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Nonlinear Flexural-Torsional Dynamic Analysis of Beams of Variable Cross Section using the Boundary Element Method: Application to the Analysis of Wind Turbine Towers

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Abstract

A boundary element method is developed for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary doubly symmetric variable cross section, undergoing moderate large deflections and twisting rotations under general boundary conditions, taking into account the effects of rotary and warping inertia.

Keywords: beams of variable cross section, nonlinear analysis, dynamic analysis, flexural-torsional analysis, wind turbine towers, boundary element method.

1 Introduction

In engineering practice the dynamic analysis of beam-like continuous systems is frequently encountered. This analysis becomes more complicated in cases of beams of variable cross section which are used in many structural applications in an attempt to achieve a better distribution of rigidity and weight as well as to satisfy architectural and functional requirements. Such structures often undergo arbitrary external dynamic loading, leading to the formulation of the flexural-torsional vibration problem. Moreover, since requirement of weight saving is a major aspect in the design of structures, thin-walled elements of low flexural and/or torsional stiffness are extensively used. Treating displacements and angles of rotation of these elements as being small, leads in many cases to inadequate prediction of the dynamic response; hence the occurring nonlinear effects should be taken into account.

The objective of this paper is to present a general formulation for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary variable simply or multiply connected cross section, undergoing moderate large deflections and twisting rotations [1] under general boundary conditions, taking into account the effect of rotary and warping inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading in both directions as well as to twisting and/or axial loading. Four boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to the angle of twist and solved using the analog equation method (AEM) [2], a BEM based method. Application of the boundary element technique yields a system of nonlinear coupled differential-algebraic equations (DAE) of motion, which is solved iteratively using the Petzold-Gear backward differentiation formula (BDF) [3], a linear multistep method for differential equations coupled to algebraic equations. The torsional warping function and the geometric constants of the cross section are evaluated employing a pure BEM approach, i.e. only boundary discretization of the cross section is used. The essential features and novel aspects of the present formulation are summarized as follows:

i. The cross section is an arbitrarily shaped thin- or thick-walled doubly symmetric one. The formulation does not stand on the assumption of a thin-walled structure and therefore the cross section's torsional and warping rigidities are evaluated "exactly" in a numerical sense.
ii. The beam is subjected to arbitrarily distributed or concentrated transverse loads and bending moments in both directions, twisting and warping moments as well as axial loading.
iii. The beam is supported by the most general boundary conditions including elastic support or restraint.
iv. The effects of rotary and warping inertia are taken into account on the nonlinear flexural-torsional dynamic analysis of beams of variable cross section subjected to arbitrary loading and boundary conditions.
v. The proposed model takes into account all the coupling effects of bending, axial and torsional response of the beam as well as the shortening effect.
vi. The proposed method employs a BEM approach (requiring boundary discretization for the cross sectional analysis) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Subsequently, the developed model is applied to the nonlinear dynamic analysis of the tower of wind turbine structures which is usually formed as a truncated conical steel shell of variable thickness (tubular variable cross section). Analysis of such structures with beam elements is a convenient and effective method which is widely employed due to its simplicity combined with ease of result interpretation. Nevertheless because of the specific nature of the aforementioned structures, the analysis must be rigorous and inaccuracies arising from simplified analyses or simplifications in the formulation of the employed elements should be avoided. Load cases as wind loading or seismic excitations [4,5] necessitate a rigorous dynamic analysis, while an accurate evaluation of the dynamic characteristics of the tower is mandatory in order to avoid undesirable resonance phenomena between the tower and the mechanical part [4,5]. The new trend in design of wind turbine units includes the construction of significantly higher towers so as to increase the amount of the exploited energy. However in such cases, nonlinear behavior may arise due to high slenderness of the tower combined with the increase of axial loading due to weight of the larger mechanical parts, which affects the analysis. Through application of the proposed model useful conclusions can be drawn concerning the above described aspects.

2 Statement of the problem

Let us consider a beam of length l (Figure 1), of arbitrary doubly symmetric variable cross section of area $A = A(x)$. The homogeneous isotropic and linearly elastic material of the beam's cross section, with modulus of elasticity E, shear modulus G and Poisson's ratio ν occupies the two dimensional multiply connected region Ω of the y,z plane and is bounded by the Γ_j($j = 1,2,...,K$) boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Figure 1 $Cxyz$ is the principal bending coordinate system through the cross section's centroid C. The beam is subjected to the combined action of the arbitrarily distributed or concentrated time dependent axial loading $p_x = p_x(x,t)$, twisting moment $m_x = m_x(x,t)$ and warping moment $m_w = m_w(x,t)$ along x direction, transverse loading $p_y = p_y(x,t)$, $p_z = p_z(x,t)$ along the y and z directions, applied at distances y_{p_y}, z_{p_y} and y_{p_z}, z_{p_z}, with respect to the Cyz centroid system of axes (Figure 1b), respectively, as well as bending moments $m_z = m_z(x,t)$ and $m_y = m_y(x,t)$ acting along the y and z directions, respectively .

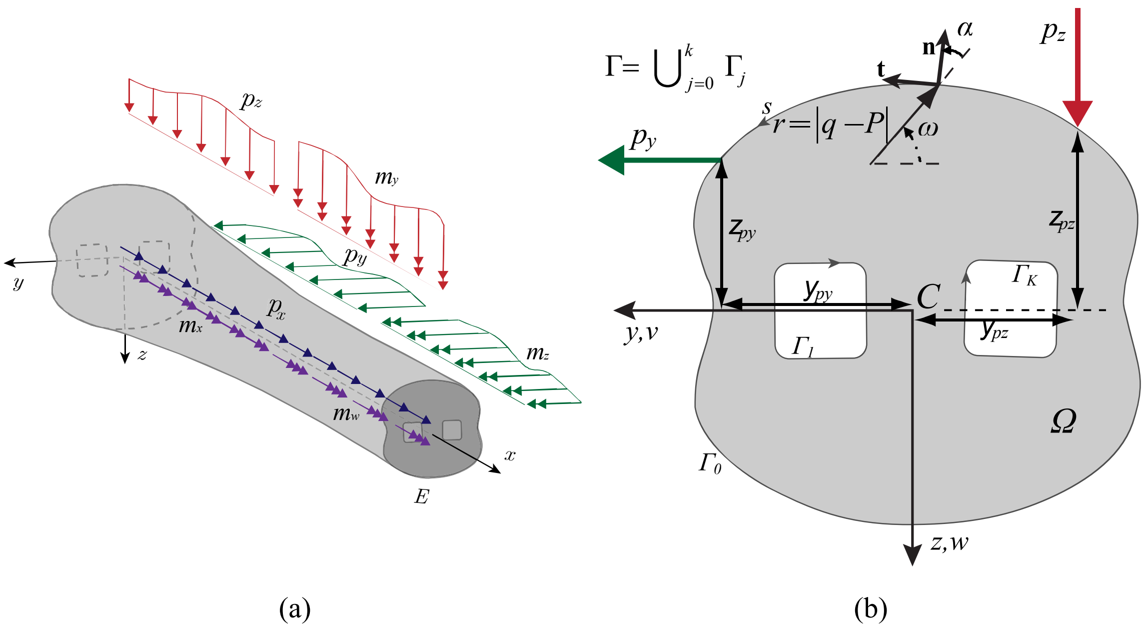

Figure 1: Beam of variable cross section (a) occupying the two-dimensional region Ω (b).

