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Paper 9

Thin-Walled Structures: Warping Modes

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Abstract

A thin-walled beam model that considers higher order effects is presented in this paper. The beam displacement field is approximated through a linear combination of products between a set of linear independent functions, which are defined over the beam cross section, and the associated amplitudes that are only dependent on the beam axis. The beam model governing equations are then obtained through the integration over the cross section of the corresponding elasticity equations weighted by the cross section approximation functions. A set of uncoupled beam deformation modes are obtained from a non linear eigenvalue problem that stems directly from the general solution of the differential homogeneous equilibrium equations. The classic deformation modes are naturally obtained, being associated with a null eigenvalue, which requires an adequate computation of a Jordan chain, whereas the higher order modes correspond to the non null eigenvalues, which allows the measurement of the mode decay along the beam axis. A numerical example is presented in order to verify the model capabilities to simulate the non classic effects associated with thin-walled beam higher order deformation modes.

Keywords: thin-walled beam model, warping modes.

1 Introduction

The structural analysis of thin-walled prismatic structures through one-dimensional models requires the consideration of higher order deformation modes in order to accurately represent its three dimensional structural behaviour. In fact, being those models derived by reducing the three-dimensional elasticity equations to a set of equations defined along the member axis through an appropriate projection of the displacement field, the definition of a convenient set of basis functions is essential to capture the 3D structural behaviour.

A thin-walled beam model that assumes the cross section in-plane rigid but considers the out-of-plane warping and the "membrane" shear deformation is presented in this paper.

The warping of thin-walled structures with open cross sections was defined by [1] through the definition of a new coordinate, the sectorial coordinate, for the approximation of the cross section displacement along the beam axis. The cross section was assumed to be in-plane undeformable and the shear deformation of the middle surface was neglected. A theory for the non uniform torsion of closed cross section was derived by [2, 3], considering the shear strain of the cross section midline to be given by the *Saint-Venant* uniform torsion theory. This assumption is conceptually identical to the one made when adopting the *Euler-Bernoulli* theory for the non uniform bending of beams. A more accurate theory for the non-uniform torsion of closed thin-walled beams was derived in [4, 5], where the axial displacement field is defined considering anh additional term representing the cross section warping, which has an amplitude variable along the beam axis that is not related with the cross section angle of torsion; reference should be made that this approach was also considered in [1] for solid sections. This theory is established and known as Benscoter theory, [7]. The cross section "classic" warping are made orthogonal to the flexural modes by considering an adequate position for measuring the sectorial coordinate that defines the warping function, in a form similar to the admitting the beam flexure around the principal axes.

A general theory for the analysis of thin-walled beams applied to a cross section either with an open, closed or branched midline profile was presented in [8, 9, 10]. The theory relies on the assumption of the *Vlassov* hypothesis, considering only the shear strain corresponding to the *Saint-Venant* torsion theory for the closed cross section, [3]. The displacement field is approximated in terms of the derivative of its tangential components along the beam axis, being the axial displacements obtained through the hypothesis of neglecting the membrane shear strain, corresponding to a procedure already adopted by [6]. However, an orthogonality criterion similar to the classic warping was not possible to derived; an uncoupling procedure based on an attempt do diagonalize the beam governing equations through generalised eigenvalue problems was adopted instead.

The generalised beam theory (**gbt**) developed by [11, 12] allows to define the cross section deformation field taking in account the cross section discretization in terms of axial displacements by considering the hypothesis of neglecting the shear at the middle surface. The approach considers the approximation of the axial displacement field, obtaining the transverse displacement as a consequence of neglecting the shear strain. This formulation was extended for considering the shear-lag effect by adopting a set of shear-lag warping modes, [13, 14]. The initial formulation of the **gbt** has however some restrictions, being one of them the application to cross sections with a generic midline profile geometry. However, further developments of the theory were made, allowing its application to cross sections with an open, closed or branched geometry, [15, 16, 17, 18]. Regarding the uncoupling between deformation modes, the theory considers the diagonalization of the beam governing equations as a criteria, adopting towards this end a set of justified generalised eigenvalues problems.

