1(\Omega)$ of equation

$$a(u^*, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where

$$\begin{aligned} a(u,v) &:= \int_\Omega \left(\sum_{i,j=1}^{2} a_{ij}\, \partial_i u\, \partial_j v + \sum_{i=1}^{2} a_i\, \partial_i u\, v + a\, u\, v \right) d\boldsymbol{x}, \\ (f,v) &:= \int_\Omega f\, v\, d\boldsymbol{x}. \end{aligned} \tag{2}$$

Let us the coefficients of Equation (2) fulfil the conditions from [11, p. 97–98] which assure the existence of the weak solution.

The nonconforming approximation $u_h^* \in V_h$ of the solution u^* is defined by the variational equation

$$a_h(u_h^*, v_h) = (f, v_h) \quad \forall v_h \in V_h, \tag{3}$$

with

$$a_h(u_h, v_h) := \sum_{Q \in \mathcal{Q}_h} \int_Q \left(\sum_{i,j=1}^{2} a_{ij}\, \partial_i u_h\, \partial_j v_h + \sum_{i=1}^{2} a_i\, \partial_i u_h\, v_h + a\, u_h\, v_h \right) d\boldsymbol{x}. \tag{4}$$

In the next section we construct the finite element space V_h, where we employ theorems from [2, 12] and show that for a sufficiently small mesh size, h, the approximating variational equation (3) is uniquely solvable.

3 Finite element RQ6

The RQ6 finite element is a nonconforming plane quadrilateral element (Figure 1), developed directly in Cartesian coordinates and characterized by the satisfaction of the so called weak continuity of displacement [7], which is mathematically expressed by Equation (8). The element is fully described in Cheung et al. [7]. There the convex quadrilaterals were only considered. The following abbreviations will be used in the paper:

$$\boldsymbol{X} := \begin{bmatrix} 1 & x & y & x\,y & x^2 & y^2 \end{bmatrix}^T, \; \alpha := \begin{bmatrix} \alpha_1 & \ldots & \alpha_6 \end{bmatrix}^T, \; \boldsymbol{q} := \begin{bmatrix} u_1 \ldots u_4 & \alpha_5 & \alpha_6 \end{bmatrix}^T$$

$$A := \begin{bmatrix} 1 & x_1 & y_1 & x_1\,y_1 & x_1^2 & y_1^2 \\ 1 & x_2 & y_2 & x_2\,y_2 & x_2^2 & y_2^2 \\ 1 & x_3 & y_3 & x_3\,y_3 & x_3^2 & y_3^2 \\ 1 & x_4 & y_4 & x_4\,y_4 & x_4^2 & y_4^2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Let Q be a quadrilateral with border ∂Q, whose outer normal components in x and y directions are denoted by n_1 and n_2, respectively. The area of the quadrilateral

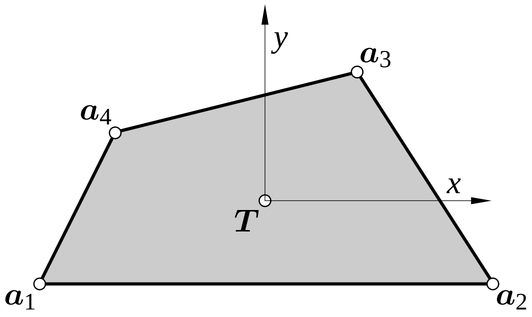

Figure 1: Quadrilateral finite element RQ6 [7]

is denoted by $|Q|$. Let the origin of the Cartesian coordinate system (x, y) be the geometric center of the quadrilateral, T (Figure 1). Let v_h denote the polynomial approximation function on Q as proposed in [7] and defined by

$$v_h := w_h + \Lambda_h, \tag{6}$$

where

$$w_h|_Q := \boldsymbol{X}^T A^{-1} \boldsymbol{q}, \quad \Lambda_h|_Q := \lambda_1 x + \lambda_2 y. \tag{7}$$

The constants λ_1 and λ_2 are determined from the equations [7]

$$\int_{\partial Q} (v_h - \widetilde{w_h}) \, n_i \, ds = 0, \quad 1 \le i \le 2, \tag{8}$$

describing the conditions of weak continuity. Function $\widetilde{w_h}$ denotes a piecewise linear function on border ∂Q, well defined and continuous on the border $\cup_{\mathcal{Q}_h} \partial Q$, obtained by the interpolation through the nodal values of function w_h, i.e. $q_i = w_h(\boldsymbol{a}_i) = \widetilde{w_h}(\boldsymbol{a}_i), 1 \le i \le 4$. In what follows it is found convenient to define an arbitrary piecewise linear function $\widetilde{v}$ on border ∂Q interpolated in the same way as $\widetilde{w_h}$, i.e. by the function values at the nodal points

$$\widetilde{v}|_{\overline{ij}} := v(\boldsymbol{a}_i)(1-t) + v(\boldsymbol{a}_j)t, \quad t \in [0, 1], \tag{9}$$

where $\overline{ij}$ denotes the side on the border ∂Q, spanning the nodes $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$. With the constants λ_1 and λ_2 obtained from (8), the finite element automatically passes Irons' numerical patch test [4, 7, 5]. Inserting v_h from Equation (6) into Equation (8) gives

$$\lambda_i = \frac{1}{|Q|} \left(\int_{\partial Q} \widetilde{w_h} \, n_i \, ds - \int_Q \partial_i w_h \, d\boldsymbol{x} \right) = \frac{1}{|Q|} \int_{\partial Q} (\widetilde{w_h} - w_h) \, n_i \, ds, \quad 1 \le i \le 2. \tag{10}$$

3.1 Finite element (Q, P_Q, Φ_Q)

In order to use the interpolation theory derived in [8, 13], we have first to define the finite element as a triple (Q, P_Q, Φ_Q)[13] and construct the interpolation operator I_h.