Under the action of the aforementioned loading, the displacement field of an arbitrary point of the cross section can be derived with respect to those of the centroid as [1]

$$\bar{u}(x,y,z,t) = u(x,t) - y\theta_z(x,t) + z\theta_y(x,t) + \theta'_x(x,t)\varphi_S^P \tag{1a}$$

$$\bar{v}(x,y,z,t) = v(x,t) - z\,sin\big(\theta_x(x,t)\big) - y\big[1 - cos\big(\theta_x(x,t)\big)\big] \tag{1b}$$

$$\bar{w}(x,y,z,t) = w(x,t) + y\,sin\big(\theta_x(x,t)\big) - z\big[1 - cos\big(\theta_x(x,t)\big)\big] \tag{1c}$$

$$\theta_y(x,t) = v'(x,t)\,sin\big(\theta_x(x,t)\big) - w'(x,t)\,cos\big(\theta_x(x,t)\big) \tag{1d}$$

$$\theta_z(x,t) = v'(x,t)\,cos\big(\theta_x(x,t)\big) + w'(x,t)\,sin\big(\theta_x(x,t)\big) \tag{1e}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the $Cxyz$ centroid system of axes; $u(x,t) = \frac{1}{A}\int_A \bar{u}(x,y,z,t)\,dA$ denotes the average axial displacement of the cross section [6] and $v = v(x,t)$, $w = w(x,t)$ are the corresponding components of the centroid C; $\theta_z(x,t)$, $\theta_y(x,t)$ are the angles of rotation of the cross section due to bending, with respect to its centroid; $\theta'_x(x,t)$ denotes the rate of change of the angle of twist $\theta_x(x,t)$ regarded as the torsional curvature and $\varphi_S^P = \varphi_S^P(x,y,z)$ is the primary warping function with respect to the shear centre (coinciding with its centroid) [7].

Employing the strain-displacement relations of the three-dimensional elasticity for moderate displacements the strain components can be written as

$$\varepsilon_{xx} = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \bar{v}}{\partial x}\right)^2 + \left(\frac{\partial \bar{w}}{\partial x}\right)^2\right] \tag{2a}$$

$$\gamma_{xy} = \frac{\partial \bar{v}}{\partial x} + \frac{\partial \bar{u}}{\partial y} + \left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial y}\right) \tag{2b}$$

$$\gamma_{xz} = \frac{\partial \bar{w}}{\partial x} + \frac{\partial \bar{u}}{\partial z} + \left(\frac{\partial \bar{v}}{\partial x}\frac{\partial \bar{v}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\frac{\partial \bar{w}}{\partial z}\right) \tag{2c}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = \gamma_{yz} = 0 \tag{2d}$$

where it has been assumed that for moderate displacements $(\partial\bar{u}/\partial x)^2 \ll \partial\bar{u}/\partial x$, $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial z) \ll \partial\bar{u}/\partial x + \partial\bar{u}/\partial z$, $(\partial\bar{u}/\partial x)(\partial\bar{u}/\partial y) \ll \partial\bar{u}/\partial x + \partial\bar{u}/\partial y$. Substituting the displacement components (1) to the strain-displacement relations (2), the strain components can be written as

$$\varepsilon_{xx} = u' + z\left(v''\,sin\,\theta_x - w''\,cos\,\theta_x\right) - y\left(v''\,cos\,\theta_x + w''\,sin\,\theta_x\right) + \theta''_x\varphi_S^P +$$

$$+\frac{1}{2}\left[v'^2+w'^2+\left(y^2+z^2\right)\left(\theta_x'\right)^2\right] \tag{3a}$$

$$\gamma_{xy}=2\varepsilon_{xy}=\left(\frac{\partial\varphi_S^P}{\partial y}-z\right)\theta_x' \tag{3b}$$

$$\gamma_{xz}=2\varepsilon_{xz}=\left(\frac{\partial\varphi_S^P}{\partial z}+y\right)\theta_x' \tag{3c}$$

Considering strains to be small and employing the second Piola-Kirchhoff stress tensor, the non vanishing stress components are defined in terms of the strain ones as

$$\begin{Bmatrix} S_{xx} \\ S_{xy} \\ S_{xz} \end{Bmatrix}=\begin{bmatrix} E^* & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \gamma_{xy} \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where E^* is obtained from Hooke's stress-strain law as $E^*=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$. If the assumption of plane stress condition is made, the above expression is reduced in $E^*=\frac{E}{1-\nu^2}$ [8], while in beam formulations, E is frequently considered instead of E^* ($E^*\approx E$) [8,9]. This last consideration has been followed throughout the paper, while any other reasonable expression of E^* could also be used without any difficulty. Substituting Equations (3) into Equations (4), the stress components are obtained as

$$S_{xx}=E\left\{u'+z\left(v''\sin\theta_x-w''\cos\theta_x\right)-y\left(v''\cos\theta_x+w''\sin\theta_x\right)+\theta_x''\varphi_S^P+\right.$$

$$\left.+\frac{1}{2}\left[v'^2+w'^2+\left(y^2+z^2\right)\left(\theta_x'\right)^2\right]\right\} \tag{5a}$$

$$S_{xy}=G\theta_x'\left(\frac{\partial\varphi_S^P}{\partial y}-z\right) \tag{5b}$$

$$S_{xz}=G\theta_x'\left(\frac{\partial\varphi_S^P}{\partial z}+y\right) \tag{5c}$$

