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Topology Optimisation of Bodies in Unilateral Contact by Maximizing the Potential Energy

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Abstract

The bottle-neck in topology optimization of non-linear structural problems, such as contact problems, is to solve the state equations and the adjoint equations. By choosing the potential energy as objective the latter equations are not needed in the sensitivity analysis. In this work we perform topology optimization of contact problems including non-zero prescribed displacements by maximizing the potential energy. For contact problems with zero initial contact gaps and zero prescribed displacements this is equivalent to minimizing the compliance, which is the standard approach in topology optimization. However, when the compliance is used as objective in topology optimization of contact problems an extra adjoint equation must be solved. This is not needed in the formulation presented in this work. Thus, the CPU-time is decreased by using the potential energy as objective. This demonstrated by performing topology optimization of several contact problems in both 2D and 3D. The solutions obtained by using the potential energy as objective are also compared to the solutions generated by compliance optimization.

Keywords: potential energy, unilateral contact, prescribed displacements.

1 Introduction

The modeling of the boundary conditions is crucial in topology optimization. Small changes in these conditions will typically imply new optimal layouts of material. In many situations the design domain is connected to an assembly of components via contact interfaces. In order to generate proper layouts for this type of design domains one must treat these contact interfaces accurately in the topology optimization procedure. Such numerical optimization approaches have been developed by the author in a number of recent papers [1, 2, 3]. In all these works, an adjoint equation is utilized in

order to perform the sensitivity analysis of the objective. This linear system of equations is one of the two major time consuming parts in the algorithm. The other part is of course the linear equations appearing in the contact algorithm.

One way of improving the numerical performance of the algorithm would be to get rid of the adjoint equation. This is possible by taking the potential energy as the objective function instead of the compliance. The choice of taking the potential energy instead of using the compliance was recently discussed in Klarbring and Strömberg [4] for linear elastic systems. In this work the potential energy is taken as objective for linear elastic bodies in unilateral contact. It is demonstrated that the CPU-time is reduced as much as 25% by applying this approach compared to the adjoint approach presented previously.

2 Governing equations

Let us consider a system of bodies which are parameterized with the SIMP-model. The design parameters ρ_e are collected in $\boldsymbol{\rho}$. The stiffness matrix of the system is obtained by the following assembly procedure:

$$\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{\rho}) = \bigcap_{e=1}^{n_{\text{el}}} \rho_e^n \boldsymbol{k}_e, \tag{1}$$

where $\boldsymbol{k}_e$ is an element stiffness matrix, n is the SIMP-factor, $\bigcap$ represents an assembly operator and n_{el} is the number of elements. The system of bodies is subjected to external forces $\boldsymbol{F}$, prescribed displacements $\boldsymbol{\delta}$ and unilateral contact conditions.

The prescribed displacements are represented by

$$\boldsymbol{D}\boldsymbol{d} = \boldsymbol{\delta}, \tag{2}$$

where $\boldsymbol{d}$ is the displacement vector and $\boldsymbol{D}$ is a matrix representing orientations of the prescribed displacements.

The unilateral contact conditions are given by

$$\boldsymbol{C}\boldsymbol{d} \leq \boldsymbol{g}, \tag{3}$$

where $\boldsymbol{C}$ contains contact normals and $\boldsymbol{g}$ is a vector of initial contact gaps.

The state of equilibrium of the system defined above is obtained by minimizing the potential energy. The potential energy of the system reads

$$\Pi(\boldsymbol{\rho}, \boldsymbol{d}) = \frac{1}{2}\boldsymbol{d}^T \boldsymbol{K}(\boldsymbol{\rho})\boldsymbol{d} - \boldsymbol{F}^T \boldsymbol{d}. \tag{4}$$

Thus, for a given density distribution $\boldsymbol{\rho} = \hat{\boldsymbol{\rho}}$, the equilibrium state is found by solving

$$\begin{cases} \min\limits_{\boldsymbol{d}} \Pi(\hat{\boldsymbol{\rho}}, \boldsymbol{d}) \\ \text{s.t.} \begin{cases} \boldsymbol{D}\boldsymbol{d} - \boldsymbol{\delta} = \boldsymbol{0} \\ \boldsymbol{C}\boldsymbol{d} - \boldsymbol{g} \leq \boldsymbol{0}. \end{cases} \end{cases} \tag{5}$$