In the triple (Q, P_Q, Φ_Q), the sets Q, P_Q and Φ_Q denote the quadrilateral, the shape functions and the degrees of freedom, respectively. Invoking Equations (6) and (10) and the piecewise linearity of the difference $\Lambda_h = v_h - w_h$, we have $\Lambda_h|_{\partial Q} = \widetilde{\Lambda_h}|_{\partial Q}$, $v_h = w_h + \Lambda_h$, $\widetilde{v_h} = \widetilde{w_h} + \widetilde{\Lambda_h}$, and consequently

$$(\widetilde{v_h} - v_h)|_{\partial Q} = (\widetilde{w_h} - w_h)|_{\partial Q}. \tag{11}$$

Using Equation (11) and considering the linearity of the difference $\Lambda_h|_Q = v_h|_Q - w_h|_Q$, we can introduce two sets of linear functionals, $\Sigma_Q := \{\varphi_i^Q, 1 \leq i \leq 6\}$ and $\Phi_Q := \{\phi_i^Q, 1 \leq i \leq 6\}$, as

$$\begin{aligned} \varphi_i^Q(w_h) &:= w_h(\boldsymbol{a}_i) = q_i = \\ &= v_h(\boldsymbol{a}_i) - \frac{1}{|Q|}\left(\int_{\partial Q} (\widetilde{v_h} - v_h)\,(n_1\,x_i + n_2\,y_i)\,ds\right) \\ &=: \phi_i^Q(v_h), \quad 1 \leq i \leq 4, \end{aligned} \tag{12a}$$

$$\begin{aligned} \varphi_{i+4}^Q(w_h) &:= \frac{1}{2\,|Q|}\int_Q \partial_i(\partial_i w_h)\,d\boldsymbol{x} = q_{i+4} = \frac{1}{2\,|Q|}\int_Q \partial_i(\partial_i v_h)\,d\boldsymbol{x} \\ &=: \phi_{i+4}^Q(v_h), \quad 1 \leq i \leq 2. \end{aligned} \tag{12b}$$

After Stummel [12] the sequence of functions $\{v_h\}$ corresponding to the sequence of divisions $\{\mathcal{Q}_h\}$ fulfills the *weak continuity condition*, if and only if there exists some constant $c_\theta > 0$ such that the inequality

$$|v_h|_Q(\boldsymbol{x}) - v_h|_{Q'}(\boldsymbol{x})| \leq c_\theta\, h \sup_{Q \in \mathcal{Q}_h(\boldsymbol{x})} |v_h|_{1,\infty,Q} \tag{13}$$

holds uniformly for all quadrilaterals $Q, Q' \in \mathcal{Q}_h(\boldsymbol{x}) := \{Q \in \mathcal{Q}_h, \boldsymbol{x} \in Q\}$ for each point $\boldsymbol{x} \in \cup_{\mathcal{Q}_h} \partial Q$ and all divisions $\mathcal{Q}_h$. Considering the choice of the degrees of freedom (12a) and Condition 1, we easily see that the sequence of the functions $\{v_h\}$ along with their derivatives fulfills the weak continuity condition. For the finite element to be well defined, it must be unisolvent [13, p. 78]. Thus the unisolvence of the finite element (Q, P_Q, Φ_Q) must be checked first. Following the definition of Ciarlet [13, p. 78], the set Φ_Q is the P_Q-unisolvent, if and only if for any given real scalars $\alpha_i, 1 \leq i \leq 6$, there exists a unique function $p \in P_Q$, which satisfies $\phi_i^Q(p) = \alpha_i, 1 \leq i \leq 6$. It turns out, however, that it is easier to prove the unisolvence of the set Σ_Q (the criterion is simply the non-singularity of the matrix A) than of the set Φ_Q. The following Lemma 1, which says that criterion for the unisolvence of the sets Σ_Q and Φ_Q is the same, resolves the problem.

Lemma 1. *Let the space P'_Q denote an algebraic dual of space P_Q. The set Φ_Q is the base for P'_Q, if and only if the set Σ_Q is the base for P'_Q.*

Proof. Let the set Σ_Q be the base for P'_Q. Since $\varphi_i^Q(w_h) = \phi_i^Q(v_h) = 0$ for $1 \leq i \leq 6$, we have $w_h|_Q = 0$. Hence we can conclude from Equations (6) and (10) that $v_h|_Q = 0$. Lemma 3.1.4 in [8] then guarantees that the set Φ_Q is the base. Similarly, invoking Equation (11), we can prove that if the set Φ_Q is the base, the set Σ_Q is also the base. □

Since the set Σ_Q is the base for P'_Q and, consequently, P_Q is unisolvent if and only if the matrix A is nonsingular, the criterion for the unisolvence of the set Φ_Q is, according to Lemma 1, the nonsingularity of the matrix A. Let $\mathcal{N}_h$, $\mathcal{Q}_h$, $\mathcal{Q}_h(\boldsymbol{a})$ and $Q_1(\boldsymbol{a})$ denote the set of all vertices, the set of all quadrilaterals, the set of quadrilaterals with common vertex $\boldsymbol{a}$ and the first quadrilateral from $\mathcal{Q}_h(\boldsymbol{a})$, respectively. For a quadrilateral Q with nodes $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ and $\boldsymbol{a}_4$, we define functionals $\phi^Q_{\boldsymbol{a}_i} := \phi^Q_i$ for $1 \le i \le 4$.

$$\begin{aligned} X_h := \{ &\forall Q \in \mathcal{Q}_h \quad v_h|_Q \in P_Q, \\ &\forall \boldsymbol{a} \in \mathcal{N}_h, \forall Q_i, Q_j \in \mathcal{Q}_h(\boldsymbol{a}) \quad \phi^{Q_i}_{\boldsymbol{a}}(v_h|_{Q_i}) = \phi^{Q_j}_{\boldsymbol{a}}(v_h|_{Q_j})\} \end{aligned} \tag{14}$$

and the related set of linear functionals (degrees of freedom)

$$\Phi_h := \{\phi_{\boldsymbol{a}} = \phi^{Q_1(\boldsymbol{a})}_{\boldsymbol{a}}, \boldsymbol{a} \in \mathcal{N}_h\} \cup \{\phi^Q_5, Q \in \mathcal{Q}_h\} \cup \{\phi^Q_6, Q \in \mathcal{Q}_h\}.$$

Using the finite element space X_h we can define the finite element space V_h as

$$V_h := \{v_h \in X_h, \quad \forall \boldsymbol{a} \in \partial\Omega \quad \phi^{Q_1(\boldsymbol{a})}_{\boldsymbol{a}}(v_h) = 0\}. \tag{15}$$

Next we employ the dual functions $p_{\boldsymbol{a}}$, p^Q_l from V_h for functionals $\phi_{\boldsymbol{a}}$, ϕ^Q_l on the open set $\Omega_h = \Omega - \cup_{Q\in\mathcal{Q}_h}\partial Q$ and define the interpolation operator

$$I_h : v \mapsto \sum_{\boldsymbol{a}\in\mathcal{N}_h} \phi_{\boldsymbol{a}}(v)\, p_{\boldsymbol{a}} + \sum_{Q\in\mathcal{Q}_h, l=5,6} \phi^Q_l(v)\, p^Q_l.$$

Remark 1. In defining the approximating space v_h, we employ the boundary conditions $\phi^{Q_1(\boldsymbol{a})}_{\boldsymbol{a}}(v_h) = 0$ rather than the more traditional conditions $\varphi^{Q_1(\boldsymbol{a})}_{\boldsymbol{a}}(v_h) = 0$. Since the function v_h is weak continuous, both conditions describe the same Dirichlet boundary conditions in the limit $h \mapsto 0$.