In order to establish the nonlinear equations of motion, the principle of virtual work

$$\delta W_{int}+\delta W_{mass}=\delta W_{ext} \tag{6}$$

where

$$\delta W_{int} = \int_V \left(S_{xx}\delta\varepsilon_{xx} + S_{xy}\delta\gamma_{xy} + S_{xz}\delta\gamma_{xz} \right) dV \tag{7a}$$

$$\delta W_{mass} = \int_V \rho \left(\ddot{\bar{u}}\delta\bar{u} + \ddot{\bar{v}}\delta\bar{v} + \ddot{\bar{w}}\delta\bar{w} \right) dV \tag{7b}$$

$$\delta W_{ext} = \int_L \left(p_x\delta u + p_y\delta\tilde{v}_{p_y} + p_z\delta\tilde{w}_{p_z} + m_y\delta\theta_y + m_z\delta\theta_z + m_x\delta\theta_x + m_w\delta\theta'_x \right) dx +$$

$$+ \left[N_b\delta u + R_{by}\delta v + R_{bz}\delta w + M_{bz}\delta\theta_z + M_{by}\delta\theta_y + M_{bt}\delta\theta_x + M_{bw}\delta\theta'_x \right]_0^l \tag{7c}$$

under a Total Lagrangian formulation, is employed. In the above equations, $\delta(\cdot)$ denotes virtual quantities, $(\dot{\ })$ denotes differentiation with respect to time, V is the volume of the beam in the undeformed configuration, $\tilde{v}_{p_y}$, $\tilde{v}_{p_z}$ are the transverse displacements of the points where the loads p_y, p_z, respectively, are applied and $N_b, R_{by}, R_{bz}, M_{bz}, M_{by}, M_{bt}, M_{bw}$ are the externally applied forces and moments at the beam ends. It is worth here noting that the aforementioned expression of the external work (Equation (7c)) takes into account the change of the eccentricity of the external transverse loading, arising from the cross section torsional rotation, inducing additional (positive or negative) torsional moment. This effect may influence substantially the torsional response of the beam. Moreover, the stress resultants of the beam can be defined as

$$N = \int_\Omega S_{xx} d\Omega \tag{9a}$$

$$M_y = \int_\Omega S_{xx} z d\Omega \tag{9b}$$

$$M_z = -\int_\Omega S_{xx} y d\Omega \tag{9c}$$

$$M_t^P = \int_\Omega \left[S_{xy} \left(\frac{\partial\varphi_S^P}{\partial y} - z \right) + S_{xz} \left(\frac{\partial\varphi_S^P}{\partial z} + y \right) \right] d\Omega \tag{9d}$$

$$M_w = -\int_\Omega S_{xx}\varphi_S^P d\Omega \tag{9f}$$

$$M_R = \int_\Omega S_{xx} \left(y^2 + z^2 \right) d\Omega \tag{9g}$$

where M_t^P is the primary twisting moment [7] resulting from the primary shear stress distribution S_{xy}, S_{xz}, M_w is the warping moment due to torsional curvature and M_R is a higher order stress resultant [1]. Substituting the expressions of the stress components (5) into Equations (9), the stress resultants are obtained as

$$N = EA \left[u' + \frac{1}{2} \left(v'^2 + w'^2 + \frac{I_P}{A}\theta_x'^2 \right) \right] \tag{10a}$$

$$M_y = -EI_y\left(w''\cos\theta_x - v''\sin\theta_x\right) \tag{10b}$$

$$M_z = EI_z\left(v''\cos\theta_x + w''\sin\theta_x\right) \tag{10c}$$

$$M_t^P = GI_t\theta_x' \tag{10d}$$

$$M_w = -EC_S\theta_x'' \tag{10e}$$

$$M_R = N\frac{I_p}{A} + \frac{1}{2}E\left(I_R - \frac{I_p^2}{A}\right)\theta_x'^2 \tag{10f}$$

where the area A, the principal moments of inertia I_y, I_z, the polar moment of inertia I_p with respect to the cross section's centroid, the torsion constant I_t, the warping constant C_S and the fourth moment of inertia I_R, with respect to the shear centre (coinciding with the centroid), are given as

$$A = A(x) = \int_\Omega d\Omega \tag{11a}$$

$$I_y = I_y(x) = \int_\Omega z^2 d\Omega \tag{11b}$$

$$I_z = I_z(x) = \int_\Omega y^2 d\Omega \tag{11c}$$

$$I_p = I_p(x) = \int_\Omega \left(y^2 + z^2\right) d\Omega \tag{11d}$$

$$I_R = I_R(x) = \int_\Omega \left(y^2 + z^2\right)^2 d\Omega \tag{11e}$$

$$I_t = I_t(x) = \int_\Omega \left(y^2 + z^2 + y\frac{\partial\varphi_S^P}{\partial z} - z\frac{\partial\varphi_S^P}{\partial y}\right) d\Omega \tag{11f}$$

$$C_S = C_S(x) = \int_\Omega \left(\varphi_S^P\right)^2 d\Omega \tag{11g}$$

Using the expressions of strain obtained in equations (3), the definitions of the stress resultants given in Equations (9) and applying the principle of virtual work (Equation (6)), the equations of motion of the beam can be derived as

$$-N' + \rho A\ddot{u} = p_x \tag{12a}$$

$$\begin{aligned}
&-\left(Nv'\right)' + \left(M_z\cos\theta_x + M_y\sin\theta_x\right)'' + \\
&\quad +\rho A\ddot{v} - \rho I_z\ddot{v}'' - \rho I_z'\ddot{v}' + \rho I_z\left[-w''\ddot{\theta}_x - w'\ddot{\theta}_x' - 2\dot{w}'\dot{\theta}_x' + \left(v''\dot{\theta}_x + 2v'\dot{\theta}_x' - 2\dot{w}''\right)\dot{\theta}_x\right] - \\
&\quad -\rho I_z'\left[w'\ddot{\theta}_x + \left(2\dot{w}' - v'\dot{\theta}_x\right)\dot{\theta}_x\right] + \rho\left(I_y - I_z\right)\left[\ddot{w}'\theta_x' - \left(v''\theta_x + v'\theta_x'\right)\ddot{\theta}_x - \right. \\
&\quad \left. -2\left(\dot{v}'\theta_x' + \dot{v}''\theta_x\right)\dot{\theta}_x + \left(\ddot{w}'' - \ddot{v}''\theta_x - \ddot{\theta}_x'v' - 2\ddot{v}'\theta_x' - 2\dot{v}'\dot{\theta}_x'\right)\theta_x\right] - \rho\left(I_y' - I_z'\right)\cdot
\end{aligned}$$

$$\cdot\left(\ddot{v}'\theta_x+\ddot{\theta}_x v'+2\dot{v}'\dot{\theta}_x-\ddot{w}'\right)\theta_x=p_y-\left(1-\frac{1}{2}\theta_x^2\right)m_z'-\left(\theta_x-\frac{1}{6}\theta_x^3\right)m_y'+$$