The corresponding KKT-conditions read

$$\boldsymbol{K}(\hat{\boldsymbol{\rho}})\boldsymbol{d} - \boldsymbol{F} + \boldsymbol{D}^T\boldsymbol{\gamma} + \boldsymbol{C}^T\boldsymbol{\lambda} = \boldsymbol{0}, \tag{6a}$$

$$\boldsymbol{D}\boldsymbol{d} - \boldsymbol{\delta} = \boldsymbol{0}, \tag{6b}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0}, \tag{6c}$$

$$\boldsymbol{C}\boldsymbol{d} - \boldsymbol{g} \leq \boldsymbol{0}, \tag{6d}$$

$$\boldsymbol{\lambda} \circ (\boldsymbol{C}\boldsymbol{d} - \boldsymbol{g}) = \boldsymbol{0}. \tag{6e}$$

Here, $\boldsymbol{\gamma}$ is a vector of Lagrange multipliers which can be interpreted as forces that enforce the prescribed displacement. Furthermore, the Lagrange multipliers in $\boldsymbol{\lambda}$ are contact forces that are governed by the three latter constraints known as Signorini's contact conditions.

By solving the KKT-conditions in (6) we can obtain $\boldsymbol{d}$=$\boldsymbol{d}(\boldsymbol{\rho})$, $\boldsymbol{\gamma}$=$\boldsymbol{\rho}(\boldsymbol{d})$ and $\boldsymbol{\lambda}$=$\boldsymbol{\lambda}(\boldsymbol{\rho})$. Explicitly this is done by a Newton method. Details can be found in Strömberg [5].

3 Optimization problem

For the system presented in the previous section, we maximize the potential energy for the nested problem. That is,

$$\begin{cases} \max\limits_{\boldsymbol{\rho}} \Pi(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho})) \\ \text{s.t.} \begin{cases} \sum\limits_{e=1}^{n_{\text{el}}} V_e \rho_e = \hat{V} \\ \boldsymbol{\epsilon} \leq \boldsymbol{\rho} \leq \boldsymbol{1}, \end{cases} \end{cases} \tag{7}$$

where V_e represents the volume of element e for $\rho_e = 1$, $\hat{V}$ is the total amount of material to be distributed, $\boldsymbol{\epsilon}$=$\{\epsilon, \ldots, \epsilon\}^T$ is a vector of small numbers ϵ and $\boldsymbol{1}$=$\{1, \ldots, 1\}^T$.

The objective function in (7) can be interpreted by inserting the KKT-conditions from (6) into $\Pi(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho}))$. This yields

$$\Pi(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho})) = -\frac{1}{2}\boldsymbol{F}^T\boldsymbol{d} - \frac{1}{2}\boldsymbol{\gamma}^T\boldsymbol{\delta} - \frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{g}. \tag{8}$$

Thus, maximizing the potential energy is equivalent to minimizing

$$\boldsymbol{F}^T\boldsymbol{d} + \boldsymbol{\gamma}^T\boldsymbol{\delta} + \boldsymbol{\lambda}^T\boldsymbol{g}.$$

The first term is the definition of the well-known compliance, the second term implies that the reaction force $-\boldsymbol{\gamma}$ is maximized. Finally, the third term implies that the contact force λ_i is minimized for $g_i > 0$ and maximized when $g_i < 0$. Of course, for $\boldsymbol{\delta}$=$\boldsymbol{0}$ and $\boldsymbol{g}$=$\boldsymbol{0}$, the established compliance optimization problem is recovered.