3.2 Unisolvence

In order for the unisolvence to be achieved we set

Condition 1. Let h_Q and ℓ_Q denote the diameter of quadrilateral Q and the length of the shortest side on the border ∂Q, respectively. Suppose that Q is star–shaped with respect to the ball with radius $\rho_Q := \sup\{\mathrm{diam}(S), \text{ball } S \subset Q, \forall x \in Q\, \forall y \in S\, \forall \lambda \in [0,1] \Rightarrow (1-\lambda)\, x + \lambda\, y \in Q\}$. Then we can define the chunkiness parameter $c_\rho := \frac{h_Q}{\rho_Q}$ and parameter $c_\ell := \frac{h_Q}{\ell_Q}$. We assume that the constant γ exists such that the following inequality holds

$$\max\left(\cup_{Q\in\mathcal{Q}_h} \max(c_\rho, c_\ell)\right) \le \gamma. \tag{16}$$

If Condition 1 is satisfied, there exists the quadriangulation $\mathcal{Q}_Q$ of quadrilateral Q such that the sets Σ_Q and Φ_Q become P_Q unisolvent (see Lemma 1). In what follows, we show the existence of such a quadriangulation $\mathcal{Q}_Q$, for which the absolute value of determinant $|A|$ is bounded from below for any quadrilateral. In order to simplify the notation, we will use some abbreviations. Let us denote the nodes of a quadrilateral with bold numbers only (see Figures 2 and 3). Next, let us denote the quadrilateral with nodes $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and $\boldsymbol{l}$ and the absolute value of determinant A of that quadrilateral by $Q_{i,j,k,l}$ and $|A|_{i,j,k,l}$, respectively. The quadrilateral Q with a singular matrix A will be referred to as *a non-unisolvent quadrilateral*. Finally, let us define the function of determinant of quadrilateral $Q_{1,2,3,4}$ of the triangular shape and all quadrilaterals obtained by the sequence of subdivisions (shown in Figure 3) by

$$\begin{aligned}\nmid A \nmid_{1,2,3,4} := \max(&|A|_{1,2,3,4}, \min(|A|_{2,5,4,1}, |A|_{4,5,2,3}),\\ &\min(|A|_{5,6,1,2}, |A|_{1,6,5,4}, |A|_{5,7,3,4}, |A|_{3,7,5,2})).\end{aligned}$$

Let us explain further the physical meaning of the last abbreviation. From the above definition it is easy to see that $\nmid A \nmid_{1,2,3,4}$ does not vanish when one and only one of the following cases occurs:

- quadrilateral $Q_{1,2,3,4}$ is unisolvent,
- quadrilateral $Q_{1,2,3,4}$ is non-unisolvent, but the two quadrilaterals, $Q_{2,5,4,1}$ and $Q_{4,5,2,3}$, obtained by the first subdivision of quadrilateral $Q_{1,2,3,4}$, are unisolvent,
- quadrilateral $Q_{1,2,3,4}$ as well as quadrilaterals $Q_{2,5,4,1}$ and $Q_{4,5,2,3}$ are non-unisolvent, but the quadrilaterals $Q_{5,6,1,2}$, $Q_{1,6,5,4}$, $Q_{5,7,3,4}$ and $Q_{3,7,5,2}$, obtained by the second subdivision of quadrilaterals $Q_{2,5,4,1}$ and $Q_{4,5,2,3}$, are unisolvent.

Lemma 2. *Let Condition 1 hold. Then there exists the quadriangulation (shown in Figure 2) and the constant $c_d = f(c_\rho, c_\ell)$ such that the following inequality holds:*

$$\max(|A|_{1,2,3,4}, \min(\nmid A \nmid_{1,6,5,4}, \nmid A \nmid_{5,6,1,2}, \nmid A \nmid_{3,7,5,2}, \nmid A \nmid_{5,7,3,4})) \geq c_d\, h_Q^4 \qquad (17)$$

with the centre $\mathbf{5}$ of the inserted circle and midpoints $\mathbf{6}$ and $\mathbf{7}$ of the lines with endpoints $\mathbf{1}$, $\mathbf{5}$ and $\mathbf{3}$, $\mathbf{5}$, respectively.

In order to prove Lemma 2, we first divide the quadrilateral Q into triangular quadrilaterals, i.e. the quadrilaterals of a triangle shape (Figure 2). The closed formula of the translation invariant determinant of the matrix A in Equation (5) of the triangular shaped quadrilateral with nodes $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$ (Figure 3) is conveniently obtained in the polar coordinate system and reads

$$|A| = -a \left(\frac{b}{2}\right)^3 \sin 2\psi_0 \, \sin\gamma. \qquad (18)$$

Let us first sketch the main ideas of the proof. There are two reasons for subdividing the quadrilateral Q into quadrilaterals of a triangle shape. The first one is the convenience of a simple form of the determinant of the matrix A demonstrated by Equation (18). The second one is that the same triangular shape of all subsequently subdivided quadrilaterals is conserved. So, in the subdivision process, the simple form of the determinant $|A|$ is retained. Using Condition 1, Equation (18) and the help of

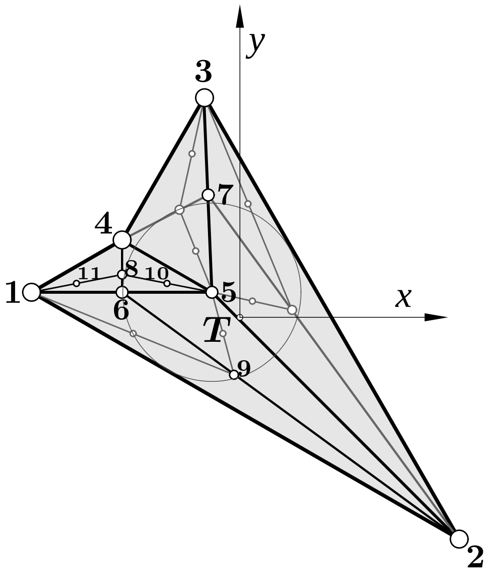

Figure 2: Quadriangulation of a quadrilateral

Figure 3, one can easily conclude that only the term $\sin 2\psi_0$ can make the matrix A singular; i.e. if $\sin 2\psi_0$ vanishes, we have to perform the subdivision of quadrilateral $Q_{1,2,3,4}$ into two quadrilaterals $Q_{2,5,4,1}$ and $Q_{4,5,2,3}$ (as displayed in Figure 3). If a newly obtained $\sin 2\psi_0'$ vanishes again, we have to perform another subdivision of each of the quadrilaterals $Q_{2,5,4,1}$ and $Q_{4,5,2,3}$ into a new pair of quadrilaterals $Q_{5,6,1,2}$, $Q_{1,6,5,4}$ and $Q_{3,7,5,2}$ $Q_{5,7,3,4}$ (Figure 3). It suffices to prove that there exists a strictly positive constant c independent on both meshing and h, but dependent on constant γ from Condition 1 such that one and the only one of the following three situations is possible:

$$|\sin 2\psi_0| \geq c(\gamma); \tag{19a}$$

$$|\sin 2\psi_0| < c(\gamma), \text{ but } |\sin 2\psi_0'| \geq c(\gamma); \tag{19b}$$

$$\begin{aligned} &|\sin 2\psi_0| < c(\gamma) \text{ and } |\sin 2\psi_0'| < c(\gamma), \text{ but} \\ &|\sin 2\psi_0''| \geq c(\gamma) \text{ and } |\sin 2\psi"_0| \geq c(\gamma). \end{aligned} \tag{19c}$$

Since there exists a subdivision of an arbitrary quadrilateral of the triangle shape in Figure 3 such that at least one of the terms $|\sin 2\psi_0|$, $|\sin 2\psi_0'|$, $|\sin 2\psi_0''|$ and $|\sin 2\psi"_0|$ is bounded from bellow by a constant $c = f(\gamma)$ and since the lengths of all sides of the quadrilateral as well as the sines of all internal angles in the triangles in any of the three subdivisions are bounded from bellow, there exists a constant $c = f(\gamma)$ such that the absolute values of the determinant $|A|$ are bounded from bellow by $c\,h_Q^4$ for all quadrilaterals in at least one of the three possible subdivisions. Now we proceed with the technical part of the proof. With the help of Figure 3 we have

$$\psi_0' = \psi_0 + \gamma', \; \psi_0'' = \psi_0' + \pi + \beta'', \; \psi_0 = \psi_0'' + \alpha' - 2\,\pi$$

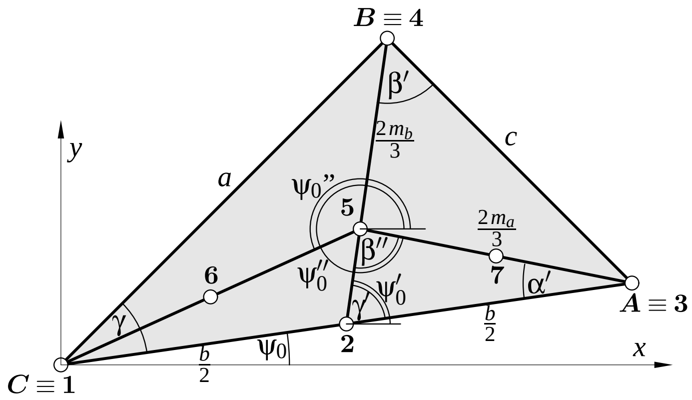

Figure 3: Successive subdivisions of a triangle

and, consequently,

$$
\begin{aligned}
\sin 2(\psi_0' - \psi_0) &= \sin 2\gamma' = \sin 2\psi_0' \, \cos 2\psi_0 - \sin 2\psi_0 \, \cos 2\psi_0', \\
\sin 2(\psi_0'' - \psi_0') &= \sin 2\beta'' = \sin 2\psi_0'' \, \cos 2\psi_0' - \sin 2\psi_0' \, \cos 2\psi_0'', \\
\sin 2(\psi_0 - \psi_0'') &= \sin 2\alpha' = \sin 2\psi_0 \, \cos 2\psi_0'' - \sin 2\psi_0'' \, \cos 2\psi_0.
\end{aligned}
$$

After introducing notations $s_0 = |\sin 2\psi_0|$, $s_0' = |\sin 2\psi_0'|$, $s_0'' = |\sin 2\psi_0''|$, the area $\triangle_{2,3,5}$ and the radius of the outer circle, $R_{2,3,5}$, of the triangle with nodes **2**, **3** and **5**, we sum the above equations and make an estimate

$$
\begin{aligned}
\sin 2\gamma' + \sin 2\beta'' + \sin 2\alpha' &= \frac{2\,\triangle_{2,3,5}}{R^2_{2,3,5}} = \frac{\triangle_{1,3,4}}{3\,R^2_{2,3,5}} = \frac{\triangle_{1,3,4}}{3\left(\frac{\frac{b}{2}\,\frac{m_b}{3}\,\frac{2\,m_a}{3}}{4\,\triangle_{2,3,5}}\right)^2} = \frac{12\,\triangle^3_{1,3,4}}{b^2\,m_a^2\,m_b^2} \\
&\le 2\,(s_0 + s_0' + s_0'') \le 6\,\max(s_0, s_0', s_0''),
\end{aligned}
$$

yielding

$$
\max(s_0, s_0', s_0'') \ge \frac{2\,\triangle^3_{1,3,4}}{b^2\,m_a^2\,m_b^2}. \tag{20}
$$

Using Condition 1 and the result of the last inequality (20), one can confirm the validity of Equations (19). Let ρ_T denote the radius of the incircle of the triangle with nodes **1**, **3** and **4**. Considering inequality (20) gives

$$
\begin{aligned}
&\max(|A|_{3,7,5,2}, |A|_{4,5,2,3}, |A|_{1,2,3,4}) \ge \\
&\frac{2\,\triangle^3_{1,3,4}}{b^2\,m_a^2\,m_b^2}\,\min\left(|\sin\alpha'|\,\frac{b}{2}\,\left(\frac{m_a}{3}\right)^3, |\sin\beta'|\,c\,\frac{m_b^3}{9}, |\sin\gamma|\,a\,\left(\frac{b}{2}\right)^3\right) \\
&= \min\left(\frac{\triangle^4_{1,3,4}}{162\,b^2\,m_b^2}, \frac{\triangle^4_{1,3,4}}{9\,m_a^2\,b^2}, \frac{\triangle^4_{1,3,4}}{4\,m_a^2\,m_b^2}\right) \ge \frac{\triangle^4_{1,3,4}}{162\,s^4} = \frac{1}{2}\,\left(\frac{\rho_T}{3}\right)^4.
\end{aligned} \tag{21}
$$