Several other beam formulations accounting for the warping of thin-walled structures have been developed. Although, some of them were developed towards the application to specific structural behaviours, it is worth to mention some of the respective concepts.

A thin-walled beam formulation for anisotropic materials considering the out of plane warping of beam cross section but assuming its in-plane rigidity was presented by [19]. A set of orthogonal functions for the shear dependent warping of thin-walled beams was derived. These warping functions were derived from considering the axial displacement defined by the product of two unknown functions, one defined on the beam axis and the other the interpolation over the beam cross section. The derivative of the transverse displacements, two translational and one torsion rotation, is assumed to be obtained from the unknown function associated with the axial displacement, being however associated with a basis function that corresponds to a parameter yet undetermined, independent of the cross section plane coordinates. Under these conditions, a weak form of the equilibrium equation is derived, which leads to a generalised eigenvalue problem, allowing to obtain a set of orthogonal warping functions, defining the so called *eigenwarpings*. An "improved Bernoulli model" and an "improved Saint-Venant model" have been defined by respectively adding to the Bernoulli model (linear axial displacement) and to the Saint-Venant model (linear axial stress distribution), a series expansion of these eigenwarpings.

The warping of thin-walled beams was also taken in account in [20] by considering an approximation of the thin-walled axial displacement field through a translation of the origin, a rotation of the average plane of bending and a linear combination of basis functions only dependent of the cross section coordinate. The basis functions are assumed to be orthogonal to the constant and linear functions associated with the origin translation and the average plane rotation, respectively. Comparatively to the model of [19], the formulation of [20] derives the system of governing equations from the assumptions made on the complete description of the displacement field, whereas in [19] the solution of warping is obtained separately, for a specific displacement interpolation.

A thin walled beam model applicable to cross section with both open and closed midline profile was presented in [21]. The model assumes the cross section in-plane rigid but considers the membrane shear deformation. A shear flexible element considering warping was presented by [22], the model is applicable to thin-walled beams with an open profile of arbitrary geometry. In terms ok kinematics the formulation considers the axial displacements of the cross section to be defined through the linear combination of the axial displacement of the centroid, the cross section rotations due to t flexure and the warping function defined through the sectorial coordinate. A finite element is derived considering the approximation of the displacement, rotation and warping through linear, quadratic and cubic functions.

A formulation for the analysis of thin-walled rectangular cross section submitted to torsion is developed in [23] in the sequel of a previous work, [24]. A set of orthogonal basis functions is adopted for the axial displacement of the cross section webs, being the displacement of the flanges obtained according to compatibility requirements. An equilibrium equation governing the warping of the walls is established in terms of the basis functions coordinates, being a torsional and a generalised warping stiffness matrix derived. This equation is proven to be independent of the twist angle. On the other hand, an equation similar to the non uniform torsion theory is derived, being the angle of twist coupled with the warping functions parameters.

A beam model aiming to establish a procedure for obtaining warping functions was presented in [25, 26, 27, 28, 29]. This model considers a division of the cross section into rectilinear elements, having the displacements a linear variation along each wall element. The axial displacements are obtained adding to the Bernoulli displacements a linear combination of these additional warpings. Hence, a redundancy can exist since it is possible to derive a combination of additional warping that correspond to the classic displacements, being the same argument applicable to the warping defined through the sectorial coordinate introduced by Vlassov.

A development of thin-walled beam models to the analysis of bridge structures was presented in [33] and [34, 35] considering some of the formulations reported in [31, 32].

The thin-walled beam model presented in this papers allows to obtain a set of deformation modes representing the cross section warping in a systematic procedure through the homogeneous solution of the beam differential equilibrium equations. These modes are uncoupled, allowing to separate the thin-wall structural behaviour. The classic equations are retrieved side by side with a set of governing equations representing higher order deformations. The shear deformation of the middle surface is included given the flexibility considered for the approximation of the displacement field; the displacement components are approximated independently, which allows the model to be applied to generic cross section shape.