$$+\theta_x\theta_x'm_z-\left(\theta_x'-\frac{1}{2}\theta_x^2\theta_x'\right)m_y \tag{12b}$$

$$-\left(Nw'\right)'+\left(M_z\sin\theta_x-M_y\cos\theta_x\right)''+$$

$$+\rho A\,\ddot{w}-\rho I_y\ddot{w}''-\rho I_y'\ddot{w}'+\rho I_y\left[v''\ddot{\theta}_x+v'\ddot{\theta}_x'+2\dot{v}'\dot{\theta}_x'+\left(w''\dot{\theta}_x+2w'\dot{\theta}_x'+2\dot{v}''\right)\dot{\theta}_x\right]+$$

$$+\rho I_y'\left[v'\ddot{\theta}_x+\left(w'\dot{\theta}_x+2\dot{v}'\right)\dot{\theta}_x\right]+\rho\left(I_y-I_z\right)\left[\ddot{v}'\theta_x'+\left(w''\theta_x+w'\theta_x'\right)\ddot{\theta}_x+\right.$$

$$\left.+2\left(\dot{w}'\theta_x'+\theta_x\dot{w}''\right)\dot{\theta}_x+\left(\ddot{v}''+\ddot{w}''\theta_x+\ddot{\theta}_x'w'+2\ddot{w}'\theta_x'+2\dot{w}'\dot{\theta}_x'\right)\theta_x\right]+\rho\left(I_y'-I_z'\right)\cdot$$

$$\cdot\left(\ddot{w}'\theta_x+\ddot{\theta}_x w'+2\dot{w}'\dot{\theta}_x+\ddot{v}'\right)\theta_x=p_z+\left(1-\frac{1}{2}\theta_x^2\right)m_y'-\left(\theta_x-\frac{1}{6}\theta_x^3\right)m_z'-$$

$$-\theta_x\theta_x'm_y-\left(\theta_x'-\frac{1}{2}\theta_x^2\theta_x'\right)m_z \tag{12c}$$

$$M_y\left(w''\sin\theta_x+v''\cos\theta_x\right)+M_y\left(w''\cos\theta_x-v''\sin\theta_x\right)-M_t^{P\prime}-M_w''-\left(M_R\theta_x'\right)'+$$

$$+\rho I_p\ddot{\theta}_x-\rho C_S\ddot{\theta}_x''-\rho C_S'\ddot{\theta}_x'+\rho I_y\left(v'^2\ddot{\theta}_x+2v'\dot{v}'\dot{\theta}_x-v'\ddot{w}'\right)+$$

$$+\rho I_z\left(w'^2\ddot{\theta}_x+2w'\dot{w}'\dot{\theta}_x+w'\ddot{v}'\right)+\rho\left(I_y-I_z\right)\left(v'\ddot{v}'-w'\ddot{w}'\right)\theta_x=$$

$$=m_x+p_z y_{p_z}\cos\theta_x-p_y z_{p_y}\cos\theta_x-p_z z_{p_z}\sin\theta_x-p_y y_{p_y}\sin\theta_x+$$

$$+m_w'+\left[v'\left(1-\frac{1}{2}\theta_x^{\,2}\right)+w'\theta_x\right]m_y+\left[w'\left(1-\frac{1}{2}\theta_x^{\,2}\right)-v'\theta_x\right]m_z \tag{12d}$$

where the expressions of the stress resultants are given from equations (10). Equations (12) are coupled and highly complicated and can be simplified if the approximate expressions [1]

$$\cos\theta_x\approx 1-\frac{\theta_x^{\,2}}{2!}=1-\frac{\theta_x^{\,2}}{2} \tag{13a}$$

$$\sin\theta_x\approx\theta_x-\frac{\theta_x^{\,3}}{3!}=\theta_x-\frac{\theta_x^{\,3}}{6} \tag{13b}$$

are employed. Thus, using the aforementioned approximations, employing the expressions of the stress resultants (Equations (10)) and ignoring nonlinear terms of the fourth or greater order [1], the governing partial differential equations of motion for the beam in terms of the kinematical components can be written as

$$-E\left(Au''+A'u'\right)-E\left[A\left(v'v''+w'w''\right)+\frac{1}{2}A'\left(v'^2+w'^2\right)+\left(I_p\theta_x''+\frac{1}{2}I_p'\theta_x'\right)\theta_x'\right]+$$
$$+\rho A\ddot{u}=p_x \tag{14a}$$

$$E\left(I_z v'''' + 2I_z' v''' + I_z'' v''\right) - Nv'' - N'v' +$$
$$+E\left(I_y-I_z\right)\left[\left(v''''\theta_x - w'''' + 4v'''\theta_x' + 2v''\theta_x''\right)\theta_x + 2\left(v''\theta_x' - w'''\right)\theta_x' - w''\theta_x''\right]+$$
$$+E\left(I_y'-I_z'\right)\left[2\left(v'''\theta_x - w''' + 2v''\theta_x'\right)\theta_x - 2w''\theta_x'\right] + E\left(I_y''-I_z''\right)\cdot$$
$$\cdot\left(v''\theta_x - w''\right)\theta_x + \rho A\ddot{v} - \rho I_z\ddot{v}'' - \rho I_z'\ddot{v}' + \rho I_z\left[-w''\ddot{\theta}_x - w'\ddot{\theta}_x' - 2\dot{w}'\dot{\theta}_x' +\right.$$
$$\left.+\left(v''\dot{\theta}_x + 2v'\dot{\theta}_x' - 2\dot{w}''\right)\dot{\theta}_x\right] - \rho I_z'\left[w'\ddot{\theta}_x + \left(2\dot{w}' - v'\dot{\theta}_x\right)\dot{\theta}_x\right] + \rho\left(I_y-I_z\right)\left[\ddot{w}'\theta_x' -\right.$$
$$\left.-\left(v''\theta_x + v'\theta_x'\right)\ddot{\theta}_x - 2\left(\dot{v}'\theta_x' + \dot{v}''\theta_x\right)\dot{\theta}_x + \left(\ddot{w}'' - \ddot{v}''\theta_x - \ddot{\theta}_x' v' - 2\ddot{v}'\theta_x' - 2\dot{v}'\dot{\theta}_x'\right)\theta_x\right] -$$
$$-\rho\left(I_y'-I_z'\right)\left(\ddot{v}'\theta_x + \ddot{\theta}_x v' + 2\dot{v}'\dot{\theta}_x - \ddot{w}'\right)\theta_x = p_y - \left(1-\frac{1}{2}\theta_x^2\right)m_z' -$$
$$-\left(\theta_x - \frac{1}{6}\theta_x^3\right)m_y' + \theta_x\theta_x' m_z - \left(\theta_x' - \frac{1}{2}\theta_x^2\theta_x'\right)m_y \tag{14b}$$