The sensitivity analysis is performed by using the corresponding Lagrangian

$$\mathcal{L}(\boldsymbol{\rho}, \boldsymbol{d}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) = \Pi(\boldsymbol{\rho}, \boldsymbol{d}) + \boldsymbol{\gamma}^T(\boldsymbol{D}\boldsymbol{d} - \boldsymbol{\delta}) + \boldsymbol{\lambda}^T(\boldsymbol{C}\boldsymbol{d} - \boldsymbol{g}). \tag{9}$$

At the state of equilibrium defined by the KKT-conditions in (6) it is clear that the Lagrangian in (9) is equivalent to the potential energy in (4), i.e.

$$\mathcal{L} = \mathcal{L}(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho}), \boldsymbol{\gamma}(\boldsymbol{\rho}), \boldsymbol{\lambda}(\boldsymbol{\rho})) = \Pi(\boldsymbol{\rho}, \boldsymbol{d}(\boldsymbol{\rho})). \tag{10}$$

This is utilized in the sensitivity analysis in following way:

$$\frac{\partial \Pi}{\partial \rho_e} = \frac{\partial \mathcal{L}}{\partial \rho_e} + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{d}}\right)^T \frac{\partial \boldsymbol{d}}{\partial \rho_e} + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}}\right)^T \frac{\partial \boldsymbol{\gamma}}{\partial \rho_e} + \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)^T \frac{\partial \boldsymbol{\lambda}}{\partial \rho_e}. \tag{11}$$

The first term in (11) equals

$$\frac{\partial \mathcal{L}}{\partial \rho_e} = \frac{1}{2} \boldsymbol{d}^T \frac{\boldsymbol{K}}{\partial \rho_e} \boldsymbol{d}, \tag{12}$$

where

$$\frac{\partial \boldsymbol{K}}{\partial \rho_e} = n \rho_e^{n-1} \boldsymbol{k}_e. \tag{13}$$

The remaining terms are all zeros by the KKT-conditons in (6). This is verified below.

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{d}} = \boldsymbol{K}\boldsymbol{d} - \boldsymbol{F} + \boldsymbol{D}^T \boldsymbol{\gamma} + \boldsymbol{C}^T \boldsymbol{\lambda} = \boldsymbol{0}, \tag{14a}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\gamma}} = \boldsymbol{D}\boldsymbol{d} - \boldsymbol{\delta} = \boldsymbol{0}, \tag{14b}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{\lambda}}\right)^T \frac{\partial \boldsymbol{\lambda}}{\partial \rho_e} = (\boldsymbol{C}\boldsymbol{d} - \boldsymbol{g})^T \frac{\partial \boldsymbol{\lambda}}{\partial \rho_e} = 0. \tag{14c}$$

Perhaps, the latter result is not obvious for $\boldsymbol{\lambda} = \boldsymbol{0}$. However, this is true by taking the derivative of the following formulation of the complementary condition in (6e):

$$\frac{\partial}{\partial \rho_e} \left(\boldsymbol{\lambda}^T (\boldsymbol{C}\boldsymbol{d} - \boldsymbol{g}) = 0 \right), \tag{15}$$

which yields

$$\left(\frac{\partial \boldsymbol{\lambda}}{\partial \rho_e}\right)^T (\boldsymbol{C}\boldsymbol{d} - \boldsymbol{g}) + \boldsymbol{\lambda}^T \boldsymbol{C} \frac{\partial \boldsymbol{d}}{\partial \rho_e} = 0. \tag{16}$$

The problem is solved by an optimality criteria approach [6, 7, 8], which is derived by performing the sensitivity analysis in an intervening variable

$$x_e = \rho_e^{-\alpha}, \quad \alpha > 0. \tag{17}$$

In such manner, one obtains the following sub-problem at an iterate $\hat{\boldsymbol{\rho}}$:

$$\begin{cases} \max\limits_{\boldsymbol{\rho}} \sum\limits_{e=1}^{n_{\text{el}}} \xi_e \rho_e^{-\alpha} \\ \text{s.t.} \begin{cases} \sum\limits_{e=1}^{n_{\text{el}}} V_e \rho_e = \hat{V} \\ \rho_e^l \leq \rho_e \leq \rho_e^u \quad (e = 1, \ldots, , n_{\text{el}}), \end{cases} \end{cases} \tag{18}$$

where

$$\xi_e = -\frac{1}{\alpha}\frac{\partial \mathcal{L}}{\partial \rho_e}\hat{\rho}_e^{\,1+\alpha}, \tag{19}$$

and ρ_e^l and ρ_e^u represent move limits. The problem in (18) is a convex separable problem which is easily solved.