2 Model formulation

A one dimensional model for the analysis of thin-walled beams in order to consider the corresponding out-of-plane warping is developed. The formulation assumes the cross section to be in-plane rigid and considers the respective shear deformation. The formulation considers the cross section to be divided into laminar elements, being the corresponding behaviour refrenced to the wall middle surface. Hence, the displacement field is approximated along the thin-walled middle surface.

2.1 Displacement field

The displacement field is defined admitting the beam cross section to be divided into n laminar elements without any geometric restriction, i.e., it is possible to deal with more than two walls converging in a node as well as to consider two consecutive aligned walls. The displacement components are defined through a set of interpolation functions independently of the corresponding direction. A local reference frame $O(x,s,n)$ is considered, being the beam longitudinal axis represented by x , whereas n represents the perpendicular direction relatively to the wall and s the running coordinate along the cross section midline profile. The middle surface is therefore defined by the cartesian pair (x,s).

The structural behaviour of the thin-walled beam is reduced to the cross section midline profile considering only the corresponding membrane behaviour inasmuch the cross section in-plane deformation is disregarded and hence the plate behaviour of the wall is neglected, having no need to consider the approximation of displacements along the perpendicular direction to the wall.

The displacement field is defined as follows:

$$u_x(x,s,n) = \tilde{u}_x(x,s) = \mathbf{\Phi}\,\mathbf{u}_x \quad \text{and} \quad u_s(x,s,n) = \tilde{u}_s(x,s) = \mathbf{\Psi}\,\mathbf{u}_s \tag{1}$$

where $\mathbf{\Phi}$ and $\mathbf{\Psi}$ represent the arrays grouping the corresponding interpolation functions, being $\tilde{u}_x$ e $\tilde{u}_s$ the respective amplitudes. Considering a set of *p* and *m* linear independent function for the interpolation functions:

$$\mathbf{\Phi} = [\phi_1, \cdots, \phi_p] \quad \text{and} \quad \mathbf{\Psi} = [\psi_1, \cdots, \phi_m] \tag{2}$$

being the respective amplitudes given by:

$$\mathbf{u}_x = [u_{x1}, \cdots, u_{xp}]^t \quad \text{and} \quad \mathbf{u}_s = [u_{s1}, \cdots, u_{sm}]^t \tag{3}$$

2.2 Deformation field

The deformation field represents the thin-wall membrane structural behaviour, being obtained by assuming the small displacement hypothesis through the following:

$$\epsilon_x(x,s,n) = \frac{\partial \tilde{u}_x}{\partial x} \tag{4}$$

$$\epsilon_s(x,s,n) = \frac{\partial \tilde{u}_s}{\partial s} \tag{5}$$

$$\gamma_{xs}(x,s,n) = \frac{\partial \tilde{u}_x}{\partial s} + \frac{\partial \tilde{u}_s}{\partial x} \tag{6}$$

By substituting the approximations defined in 1 the deformation field can be rewritten in a more compact form as follows,

$$\boldsymbol{\epsilon} = \mathbf{E}\,\mathbf{e} \tag{7}$$

being $\mathbf{E}$ and $\mathbf{e}$ defined as follows,

$$\mathbf{E} = [\mathbf{E}_1, \mathbf{E}_0], \quad \mathbf{e} = \left[\mathbf{u}', \mathbf{u}\right]^t \tag{8}$$

with

$$\mathbf{u} \,=\, [\mathbf{u}_x, \mathbf{u}_s] \quad \mathbf{E}_0 = \begin{bmatrix} \cdot & \cdot \\ \cdot & \mathbf{\Psi}_{,s} \\ \mathbf{\Phi}_{,s} & \cdot \end{bmatrix} \quad \mathbf{E}_1 = \begin{bmatrix} \mathbf{\Phi} & \cdot \\ \cdot & \cdot \\ \cdot & \mathbf{\Psi} \end{bmatrix} \tag{9}$$

where: i) the matrix $\mathbf{E}$ groups the cross section deformation modes, ii) the vector $\mathbf{e}$ represents the corresponding parameters and iii) the vector $\mathbf{u}$ correspond to the amplitudes of the displacement field approximations over the cross section.