The similar procedure is performed for the remaining triangles. In order to establish the constant $c_d = c_d(c_\rho, c_\ell)$, we have to find the triangle with the smallest radius of incircle among all admissible triangles. Using Condition 1 one can prove the estimate

$$\rho_T \geq \frac{h_Q}{(c_\rho - 1)\, c_\ell + 2\, c_\rho}.$$

Combining inequality (21) with the above inequality and inserting the result of Equation (17) gives

$$c_d = \frac{1}{162}\left(\frac{1}{(c_\rho - 1)\, c_\ell + 2\, c_\rho}\right)^4 \geq \frac{1}{162}\left(\frac{1}{(\gamma + 2)\gamma}\right)^4 .$$

Remark 2. For illustrative purposes, we give a simple example of a non-unisolvent convex quadrilateral (rotated square) which is defined by the nodal coordinates given below:

$$\boldsymbol{a}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{a}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and a more complicated example of a non-unisolvent nonconvex quadrilateral where unisolvence cannot be achieved solely by a rotation of the Cartesian coordinate system, defined by the nodal coordinates given below:

$$\boldsymbol{a}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} \frac{3}{2}\left(1 + \sqrt{3}\right) \\ \frac{1}{2}\left(-3 - \sqrt{3}\right) \end{bmatrix}, \boldsymbol{a}_3 = \begin{bmatrix} \frac{1}{2} + \frac{2}{\sqrt{3}} \\ 1 + \frac{\sqrt{3}}{2} \end{bmatrix}, \boldsymbol{a}_4 = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}.$$

4 Error estimate in the energy norm

Let h denote the greatest diameter of all quadrilaterals of quadriangulation $\mathcal{Q}_h$. Assume that Condition 1 holds. In what follows c will denote a generic constant independent of h, which may take different values at different places. For each quadrilateral Q, we introduce the quadrilateral $\hat{Q}$ with the same shape, yet with the diameter $h_{\hat{Q}}$ equal 1. Considering the second Strang Lemma [13]

$$\|u^* - u_h^*\|_{1,h} \leq c\left(\inf_{v_h \in v_h} \|u^* - v_h\|_{1,h} + \sup_{v_h \in v_h} \frac{|(f, v_h) - a_h(u^*, v_h)|}{\|v_h\|_{1,h}}\right), \qquad (22)$$

the error estimate can be splitted into two parts, i.e. the error estimate of the approximability term, and the error estimate of the consistency term. First we will derive an error estimate of the consistency term. Next we will consider the validity of the approximability condition and derive an error estimate of the approximability term.

4.1 Error estimate of the consistency term

In order to study convergence and to derive an error estimate of the consistency term, we will now employ Stummel's generalized patch test [6] supplemented by Condition 1. According to [6] and [3] we need to show that a bounded sequence of functions

$\{v_{h_i}, i \in \mathbb{N}' \subseteq \mathbb{N}\}$ in an appropriate divisions $\{\mathcal{Q}_{h_i}, i \in \mathbb{N}', h_i \to 0\}$ of domain Ω satisfies the conditions

$$\lim_{i\in\mathbb{N}'} T_j(\psi, v_{h_i}) := \lim_{i\in\mathbb{N}'} \sum_{Q\in\mathcal{Q}_h} \int_{\partial Q} \psi\, v_{hi}\, n_j\, ds = 0, \quad 1 \le j \le 2, \quad \psi \in C_0^\infty(\mathbb{R}^2), \tag{23}$$

Since the function $\widetilde{w_h}$ and, consequently, the function $\psi\,\widetilde{w_h}$ are conforming on the element boundary, the sum of the line integrals of $\psi\,\widetilde{w_h}$ over the closed boundary must vanish:

$$T_r(\psi, \widetilde{w_h}) = \sum_{Q\in\mathcal{Q}_h} \int_{\partial Q} \psi\,\widetilde{w_h}\, n_r\, ds = 0, \quad 1 \le r \le 2. \tag{24}$$

As an implication of Equation (8) we have that

$$T_r(\psi_0, v_h - \widetilde{w_h}) = \sum_{Q\in\mathcal{Q}_h} \int_{\partial Q} \psi_0\,(v_h - \widetilde{w_h})\, n_r\, ds = 0, \quad 1 \le r \le 2 \tag{25}$$

for all piecewise constant functions ψ_0 ($\psi_0|_Q = \text{const}$). Since $w_h(\boldsymbol{a}) = \widetilde{w_h}(\boldsymbol{a})$ for any vertex $\boldsymbol{a}$ of the quadrilateral Q, we get

$$|\lambda_1| \le c\, |w_h|_{2,\infty,Q}\, h, \quad |\lambda_2| \le c\, |w_h|_{2,\infty,Q}\, h. \tag{26}$$

Due to the linearity of the function $\Lambda_h|_Q$, we have $|v_h|_{2,\infty,Q} = |w_h|_{2,\infty,Q}$. Following [8] we define operators P_0 and R_0 as $P_0\, v := Q_1\, v$, $R_0\, v := v - Q_1\, v$, $\hat{R}_0\,\hat{v} := \hat{v} - \hat{Q}_1\,\hat{v}$. The functions v, $\hat{v}$, $Q_1\, v$ and $\hat{Q}_1\,\hat{v}$ are related by the affine mapping, which is scaling in the present case of F_Q with equations $v := \hat{v}\circ F_Q^{-1}$, $Q_1\, v := \hat{Q}_1\hat{v}\circ F_Q^{-1}$. Invoking Equation. (6) we write

$$T_r(\psi, v_h) = T_r(\psi, \widetilde{w_h}) + T_r(P_0\psi, v_h - \widetilde{w_h}) + T_r\left(R_0\psi, w_h - \widetilde{w_h} + \Lambda_h\right).$$

The first and the second term on the right-hand side of the above equation are zero, see Equations (24) and (25). Hence we need to proceed to estimate the third term only:

$$\begin{aligned}
|T_r(R_0\psi, w_h - \widetilde{w_h} + \Lambda_h)| &\le \sum_{Q\in\mathcal{Q}_h} \left| \int_{\partial Q} R_0\,\psi\,(w_h - \widetilde{w_h} + \Lambda_h)\, n_r\, ds \right| \\
&\le \sum_{Q\in\mathcal{Q}_h} \left| \int_{\partial Q} R_0\,\psi\,(w_h - \widetilde{w_h})\, n_r\, ds \right| + \sum_{Q\in\mathcal{Q}_h} \left| \int_{\partial Q} R_0\,\psi\,\Lambda_h\, n_r\, ds \right| \\
&\le \sum_{Q\in\mathcal{Q}_h} |R_0\psi|_{0,Q}\, |w_h - \widetilde{w_h}|_{0,Q} + \sum_{Q\in\mathcal{Q}_h} |R_0\psi|_{0,Q}\, |\Lambda_h|_{0,Q} \\
&\le c \sum_{Q\in\mathcal{Q}_h} |\psi|_{1,2,Q}\,\sqrt{h}\, |w_h|_{2,\infty,Q}\, h^2\sqrt{h} = c \sum_{Q\in\mathcal{Q}_h} |\psi|_{1,2,Q}\,\sqrt{h}\, |v_h|_{2,\infty,Q}\, h^2\sqrt{h} \\
&\le c\, h \sum_{Q\in\mathcal{Q}_h} |\psi|_{1,2,Q}\, |v_h|_{1,2,Q} \\
&\le c\, h \left(\sum_{Q\in\mathcal{Q}_h} |\psi|^2_{1,2,Q}\right)^{\frac{1}{2}} \left(\sum_{Q\in\mathcal{Q}_h} |v_h|^2_{1,2,Q}\right)^{\frac{1}{2}} = c\, h\, |\psi|_1\, |v_h|_{1,h}.
\end{aligned} \tag{27}$$