$$E\left(I_y w'''' + 2I_y' w''' + I_y'' w''\right) - Nw'' - N'w' -$$
$$-E\left(I_y-I_z\right)\left[\left(w''''\theta_x + v'''' + 4w'''\theta_x' + 2w''\theta_x''\right)\theta_x + 2\left(w''\theta_x' + v'''\right)\theta_x' + v''\theta_x''\right]-$$
$$-E\left(I_y'-I_z'\right)\left[2\left(w'''\theta_x + v''' + 2w''\theta_x'\right)\theta_x + 2v''\theta_x'\right] - E\left(I_y''-I_z''\right)\cdot$$
$$\cdot\left(w''\theta_x + v''\right)\theta_x + \rho A\,\ddot{w} - \rho I_y\ddot{w}'' - \rho I_y'\ddot{w}' + \rho I_y\left[v''\ddot{\theta}_x + v'\ddot{\theta}_x' + 2\dot{v}'\dot{\theta}_x' +\right.$$
$$\left.+\left(w''\dot{\theta}_x + 2w'\dot{\theta}_x' + 2\dot{v}''\right)\dot{\theta}_x\right] + \rho I_y'\left[v'\ddot{\theta}_x + \left(w'\dot{\theta}_x + 2\dot{v}'\right)\dot{\theta}_x\right] + \rho\left(I_y-I_z\right)\left[\ddot{v}'\theta_x' +\right.$$
$$+\left(w''\theta_x + w'\theta_x'\right)\ddot{\theta}_x + 2\left(\dot{w}'\theta_x' + \theta_x\dot{w}''\right)\dot{\theta}_x + \left(\ddot{v}'' + \ddot{w}''\theta_x + \ddot{\theta}_x' w' + 2\ddot{w}'\theta_x' + 2\dot{w}'\dot{\theta}_x'\right)\cdot$$
$$\left.\cdot\theta_x\right] + \rho\left(I_y'-I_z'\right)\left(\ddot{w}'\theta_x + \ddot{\theta}_x w' + 2\dot{w}'\dot{\theta}_x + \ddot{v}'\right)\theta_x = p_z + \left(1-\frac{1}{2}\theta_x^2\right)m_y' -$$
$$-\left(\theta_x - \frac{1}{6}\theta_x^3\right)m_z' - \theta_x\theta_x' m_y - \left(\theta_x' - \frac{1}{2}\theta_x^2\theta_x'\right)m_z \tag{14c}$$

$$E\left(C_S\theta_x'''' + 2C_S'\theta_x''' + C_S''\theta_x''\right) - G\left(I_t\theta_x'' + I_t'\theta_x'\right) - N\left[\frac{I_p}{A}\theta_x'' + \left(\frac{I_p'}{A} - \frac{I_pA'}{A^2}\right)\theta_x'\right] - N'\frac{I_p}{A}\theta_x' +$$
$$-\frac{3}{2}E\left(I_R - \frac{I_p^2}{A}\right)\theta_x''\theta_x'^2 + E\left(\frac{2I_p'I_p}{A} - \frac{A'I_p^2}{A^2} - I_R'\right)\theta_x'^3 + E\left(I_y-I_z\right)\cdot$$
$$\cdot\left[\left(v''^2 - w''^2\right)\theta_x - w''v''\right] + \rho I_p\ddot{\theta}_x - \rho C_S\ddot{\theta}_x'' - \rho C_S'\ddot{\theta}_x' + \rho I_y\left(v'^2\ddot{\theta}_x +\right.$$

$$+2v'\dot{v}'\dot{\theta}_x - v'\ddot{w}'\Big) + \rho I_z \left(w'^2 \ddot{\theta}_x + 2w'\dot{w}'\dot{\theta}_x + w'\ddot{v}' \right) + \rho \left(I_y - I_z \right) \left(v'\ddot{v}' - w'\ddot{w}' \right) \theta_x =$$

$$= m_x + p_z y_{p_z} - p_y z_{p_y} + \left(\frac{1}{2}\theta_x^2 z_{p_y} + \frac{1}{6}\theta_x^3 y_{p_y} - \theta_x y_{p_y} \right) p_y +$$

$$+ \left(\frac{1}{6}\theta_x^3 z_{p_z} - \frac{1}{2}\theta_x^2 y_{p_z} - z_{p_z}\theta_x \right) p_z + m_w' + \left[v' \left(1 - \frac{1}{2}\theta_x{}^2 \right) + w'\theta_x \right] m_y +$$

$$+ \left[w' \left(1 - \frac{1}{2}\theta_x{}^2 \right) - v'\theta_x \right] m_z \tag{14d}$$

The above governing differential equations (Equations (14)) are also subjected to the initial conditions ($x \in (0,l)$)

$$u(x,0) = u_0(x) \qquad \dot{u}(x,0) = \dot{u}_0(x) \tag{15a,b}$$

$$v(x,0) = v_0(x) \qquad \dot{v}(x,0) = \dot{v}_0(x) \tag{16a,b}$$

$$w(x,0) = w_0(x) \qquad \dot{w}(x,0) = \dot{w}_0(x) \tag{17a,b}$$

$$\theta_x(x,0) = \theta_{x0}(x) \qquad \dot{\theta}_x(x,0) = \dot{\theta}_{x0}(x) \tag{18a,b}$$

together with the corresponding boundary conditions of the problem at hand, which are given as

$$a_1 u(x,t) + \alpha_2 N_b(x,t) = \alpha_3 \tag{19}$$

$$\beta_1 v(x,t) + \beta_2 R_{by}(x,t) = \beta_3 \qquad \bar{\beta}_1 \theta_z(x,t) + \bar{\beta}_2 M_{bz}(x,t) = \bar{\beta}_3 \tag{20a,b}$$

$$\gamma_1 w(x,t) + \gamma_2 R_{bz}(x,t) = \gamma_3 \qquad \bar{\gamma}_1 \theta_y(x,t) + \bar{\gamma}_2 M_{by}(x,t) = \bar{\gamma}_3 \tag{21a,b}$$

$$\delta_1 \theta_x(x,t) + \delta_2 M_{bt}(x,t) = \delta_3 \qquad \bar{\delta}_1 \theta_x{}'(x,t) + \bar{\delta}_2 M_{bw}(x,t) = \bar{\delta}_3 \tag{22a,b}$$

at the beam ends $x = 0,l$, where R_{by}, R_{bz} and M_{bz}, M_{by} are the reactions and bending moments with respect to y, z, respectively, given by the following relations (ignoring again the nonlinear terms of the fourth or greater order)