The compatibility conditions are written as follows,

$$\mathbf{e} \,=\, \mathbb{D}^* \, \mathbf{u} \tag{10}$$

being $\mathbb{D}^*$ a compatibility differential operator defined by

$$\mathbb{D}^* \,=\, \begin{bmatrix} \boldsymbol{\partial}_x \\ \mathbf{I} \end{bmatrix} \tag{11}$$

2.3 Constitutive relations

A linear, elastic and isotropic behaviour is admitted for each thin-walled element, being considered the following constitutive relation

$$\boldsymbol{\sigma} \,=\, \mathbf{C}\,\boldsymbol{\varepsilon} \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} E^* & E^*\,\nu & \cdot \\ E^*\,\nu & E^* & \cdot \\ \cdot & \cdot & G \end{bmatrix} \tag{12}$$

where G represents the distortion *moduli* and E^* represents the elastic *moduli* for a plane state of stress, being given through

$$G \,=\, \frac{E}{2(1+\nu)} \quad \text{and} \quad E^* \,=\, \frac{E}{1-\nu^2} \tag{13}$$

It is convenient for the model formulation to have a "discrete" measure associated with the continuum variable that represents the stress field, $\boldsymbol{\sigma}$. This is the dual variable in relation to the deformation parameters $\mathbf{e}$ defined in (8). Hence, a vector of generalised internal forces is defined by weighting the stress field through the deformation modes as follows,

$$\mathrm{s} \,=\, \int_A \mathbf{E}^t \boldsymbol{\sigma}\, dA \tag{14}$$

The vector s represents a set of generalised forces, including the classic internal forces from the classic beam theories and internal forces (self-equilibrated) of higher order. For the sake of consistency with the developed formulation these generalised forces are rewritten as follows,

$$\mathrm{s} \,=\, [\mathrm{s}_1, \mathrm{s}_0\,]^t \tag{15}$$

The relation between the generalised forces s and the corresponding deformation parameters is obtained by setting the deformation energy of these discrte measures to equal the deformation energy of the corresponding continuum variables, namely the stress and deformation field. The following constitutive relation at a cross section level is then obtained

$$\mathbf{s} = \mathbf{K}\,\mathbf{e} \tag{16}$$

being the siffness matrix $\mathbf{K}$ defined as follows,

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{10} \\ \mathbf{K}_{01} & \mathbf{K}_{00} \end{bmatrix} \quad \text{and} \quad \mathbf{K}_{ij} = \int_A \mathbf{E}_i^t\,\mathbf{C}\,\mathbf{E}_j\,dA \quad \text{for} \quad i,j = 0,1,2. \tag{17}$$

2.4 Equilibrium equations

The equilibrium equations are obtained assuming valid the small displacements hypothesis through the dual relation of the compatibility conditions written in (10), being written as follows,

$$\mathbb{D}\,\mathbf{s} + \mathbf{p} = 0 \quad \text{em que } \mathbb{D} = \left[\tilde{\mathbb{D}}, -\mathbf{I}\right] \tag{18}$$

where i) $\tilde{\mathbb{D}}$ represents the equilibrium differential operator, which is adjoint of the compatibility operator $\tilde{\mathbb{D}}^*$ and ii) $\mathbf{p}$ represents the load vector.

The equilibrium equations can be written in terms of the amplitudes of the basis function that approximate the displacement field over the cross section by substituting in (18) the constitutive relations (16) and the compatibility conditions (10) obtaining the following equation,

$$\mathbb{D}\,\mathbf{K}\,\mathbb{D}^*\,\tilde{\mathbf{u}}(x) + \mathbf{p} = 0 \tag{19}$$