Thanks to Condition 1, we can employ the well known inverse inequality in the fifth inequality of the above equation. This leads to

$$|T_r(\psi, v_h)| \leq c\,h\,|\psi|_1\,|v_h|_{1,h}. \tag{28}$$

Employing the consistency error functional

$$E_h(u^*, v_h) := (f, v_h) - a_h(u^*, v_h), \tag{29}$$

using the integration by parts

$$\begin{aligned} E_h(u^*, v_h) = (f, v_h) - a_h(u^*, v_h) &= \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \left(\sum_{i,j=1}^{2} a_{ij}\,\partial_i u^*\,n_j \right) v_h\,ds \\ &= \sum_{j=1}^{2} T_j(\varphi_j, v_h), \end{aligned}$$

where $\varphi_j := \sum_{i=1}^{2} a_{ij}\,\partial_i u^*,\ 1 \leq j \leq 2$, considering inequality (28) and following Shi [2] yields the final estimate

$$|E_h(u^*, v_h)| \leq c\,h\,|u^*|_{2,h}\,|v_h|_{1,h}. \tag{30}$$

The fourth inequality in Equation (27) and the estimate given in (30) follow from the Bramble-Hilbert Lemma (4.3.8) in [8], the Sobolev Imbedding Theorem [14, Case B] and Green's formulae [15].

Remark 3. Employing (8) in $\int_Q \partial_i v_h\,d\boldsymbol{x}$ gives

$$\sum_{Q \in \mathcal{Q}_h} \int_Q \partial_i v_h\,d\boldsymbol{x} = \sum_{Q \in \mathcal{Q}_h} \int_{\partial Q} \widetilde{w_h}\,n_i\,ds = 0, \quad 1 \leq i \leq 2. \tag{31}$$

This shows that the element (Q, P_Q, Φ_T) passes Irons' patch test [3]. Due to the equality $\widetilde{w_h}(\boldsymbol{a}_i) = w_h(\boldsymbol{a}_i) = q_i$ and the second equality of Equation (31) Irons' patch test is fulfilled even in cases where nonunisolvent quadrilaterals are used.

4.2 Estimate of the approximability term

In order to verify the satisfaction of the approximability condition, we will use the interpolation theory as presented in [8, 13]. To this end, it is worth first transforming the original problem into the scaled affine space, where the interpolation theory can easier be implemented. In order to prove the satisfaction of the approximability condition, we use Theorem 4.4.4 from Brenner and Scott [8]. Let us first adapt their Definition 4.4.2 and Theorem 4.4.4 into the form suitable for our subsequent purposes.

Lemma 3. *Let (Q, P, Φ) be a finite element satisfying*

(i) Q is star-shaped with respect to some ball,

(ii) $\mathcal{P}_1 \subset P \subset W^2_\infty(Q)$ *and*

(iii) $\Phi \subset (W^2_2(Q))'$.

Suppose that $p = 2$. *Then for* $0 \le i \le 2$ *and* $v \in W^2_2(Q)$, *we have*

$$|v - I_h\, v|_{i,2,Q} \le C_{\gamma,\sigma(\hat{Q})} (\operatorname{diam} Q)^{2-i}\, |v|_{2,2,Q},$$

where $\hat{Q} := \{\frac{x}{(\operatorname{diam} Q)},\ x \in Q\}$, *parameter* γ *is introduced in Condition 1, and* $\sigma(\hat{Q})$ *is defined to be the operator norm of* $\hat{I}_h : W^2_2(\hat{Q}) \to W^2_2(\hat{Q})$.

Let us note that the interpolation operator $\hat{I}_h$ is well defined on $W^2_2(\hat{Q})$. This follows from the Cauchy–Schwarz inequality $|\int_{\hat{Q}} \hat{u}_{\hat{x}\hat{x}}\, d\hat{\boldsymbol{x}}| \le C\sqrt{\int_{\hat{Q}} \hat{u}^2_{\hat{x}\hat{x}}\, d\hat{\boldsymbol{x}}}$ and the Sobolev Imbedding Theorems. Our aim is to estimate the norm $\sigma(\hat{Q})$ of operator $\hat{I}_h : W^2_2(\hat{Q}) \to W^2_2(\hat{Q})$. For the sake of simplicity of notation, we skip the hat over all symbols. Employing the inequalities

$$\|I_h\, u\|_{2,2,Q} \le \sum_{i=1}^{6} |\phi_i(u)|\, \|p_i\|_{2,2,Q} \le \sum_{i=1}^{6} \|\phi_i\|_{W^2_2(Q)'}\, \|p_i\|_{2,2,Q}\, \|u\|_{2,2,Q}$$

we derive

$$\sigma(Q) \le \sum_{i=1}^{6} \|\phi_i\|_{W^2_2(Q)'}\, \|p_i\|_{2,2,Q}\ . \tag{32}$$

Next we show that the norm $\sigma(Q)$ is uniformly bounded for all quadrilaterals Q. As it has been shown by Ciarlet [13] and Adams [14], the identity from $W^2_2(Q)$ to $C_0(Q)$ is uniformly continuous. For $i = 1, 2, 3, 4$ we estimate

$$\begin{aligned} |\phi_i(v)| &= \left| v(\boldsymbol{a}_i) - \frac{1}{|Q|}\left(\int_{\partial Q} (\widetilde{v} - v)\, n_1\, ds\, x_i + \int_{\partial Q} (\widetilde{v} - v)\, n_2\, ds\, y_i \right) \right| \\ &\le c\, \|v\|_{0,\infty,Q} \le c\, \|v\|_{2,2,Q}, \quad 1 \le i \le 4. \end{aligned} \tag{33}$$