$$R_{by} = -E\left(I_z v''' + I_z' v'' \right) + Nv' + E\left(I_y - I_z \right) \cdot \left[w''\theta_x' + \left(w''' - v'''\theta_x - 2v''\theta_x' \right)\theta_x \right] -$$

$$-E\left(I_y' - I_z' \right)\left(v''\theta_x - w'' \right)\theta_x - \left(1 - \frac{1}{2}\theta_x^2 \right) m_z - \left(\theta_x - \frac{1}{6}\theta_x^3 \right) m_y \tag{23a}$$

$$R_{bz} = -E\left(I_y w''' + I_y' w'' \right) + Nw' + E\left(I_y - I_z \right)\left[v''\theta_x' + \left(v''' + w'''\theta_x + 2w''\theta_x' \right)\theta_x \right] +$$

$$+E\left(I_y' - I_z' \right)\left(w''\theta_x + v'' \right)\theta_x + \left(1 - \frac{1}{2}\theta_x^2 \right) m_y - \left(\theta_x - \frac{1}{6}\theta_x^3 \right) m_z \tag{23b}$$

$$M_{bz} = EI_z v'' + E\left(I_y - I_z\right)\left(\theta_x v'' - w''\right)\theta_x \tag{23c}$$

$$M_{by} = -EI_y w'' + E\left(I_y - I_z\right)\left(\theta_x w'' + v''\right)\theta_x \tag{23d}$$

while M_{bt} and M_{bw} are the torsional and warping moments at the boundaries of the beam, respectively, given as

$$M_{bt} = -EC_S\theta_x''' - EC_S'\theta_x'' + \left(GI_t + N\frac{I_P}{A}\right)\theta_x' + \frac{1}{2}E\left(I_R - \frac{I_P^2}{A}\right)\theta_x'^3 + m_w \tag{24a}$$

$$M_{bw} = -EC_S\theta_x'' \tag{24b}$$

Finally, α_k, β_k, $\bar{\beta}_k$, γ_k, $\bar{\gamma}_k$, δ_k, $\bar{\delta}_k$ $(k = 1,2,3)$ are time dependent functions specified at the boundaries of the bar ($x = 0,l$). The boundary conditions (19)-(22) are the most general boundary conditions for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1 = \beta_1 = \bar{\beta}_1 = \gamma_1 = \bar{\gamma}_1 = \delta_1 = \bar{\delta}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \delta_2 = \delta_3 = \bar{\beta}_2 = = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\delta}_2 = \bar{\delta}_3 = 0$).

3 Numerical solution

According to the precedent analysis, the nonlinear flexural-torsional dynamic analysis of beams of variable cross section under general boundary conditions and arbitrary loading reduces in establishing the kinematical components $u(x,t)$, $v(x,t)$, $w(x,t)$ and $\theta_x(x,t)$ having continuous partial derivatives up to the second and fourth order with respect to x, respectively and up to the second order with respect to t, satisfying the nonlinear initial boundary value problem described by the coupled governing differential equations of motion (Equations (14)) along the beam, the initial conditions (Equations (15)-(18)) and the boundary conditions (Equations (19)-(22)) at the beam ends $x = 0,l$. Equations (14)-(22) are solved using the analog equation method (AEM) [2], a BEM based method. Application of the boundary element technique yields a system of nonlinear coupled differential-algebraic equations (DAE) of motion, which can be solved iteratively using any efficient direct time integration scheme. In the present study the Petzold-Gear backward differentiation formula (BDF) [3] is employed, which is a linear multistep method for differential equations coupled to algebraic equations. The torsional warping function and the geometric constants of the cross section are evaluated employing a pure BEM approach, i.e. only boundary discretization of the cross section is used.

4 Numerical example

In this example, in order to demonstrate the influence of geometrical nonlinearity on the dynamic response of wind turbine towers, a steel tower of variable tubular cross section ($E = 2.1\times 10^8\ kN/m^2$, $\rho = 8.5\ tn/m^3$, $v = 0.2$, $G = 8.75\times 10^7\ kN/m^2$, $H = 120\ m$) supporting the NREL baseline 5-MW nacelle and rotor (3 blades of length $r_{bl} = 61.5\ m$) [10] is examined (Figure 2). The radius and the thickness of the tubular cross section vary linearly along the tower length according to the dimensions presented in Figure 2b. In order to take into account the inertial forces applied by the mechanical parts (nacelle, rotor and blades), an additional concentrated mass $M_c = 403.22\ tn$ [10] is added at the top of the tower. The tower is considered to be clamped at its base and is subjected to the combined action of the total weight of the structure (tower and mechanical parts) and of the time-dependent drag force $\bar{F}_D(t)$ at its top due to wind. More specifically a distributed axial load along the length of the tower and a concentrated axial force at its top are applied, which are evaluated according to the corresponding masses. It is worth here noting, that this loading is applied as a static one and the axial displacement u_{static} at the static equilibrium state is imposed as an initial condition on the tower ($u_0(x) = u_{static}$, $\dot{u}_0(x) = 0$). Moreover the drag force $\bar{F}_D(t)$ can be obtained by the following relation [10]

$$\bar{F}_D(t) = \frac{1}{2}\rho_{air} C_D A_{bl} \left(\bar{V}(t)\right)^2 \tag{25}$$

where $\rho_{air} = 1.225\times 10^{-3}\ tn/m^3$ is the air density, C_D is the drag coefficient which in this example takes the value 1.2 [10] and $\bar{V}(t)$ is the velocity of the wind which is assumed to have a uniform spatial distribution over the actuator disc. Moreover A_{bl} is the area of structure affected by the wind which is considered as the total area of the wind turbine blades. In this case $A_{bl} = 553.5\ m^2$ having assumed for simplicity that the blades have a constant width $c = 3m$. Breaking $\bar{V}(t)$ down into a mean component V_m and a fluctuating component $V(t)$, the corresponding mean and fluctuating components of $\bar{F}_D(t)$ can be obtained as

$$F_{Dm} = \frac{1}{2}\rho_{air} C_D A_{bl} V_m^2 \tag{26a}$$

$$F_D(t) = \frac{1}{2}\rho_{air} C_D A_{bl} \left(2V_m V(t) + V(t)^2\right) \tag{26b}$$