After performing the operations corresponding to the differential operators, the governing equation can be rewritten in the following format,

$$\mathbb{K}_2\,\tilde{\mathbf{u}}''(x) + \mathbb{K}_1\,\tilde{\mathbf{u}}'(x) + \mathbb{K}_0\,\tilde{\mathbf{u}}(x) = 0 \tag{20}$$

where $(') = \frac{d}{dx}$ and the coefficient matrices are obtained as follows,

$$\mathbb{K}_2 = \mathbf{K}_{11} = \int_A \mathbf{E}_1^t\,\mathbf{C}\,\mathbf{E}_1\,dA \tag{21}$$

$$\mathbb{K}_1 = \mathbf{K}_{10} - \mathbf{K}_{01}^t = \int_A \left(\mathbf{E}_1^t\,\mathbf{C}\,\mathbf{E}_0 - \mathbf{E}_0^t\,\mathbf{C}\,\mathbf{E}_1\right) dA \tag{22}$$

$$\mathbb{K}_0 = \mathbf{K}_{00} = \int_A \mathbf{E}_0^t\,\mathbf{C}\,\mathbf{E}_0\,dA \tag{23}$$

Since the approximation functions are linear independent the matrix $\mathbb{K}_2$ is verified to be non-singular. The matrix $\mathbb{K}_1$ has a skew-symmetric structure, the matrix $\mathbb{K}_0$ symmetric and singular.

The coefficient matrices, (21), (22) and (23) considering the approximations of the displacement field defined in (1) can be written by diagonal blocks as follows,

$$\mathbb{K}_2 = \begin{bmatrix} \mathbf{K}_{2,a} & \cdot \\ \cdot & \mathbf{K}_{2,t} \end{bmatrix}, \quad \mathbb{K}_1 = \begin{bmatrix} \cdot & \mathbf{K}_1 \\ -\mathbf{K}_1^t & \cdot \end{bmatrix} \quad \text{and} \quad \mathbb{K}_0 = \begin{bmatrix} \mathbf{K}_{0,a} & \cdot \\ \cdot & \mathbf{K}_{0,t} \end{bmatrix}$$

being the corresponding submatrices defined as follows,

$$\mathbf{K}_{2,a} = E^* t \int_A \mathbf{\Phi} \otimes \mathbf{\Phi} \, dA \qquad \mathbf{K}_{2,t} = G t \int_A \mathbf{\Psi} \otimes \mathbf{\Psi} \, dA$$

$$\mathbf{K}_{0,a} = G t \int_A \mathbf{\Phi}_{,s} \otimes \mathbf{\Phi}_{,s} \, dA \qquad \mathbf{K}_{0,t} = G t \int_A \mathbf{\Psi}_{,s} \otimes \mathbf{\Psi}_{,s} \, dA$$

$$\mathbf{K}_1 = G t \int_A \mathbf{\Phi}_{,s} \otimes \mathbf{\Psi} \, dA$$

3 Uncoupling procedure

3.1 General concept

The successful application of a beam model to the analysis of the three-dimensional behaviour of thin-walled beams depends on the refinement considered for the displacement field projection, which can be obtained through an enrichment of the approximation functions or by a refinement of the corresponding mesh, (cross section discretization). However, if the resulting governing equations are not properly uncoupled, the result is a tangled set of equations that although the fact of being representing the elasticity formulation of the problem, taking in account the respective reduction to a 1D model, do not allow a clear physical interpretation of the structural phenomena.

Hence, the procedure of uncoupling the beam governing equations after a proper projection of the displacement field is mandatory. In this paper, a new concept of uncoupling is put forward by taking in account the eigenvalue statement associated with the beam equations homogeneous solution and also the particular case that neglects the cross in-plane deformability, which allows to simplify the problem and define a systematic and simple procedure to obtain warping functions.

The conceptual idea of uncoupling within the framework of this beam model is to derive the corresponding set of displacement modes from a set of orthogonal solutions.