Using the Cauchy-Schwarz inequality the above estimate also holds for $i = 5, 6$:

$$|\phi_i(v)| \le c\, \|v\|_{2,2,Q}, \quad 5 \le i \le 6. \tag{34}$$

It follows that the norms $\|\phi_i\|_{W^2_2(Q)'}$ are bounded from above by a constant $c = c(\gamma)$. Due to Condition 1, the base functions p_i are also bounded. Thus we can prove the inequality, introduced as

Lemma 4.

$$\|p_i\|_{2,2,Q} \le c(\gamma), \quad 1 \le i \le 6. \tag{35}$$

Proof. Using Equation (59) in [7] we can write

$$p_i|_Q = (\boldsymbol{X} + x\,(N_{xc}\, A - N_{x0}) + y\,(N_{yc}\, A - N_{y0}))\ A^{-1}\, \boldsymbol{e}_i,$$

where

$$N_{x0} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, N_{y0} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$
$$N_{xc} = \frac{1}{|Q|} \begin{bmatrix} y_2 - y_4 & y_3 - y_1 & y_4 - y_2 & y_1 - y_3 & 0 & 0 \end{bmatrix},$$
$$N_{yc} = \frac{1}{|Q|} \begin{bmatrix} x_4 - x_2 & x_1 - x_3 & x_2 - x_4 & x_3 - x_1 & 0 & 0 \end{bmatrix},$$

which when put into $\|p_i\|_{2,2,Q}$ results in the above inequality. □

Considering the second Strang Lemma [13]

$$\|u^* - u_h^*\|_{1,h} \leq c \left(\inf_{v_h \in V_h} \|u^* - v_h\|_{1,h} + \sup_{v_h \in V_h} \frac{|(f, v_h) - a_h(u^*, v_h)|}{\|v_h\|_{1,h}} \right), \tag{36}$$

the error functional estimate (30), the estimate of the approximability term (or Lemma 3), and Equations (32), (33), (34), (35), one can readily conclude that the error in the energy norm decreases at least linearly with h. We can sum up the above conclusions in Theorem 1.

Theorem 1. *Let Condition 1 be satisfied. Then the numerical solutions u_h^* of the Dirichlet boundary value problem on convex domain Ω from Section 2, obtained by finite elements RQ6, using the presented subdividing scheme on nonunisolvent quadrilaterals, converge with $h \to 0$ to the weak solution u^*. Furthermore, the error in the energy norm decreases at least linearly with the maximum element diameter h, i.e. there exists a constant c, dependent only on constant γ from Equation* (16) *and ellipcity constants, but independent on h, such that the following inequality holds*

$$\|u^* - u_h^*\|_{1,h} \leq c\, h\, |u^*|_{2,\Omega}. \tag{37}$$

5 Error estimate in the L^2 norm

Let us suppose that the bilinear functional $a(\cdot, \cdot)$ is symmetric. Consequently, functionals $a_h(\cdot, \cdot)$ are also symmetric. Performing the same proof steps as in [10] we finally obtain the following estimate in the L^2 norm:

$$\|u^* - u_h^*\|_{0,\Omega} \leq c\, h^2\, |u^*|_{2,\Omega}. \tag{38}$$

It follows that the error in the L^2 norm decreases with h at least quadratically. We can sum up the above findings in Theorem 2.

Theorem 2. *Let the conditions of Theorem 1 hold. Assume additionally that the bilinear functional $a(\cdot, \cdot)$ is symmetric. Then the numerical solutions u_h^* of the Dirichlet boundary value problem on convex domain Ω from Section 2, obtained by finite elements RQ6, using the presented subdividing scheme on nonunisolvent quadrilaterals, converge with $h \to 0$ to the weak solution u^*. Furthermore, the error in the L^2 norm* *decreases at least quadratically with the maximum element diameter h, i.e. there exists a constant c, dependent only on constant γ from Equation* (16) *and ellipcity constants, but independent on h, such that the following inequality holds*

$$\|u^* - u_h^*\|_{0,\Omega} \leq c\,h^2\,|u^*|_{2,\Omega}. \tag{39}$$

6 Numerical results

6.1 Dirichlet boundary value problem

In order to confirm numerically the convergence results of elements RQ6, we have performed numerical tests with different types of meshes. The numerical tests have confirmed the linear convergence in the energy norm and the quadratical convergence in the L^2 norm in all cases provided that the nonunisolvent quadrilaterals have been repaired according to the procedure from Section 3.2.

6.1.1 Definition of boundary value problem

Let the boundary value problem be described by the differential equation and homogeneous boundary conditions [10]

$$(-1)\,(\partial_{20}u + \partial_{02}u) = -\triangle u = f := -2\,(-2 + x^2 + y^2), \quad u|_{\partial\Omega} = 0, \tag{40}$$

on the domain $\Omega := [-1,1] \times [-1,1]$ with solution $u^\star(x,y) \mapsto (x^2-1)\,(y^2-1)$. Alternatively, we can reformulate the strong form of the boundary value problem (40) into its weak form [6]:

$$\int_\Omega \sum_{i=1}^{2} \partial_i v\,\partial_i u\,d\boldsymbol{x} = \int_\Omega f\,v\,d\boldsymbol{x}, \quad v \in H_0^1(\Omega). \tag{41}$$

The solution of the above boundary value problem (41) is termed the weak solution and denoted by $u^* \in H_0^1(\Omega) \cap H^2(\Omega)$. In what follows we study the convergence rate of the weak numerical solutions of BVP introduced in (40) and (41) using three different mesh–generating schemes, here termed A, B and C.

6.1.2 Convex meshing schemes A and B

Scheme A In each step of the mesh formation, we first perform a domain triangulation using the algorithm described in [16] by applying the following parameters [10]:

```
box=[-1,-1;1,1];
fix=[-1,-1;1,-1;1,1;-1,1];
fd=inline('drectangle(p,-1,1,-1,1)','p');
[p,t]=distmesh2d(fd,@huniform,0.4/2^(n-1),box,fix);
```

We perform the triangulations for $\mathtt{n} = 1, \ldots, 4$. Then we divide each triangle into three quadrilaterals using the algorithm given in [17, Method 4.42 on p. 113]. The subsequent refined meshes are constructed by using the bisection dividing schemes. Finally we perform the subdividing only on the nonunisolvent quadrilaterals using the algorithm from Section 3.2 and explained in Figures 2 and 3. Please, see Figure 4 for the details of the mesh refinement for the problem under consideration.