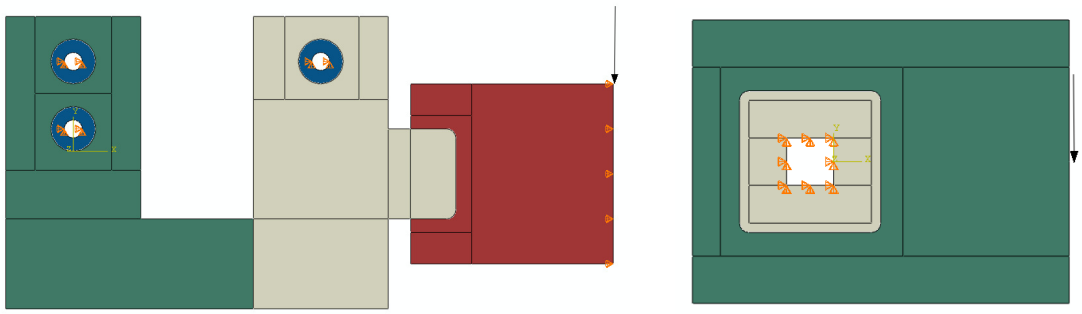

Figure 1: Two problems in two-dimensional settings. The assembly to the left consists of six bodies in unilateral contact.

4 Numerical examples

The efficiency of the approach presented in the previous sections is demonstrated here by studying three different problems, two problems in 2D and one in a three-dimensional setting. The theory presented in the previous sections is implemented by using Matlab and Intel Fortran, where the Fortran code is linked to Matlab as mex-files. The problems are solved using this implementation on a laptop with an Intel Core i7 2.67 GHz processor and a 64 bit version of Windows. The linear equation systems are solved by using the sparse Cholesky solver of Matlab. The implementation can be downloaded as a toolbox (Topo4abq) from *www.fema.se*.

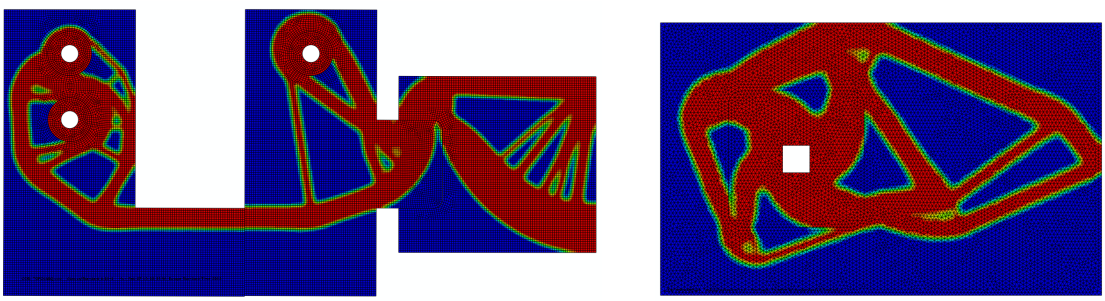

Figure 2: Problem 1 and 2 - Optimal topologies.

The first problem consists of six elastic bodies in unilateral contact, see the left part of Figure 1. The limit on the volume fraction is set to 40%. The number of elements is 25857. The problem is also solved for the compliance problem using an adjoint approach [1, 2, 3]. The optimization is run for 40 iterations. The convergence properties are shown in Figure 3. The convergence in the corresponding compliance is plotted in the left part of the figure. The right part shows the convergence in volume fraction of elements with densities greater than 0.9. Thus, after 40 iterations the volume fraction of element densities greater than 0.9 is 35%. The remaining 5% is explained by the regions of intermediate densities due to the filter. The CPU-times for both approaches are presented in Table 1. Thus, the presented approach is 21% faster than the adjoint approach for this problem.