The beam differential equation is intimately associated with the quadratic eigenvalue problem, **qep**. In fact, the general solution for equation (20) can be written in an exponential form as follows,

$$\tilde{\mathbf{u}}(x) = \tilde{\mathbf{u}}_0 \, e^{(\lambda \, x)} \tag{24}$$

The substitution of the solution (24) into the beam model equations (20) yields the following set of equations

$$\mathbf{Q}(\lambda) \, \tilde{\mathbf{u}}_0 \, e^{\lambda \, x} = 0 \quad \text{with} \quad \mathbf{Q}(\lambda) = \mathbb{K}_2 \, \lambda^2 + \mathbb{K}_1 \, \lambda + \mathbb{K}_0 \tag{25}$$

which, since $e^{\lambda x} > 0$, corresponds to the following set of algebraic equations,

$$\mathbf{Q}(\lambda)\,\tilde{\mathbf{u}}_0 = 0 \tag{26}$$

The equations (26) represent a quadratic eigenvalue problem, being λ and $\tilde{\mathbf{u}}_0$ the eigenvalue and the corresponding eigenvector, respectively. The solution (24) was already considered by [36, 37, 39] for plane elasticity problems and by [40] for beams with a solid cross section with a three-dimentional behaviour in order to provide a quantification for the *Saint-Venant* principle, being the exponent parameter λ identified as the inverse of a decay length associated with the solution.

The solution of the **qep** regarding the obtention of the corresponding eigenvectors and eigenvalues and a transformation of coordinates that guarantee the same spectrum (set of eigenvalues) is not as simple and straight foward as in the case of a the standard eigenvalue problem. A major difference in relation to the more usual eigenvalue problems, namely the standard and the generalized eigenvalue problems, is the fact that in the quadratic eigenvalue problem the number of eigenvalues to obtain is twice of the problem dimension, i.e., $(2 \times n)$, and therefore the associated eigenvectors cannot form a linearly independent set of vectors. Hence, the definition of the eigenvectors for a new base of the equations is not possible.

Essentially two approaches exist regarding the **qep** solution: a) one that considers the problem in its original form and seeks for a solvent of the corresponding set of algebraic equations and b) other that linearizes the problem in order to obtain a solution through a generalized eigenvalue problem. The use of a linearization is the usual procedure since it has a simple numerical implementation. However, careful must be taken in order to preserve the initial symmetric/skew-symetric structure of the problem.

The structure of the eigenvalue problem, i.e. the symmetry of $\mathbb{K}_2$ and $\mathbb{K}_0$ and the skew symmetry of $\mathbb{K}_1$, implies a spectrum symmetric not only in relation to the real axis but also symmetric in relation to the imaginary axis, [41]. Therefore, if λ is an eigenvalue of the problem so are the eigenvalues $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$. The linear form of the beam **qep** should therefore reproduce such spectrum location. However, one of the most common linearizations in differential equations corresponds to consider the following system of equation,

$$\mathbf{A}\,\tilde{\mathbf{w}}' - \mathbf{B}\,\tilde{\mathbf{w}} = \mathbf{0} \quad \text{with} \quad \tilde{\mathbf{w}} = \begin{bmatrix} \tilde{\mathbf{u}}'(x) & \tilde{\mathbf{u}}(x) \end{bmatrix}^t \tag{27}$$

and being the coefficient matrices $\mathbf{A}$ and $\mathbf{B}$ square matrices of dimension $2\,n$ (doubling the problem dimension) given through,

$$\mathbf{A} = \begin{bmatrix} \mathbb{K}_2 & \cdot \\ \cdot & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} \mathbb{K}_1 & \mathbb{K}_0 \\ \mathbf{I} & \cdot \end{bmatrix} \tag{28}$$

However, both matrices presented in (27) are neither Hamiltonian nor skew-Hamiltonian and hence the spectrum symmetry will not be preserved. In fact,

$$(\mathbf{A}\,\mathbf{J})^t \neq \mathbf{A}\,\mathbf{J} \quad \text{with} \quad \mathbf{J} = \begin{bmatrix} \cdot & \mathbf{I} \\ -\mathbf{I} & \cdot \end{bmatrix} \tag{29}$$