In this case the mean velocity is taken as $V_m = 20\ m/sec$. Moreover, in order to take into account the wind velocity fluctuation at the altitude of $H = 120\,m$, an artificial velocity time history is generated following the regulations of EC1, Part 1.4 [11] and applying the procedure presented in [12,13]. In Figure 4 the time history of the fluctuating component $V(t)$ is presented. In Table 1 the maximum values of tip displacements $u(H,t)$ and $w(H,t)$ and of reactions $R_{bz}(0,t)$ and $M_{by}(0,t)$ at the base of the tower taking into account or ignoring the influence of rotary inertia, are presented. In Figures 5, 6 the time histories of displacements $u(H,t)$ and $w(H,t)$, respectively, are presented. In Figures 7, 8 the time histories of reactions $R_{bz}(0,t)$ and $M_{by}(0,t)$ at the base of the tower are also shown. From these figures and table the influence of geometrical nonlinearity can be easily verified. More specifically, it can be observed that the axial loading acting on the tower reduces its stiffness and may result in the increase of the displacements and the stress resultants. This behavior is of significant importance as the tendency in the design of wind turbine towers leads to the construction of higher towers supporting larger mechanical parts. Finally, from the presented table, it can be observed that in this case the influence of rotary inertia on the kinematical components is not intense, while it is more pronounced on the stress resultants.

	Linear analysis		Nonlinear analysis	
	Ignoring rotary inertia	Full set of equations	Ignoring rotary inertia	Full set of equations
$u(H)_{max}$	–0.00554	–0.00554	–0.00774	–0.00774
$w(H)_{max}$	0.54919	0.55078	0.61149	0.61229
$M_{by}(0)_{max}$	–135086.951	–137682.309	–148436.804	–149118.321
$R_{bz}(0)_{max}$	2005.993	1688.502	1858.422	1855.295

Table 1: Maximum values of displacements $u(H,t)$ (m), $w(H,t)$ (m) and of Reactions $M_{by}(0,t)$ (kNm), $R_{bz}(0,t)$ (kN) of wind turbine tower.

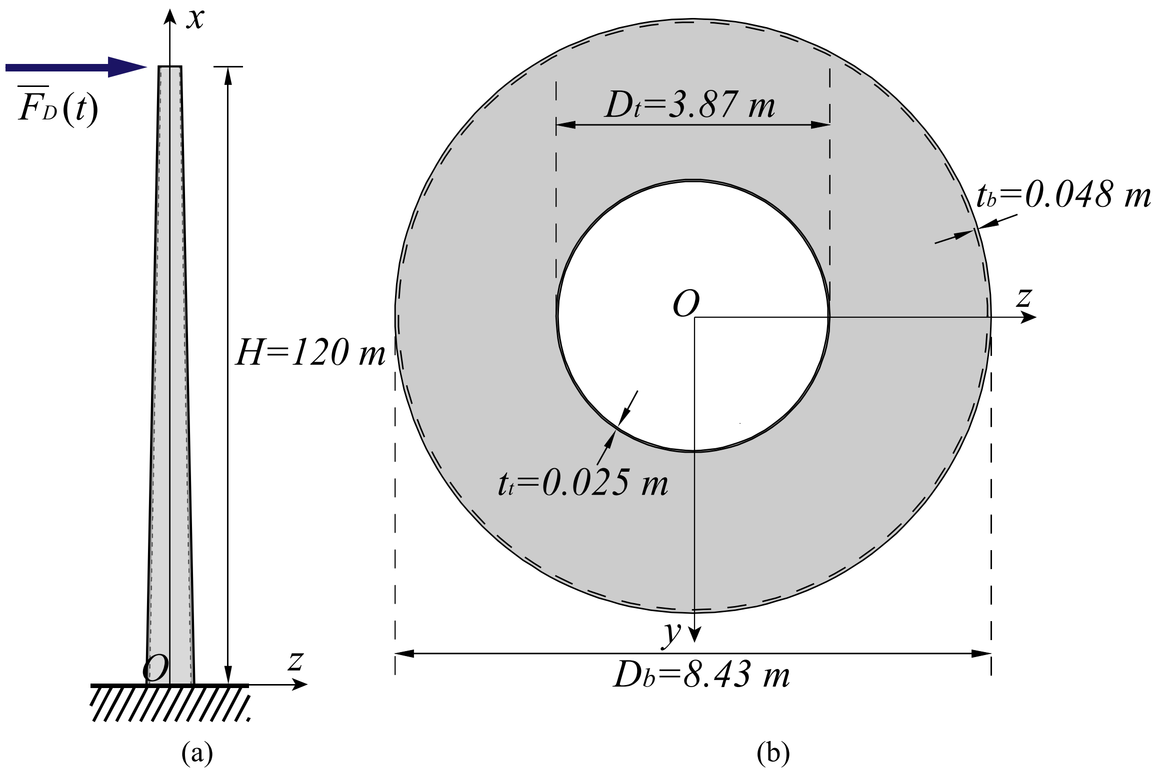

Figure 2: Wind turbine tower (a) of variable tubular cross section (b).

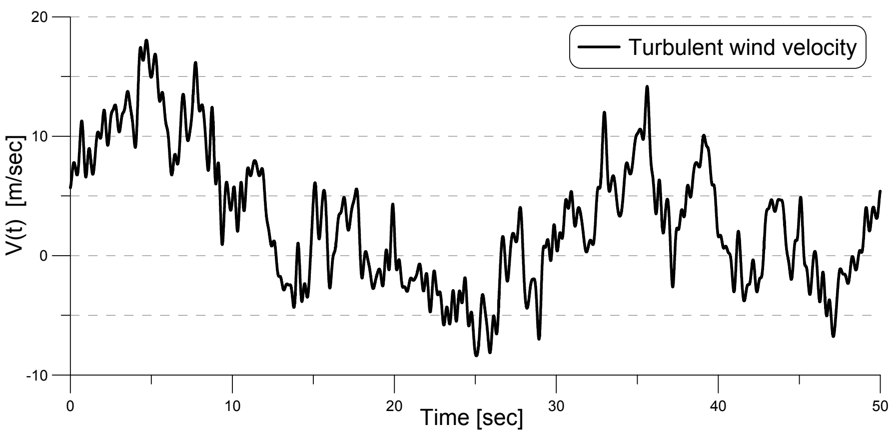

Figure 3: Time history of the fluctuating component $V(t)$ of the total wind velocity $\overline{V}(t)$ ($V_m = 20\ m/sec$).