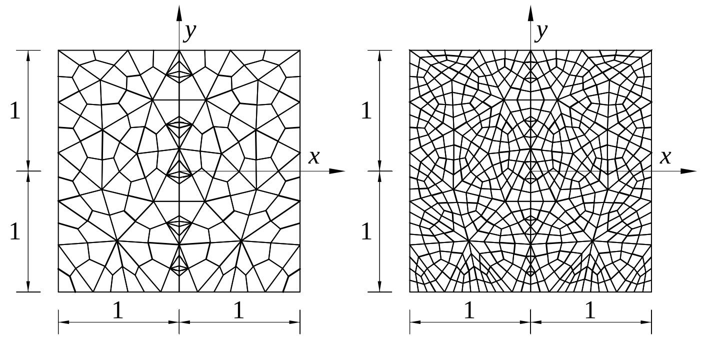

Figure 4: Bisection dividing scheme A of the domain. This scheme yields linear and quadratic convergence, respectively, in the energy and L^2 norms

Scheme B In the initial step of the mesh formation, we perform a domain triangulation using the algorithm described in [16] assuming the same parameters as above. Then we join each pair of triangles with common border into a quadrilateral and perform one step of bisection dividing scheme on the quadrilaterals. Next we divide the remaining triangles into three quadrilaterals, using the algorithm given in [17, Method 4.42 on p. 113]. The subsequent refined meshes are constructed by using the bisection dividing schemes. Finally we perform the subdividing only on the nonunisolvent quadrilaterals using the algorithm from Section 3.2 and explained in Figures 2 and 3. Please, see Figure 5 for the details of the mesh refinement.

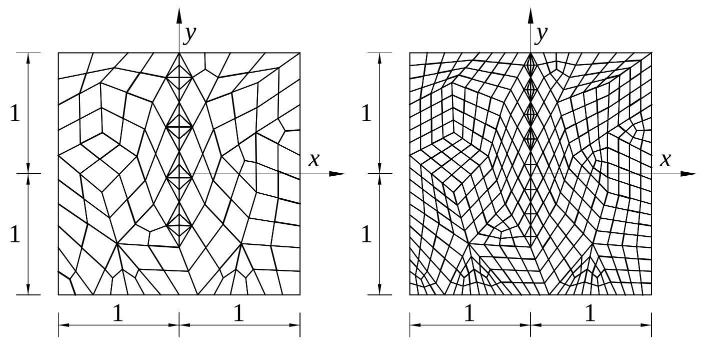

Figure 5: Bisection dividing scheme B of the domain. This scheme yields linear and quadratic convergence, respectively, in the energy and L^2 norms

6.1.3 Nonconvex scheme C

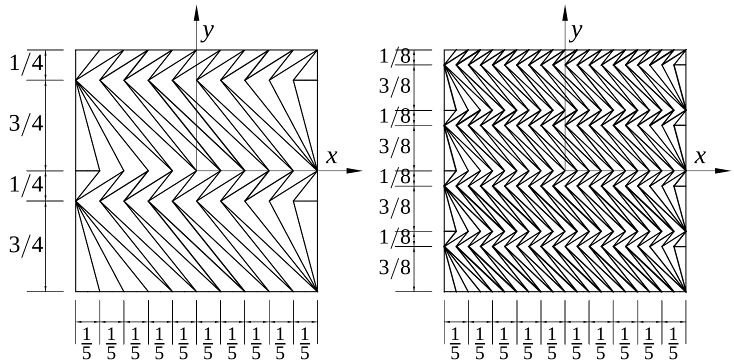

Figure 6: Dividing scheme C of the domain. The resulting initial meshes contain nonuninsolvent quadrilaterals

Scheme C This scheme results in a mesh having many nonconvex quadrilaterals. We construct the sequence of initial meshes, which could consist of many nonunisolvent quadrilaterals. The sequence of two initial meshes is shown in Figure 6. These unisolvent quadrilaterals are further subdivided as discussed in Section 3.2 and displayed in Figure 7.

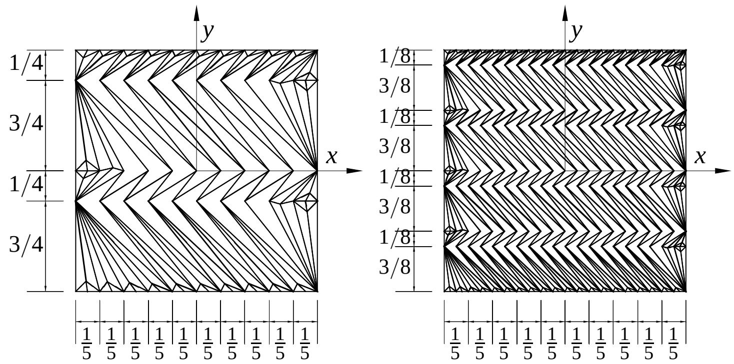

Figure 7: Dividing scheme C of the domain, corrected for nonunisolvent elements. This scheme yields linear and quadratic convergence, respectively, in the energy and L^2 norms

6.1.4 Convergence rates in energy and L^2 norms

Schemes A, B and C The characteristic convergence related results are presented in Figure 8. These results show that convergence of element RQ6 for the convex domain problem at hand is indeed linear in the energy norm and quadratic in the L^2 norm, exactly as predicted by Theorems 1 and 2.

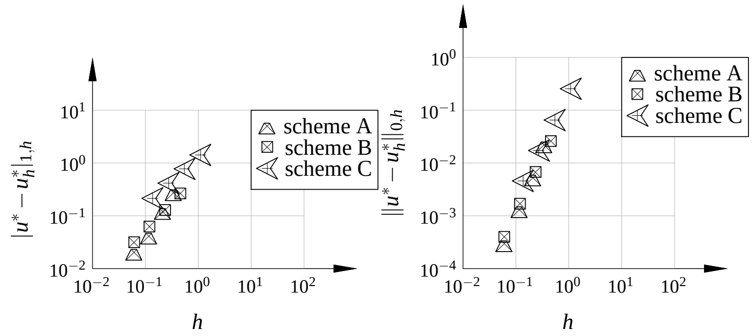

Figure 8: The error in the energy and L^2 norms for element RQ6 using properly corrected dividing schemes A, B and C

7 Conclusions

The convergence of both the convex and the nonconvex quadrilateral nonconforming finite element RQ6 in solving the Dirichlet BVP of a second order elliptic equation on convex domain was proved. The singularity of the matrix A was resolved by the specially designed quadriangulation of the nonunisolvent quadrilaterals. Assuming the standard geometric conditions and employing the quadriangulation of nonunisolvent quadrilaterals using the procedure described in Section 3.2, the linear convergence in the energy and the quadratic convergence in the L^2 norms were derived. The numerical results confirm the theoretically derived convergence rate. They also show that convergence of results depends somewhat on the meshing employed such that the results obtained by dividing schemes using only convex quadrilaterals are more accurate compared to the results with nonconvex quadrilaterals.

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