Therefore, in order to preserve the symmetric structure of the problem, the symmetric linearization of the beam differential equation adopted for this formulation yields the following associated genralised eigenvalue problem,

$$\begin{gathered} \left(\bar{\mathbf{B}} - \lambda\,\bar{\mathbf{A}}\right)\tilde{\mathbf{w}}_0 \;=\; 0 \\ \bar{\mathbf{A}} \;=\; \begin{bmatrix} \cdot & -\mathbb{K}_0 \\ \mathbb{K}_2 & \cdot \end{bmatrix} \quad \bar{\mathbf{B}} \;=\; \begin{bmatrix} \mathbb{K}_2 & \mathbb{K}_1 \\ \cdot & \mathbb{K}_2 \end{bmatrix} \end{gathered} \tag{30}$$

In fact, these matrices are respectively Hamiltonian, $\bar{\mathbf{A}}$ and skew-Hamiltonian $\bar{\mathbf{B}}$, which imply that the solution of the generalized eigenvalue problem permits to obtain a spectrum both symmetric to the real and imaginary axis.

The set of eigenvalues obtained from (26) occur in pairs of symmetric real non-null values and a null eigenvalue representing a twelve fold root, i.e. with an algebraic multiplicity of 12. The symmetric values correspond to solutions decaying along both directions along the beam longitudinal axis, whereas the null eigenvalue correspond to a polynomial solution with no decaying pattern.

3.2 Classic modes

The null eigenvalue is a twelve fold solution of equation (30), which corresponds to non-decaying solutions: six body rigid body motions and the Saint Venant classic solutions of extension, flexure and uniform torsion. Hence, the 12 corresponding eigenvectors that would represent a basis for the displacement modes associated with those solutions have to be obtained. However, the geometric multiplicity of the null eigenvalue is 4: the submatrix $\mathbb{K}_{0,t}$ (3×3) is null due to the in-plane undeformability and the submatrix $\mathbb{K}_{0,a}$ is obtained through the integration between a set of functions, having one linear dependency. Therefore, since the algebraic and thr geometric multiplicities of the null eigenvalue do not coincide, a Jordan chain block of matrices has to be obtained in order to identify the respective generalised eigenvectors, [44].

This fact corresponds to consider for the beam governing equation a general solution of the following form,

$$\tilde{\mathbf{u}}(x) \;=\; \left(\frac{x^k}{k!}\,\tilde{\mathbf{u}}_0 \,+\, \ldots, + x\,\tilde{\mathbf{u}}_{k-1} \,+\, \tilde{\mathbf{u}}_k\right)\, e^{\lambda\,x} \quad \text{with}\; \tilde{\mathbf{u}}_0 \;\neq\; 0 \tag{31}$$

where in the absence of span loads $k \;=\; 3$. According to [43] the solution (31) is valid if and only if the following equalities hold,

$$\textstyle\sum_{j=0}^{i} \dfrac{1}{j!}\,\mathbf{Q}^j(0)\,\mathbf{u}_{i-j} \;=\; 0 \quad \text{for} \quad i \;=\; 0 \cdots k. \tag{32}$$

where the superscript j in $\mathbf{Q}^{(j)}(0)$ represents the j^{th} derivative in relation to λ of the matrix $\mathbf{Q}$. The application of (32) results in the successive solution of a linear system of equations. The first set of equations, corresponding to $j = 0$ and $i = 0$, reads as follows,

$$\mathbb{K}_0\,\mathbf{u}_0 = \mathbf{0} \tag{33}$$

which solution is obtained by the null space of matrix $\mathbb{K}_0$. The vectors $\mathbf{u}_0$ are eigenvectors associated with $\lambda_0 = 0$, spanning a space of dimension β_0, i.e., the geometric multiplicity of eigenvalue λ_0. Since $\beta_0 < \alpha = 12$ and providing that $\mathbf{u}_0 \neq \mathbf{0}$ the next set of equations needs to be considered. The following set of equations is then solved,

$$\mathbb{K}_0\,\mathbf{u}_0 = \mathbf{0} \quad \text{and} \quad \mathbb{K}_1\,\mathbf{u}_0 + \mathbb{K}_0\,\mathbf{u}_1 = \mathbf{0}$$

being mandatory that the eigenvector should not be null, i.e., $\mathbf{u}_0 \neq \mathbf{0}$. For the beam homogeneous differential equations either for a two-dimensional or a three-dimensional formulation, it was verified that for achieving a convergence, i.e., $\beta = \alpha = 12$, four sets of equations are required, corresponding to 4 sets of generalised eigenvectors: $\mathbf{u}_0,\ \mathbf{u}_1,\ \mathbf{u}_2,$ and $\mathbf{u}_3$.