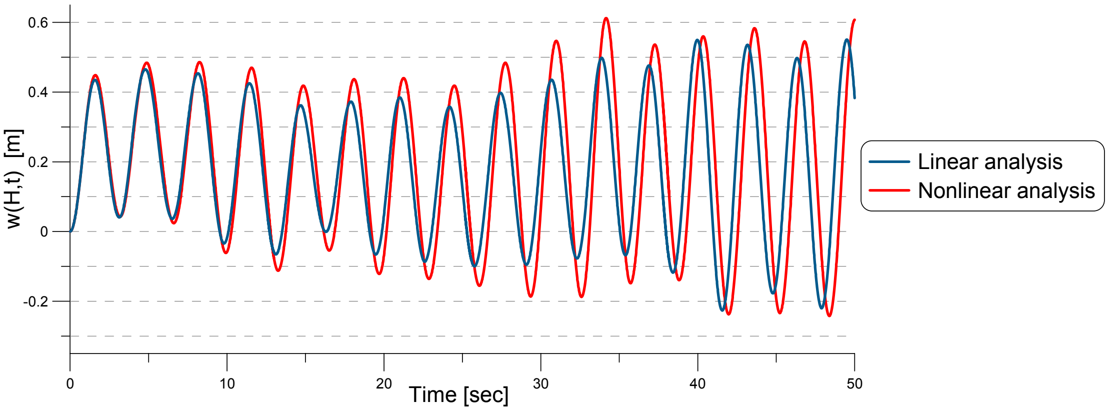

Figure 4: Time history of $w(t)$ at the tip of the wind turbine tower.

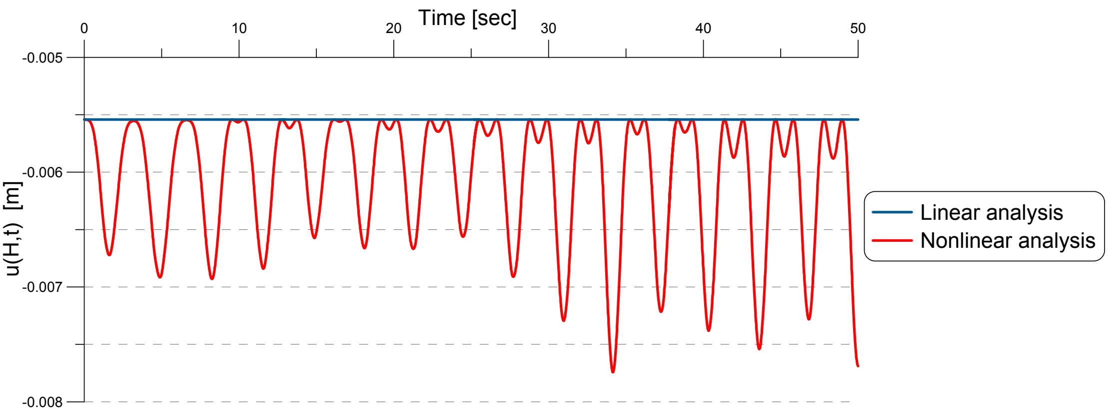

Figure 5: Time history of $u(t)$ at the tip of the wind turbine tower.

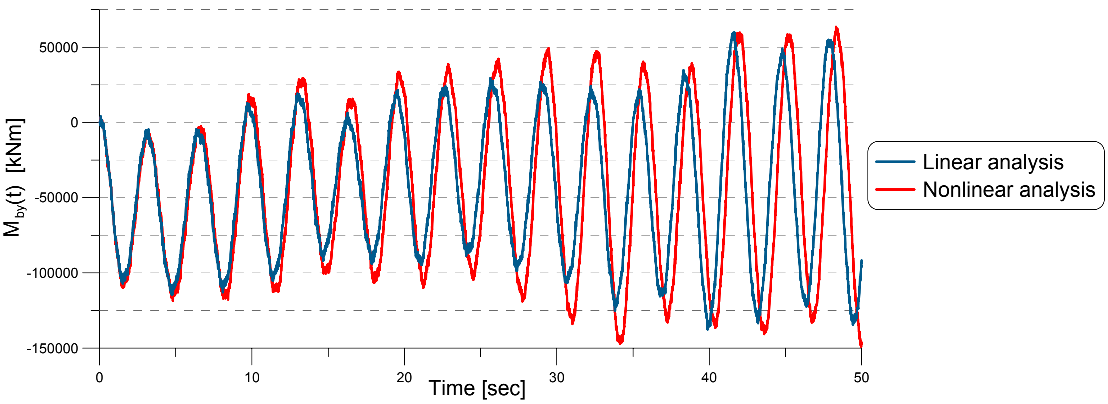

Figure 6: Time history of $M_{by}(t)$ at the base of the wind turbine tower.

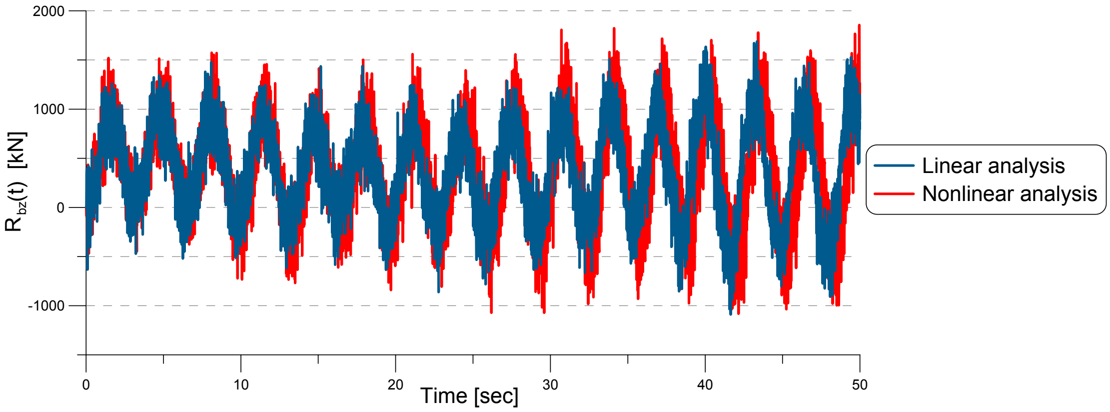

Figure 7: Time history of base shear $R_{bz}(t)$ of the wind turbine tower.

5 Conclusions

The main conclusions that can be drawn from this investigation are:

a. The numerical technique presented in this investigation is well suited for computer aided analysis of beams of arbitrary simply or multiply connected doubly symmetric variable cross section, supported by the most general boundary conditions and subjected to the combined action of arbitrarily distributed or concentrated time dependent loading.
b. The geometrical nonlinearity leads to strong coupling between the axial, torsional and bending equations of motion resulting in different dynamic response of the beam compared to the one obtained by linear analysis.
c. In the treated example, the influence of rotary and warping inertia, on the dynamic response of the tower, proved to be of minor importance.
d. The developed procedure retains most of the advantages of a BEM solution over a finite element approach, although it requires longitudinal domain discretization.

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