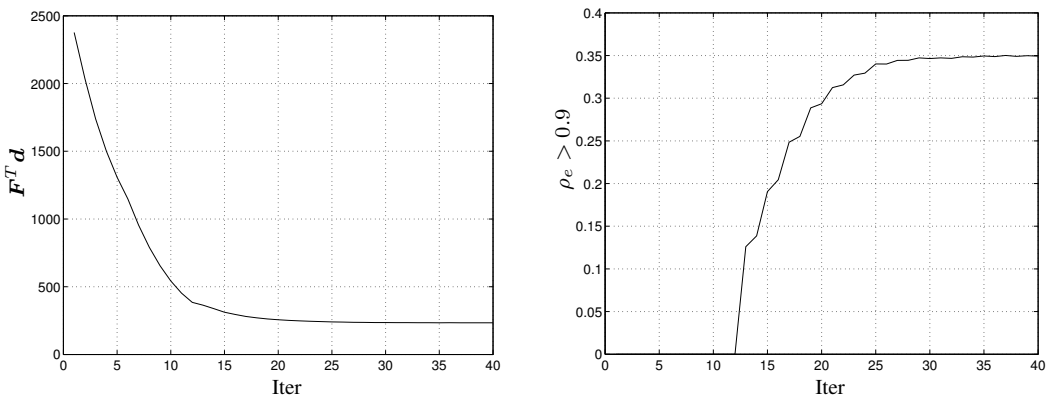

Figure 3: Problem 1 - Convergence properties.

The second problem consists of two elastic bodies in unilateral contact, see the right part of Figure 1. The limit on the volume fraction is again set to 40% and the number of elements is 20534. The convergence properties are similar to the ones presented in Figure 3, and similar CPU-times as for problem one can be found in Table 1. Now, the presented approach is 18% faster than the adjoint approach.

	CPU-time [s]	
	Compliance ([1, 2, 3])	Potential (this paper)
Problem 1	187	147
Problem 2	90	74
Problem 3	2535	1878

Table 1: Problem 1-3 - CPU-times.

The final problem, problem 3, is a three-dimensional setting of the second problem. The 2D geometry is extrude in the out-of-plane direction in order to generate the 3D geometry. The limit on the volume fraction is again set to 40%. The number of elements is 104569. Manufacturing constraints in form of draw directions (split) in the out-of-plane direction are also included. Otherwise, a hollow box structure will

be developed. The optimal solution is presented in Figure 4 and the corresponding CPU-times are presented in Table 1. The efficiency is now even more pronounced compared to the previous problem in 2D. The presented approach is 26% faster than the adjoint approach.

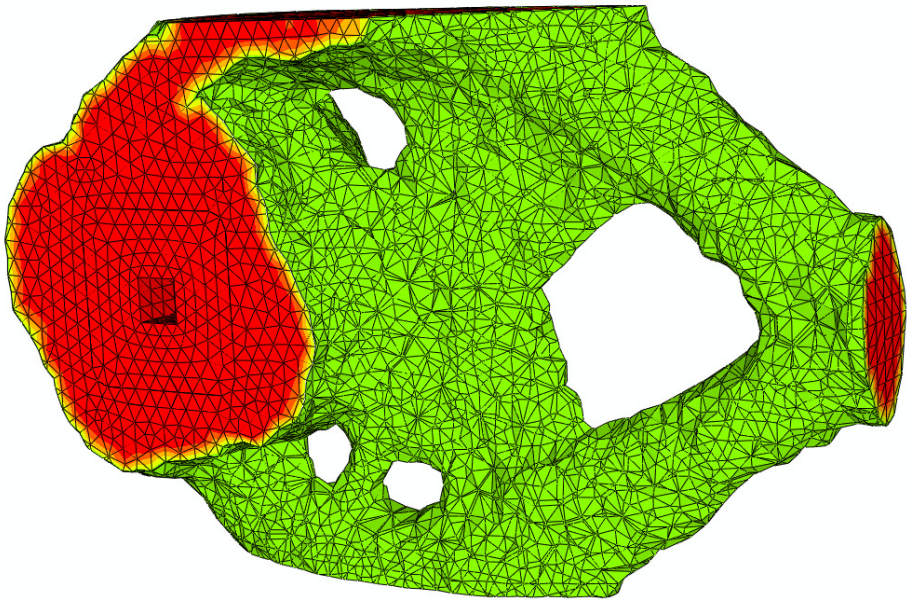

Figure 4: Problem 3 - Optimal topology.

5 Conclusions

In this work topology optimization of linear elastic bodies in unilateral contact is performed by maximizing the potential energy. In such manner no adjoint equation is needed in the sensitivity analysis. It is shown that the CPU-time is reduced as much as 25% compared to the adjoint approach developed in [1, 2, 3].

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