3.3 Application to cross sections rigid in-plane

A closer analysis of the differential beam governing equations allows to conclude that it is possible to establish a relation between the transverse displacement amplitudes and the amplitudes of the axial displacements approximation. This fact allows to write the beam governing equations exclusively in terms of the axial amplitudes, in a procedure similar to the dynamic analysis, which is well kwon as static condensation. Towards this end, considering the coefficient matrices defined in (24) and given the fact that since the in-plane cross section deformation is neglected the submatrix $\mathbf{K}_{0,t}$ becomes null, the beam governing equations are separated in the axial direction and in the transverse direction as follows,

$$\mathbf{K}_{2,a}\,\tilde{\mathbf{u}}_a^{''}(x) + \mathbf{K}_1\,\tilde{\mathbf{u}}_t^{'}(x) + \mathbf{K}_{0,a}\,\tilde{\mathbf{u}}_a(x) = \mathbf{0} \tag{34}$$

$$\mathbf{K}_{2,t}\,\tilde{\mathbf{u}}_t^{''}(x) + \mathbf{K}_1\,\tilde{\mathbf{u}}_a^{'}(x) = \mathbf{0} \tag{35}$$

Given the fact that the sets of approximation functions, $\mathbf{\Phi}$ and $\mathbf{\Psi}$, are constituted by linear independent basis functions the submatrices $\mathbf{K}_{2,a}$, $\mathbf{K}_{2,a}$ are non-singular and hence can be inverted. Considering the equation (35), it is possible to obtain a relation between the axial and traverse amplitudes of the displacement approximation functions through the integration of the equilibrium equations on the transverse direction, equation (35),

$$\tilde{\mathbf{u}}_t^{'}(x) = -\,\mathbf{K}_{2,t}^{-1}\,\mathbf{K}_1\,\tilde{\mathbf{u}}_a(x) + \mathbf{c} \tag{36}$$

The subsequent substitution on the axial equilibrium equations allows to obtain a simplified governing differential system of equations,

$$\mathbf{K}_{2,a}\,\tilde{\mathbf{u}}_a^{''}(x) + \mathbf{K}_{0,a}^{*}\,\tilde{\mathbf{u}}_a(x) = \mathbf{c}^{*} \tag{37}$$

being the matrix $\mathbf{K}_{0,a}$ defined by

$$\mathbf{K}_{0,a}^{*} = \left(\mathbf{K}_{0,a} - \mathbf{K}_1 \mathbf{K}_{2,t}^{-1} \mathbf{K}_1\right) \tilde{\mathbf{u}}_a(x) \tag{38}$$

which is a symmetric matrix. Considering a general homogeneous solution for (37) through

$$\tilde{\mathbf{u}}_a(x) = \tilde{\mathbf{u}}_{a0}\, e^{\sqrt{\mu}\, x} \tag{39}$$

a generalised eigenvalue problem allowing to obtain the warping modes for the beam cross section is obtained,

$$\left(\mu\, \mathbf{K}_{2,a} + \mathbf{K}_{0,a}^{*}\right) \tilde{\mathbf{u}}_{ao} = 0 \tag{40}$$

The eigenvectors obtained by (40) allows to obtain a general homogeneous uncoupled solution of the beam governing equations for the higher order deformation modes, i.e., modes with a decaying behaviour along the beam axis. In order to define an orthogonal displacement basis for the displacement field that allows the definition of warping modes, the eigenvectors of (40) must be swept from the set of generalised eigenvectors previously defined in section 3.2.

4 Beam model implementation

The approximation of the displacement filed over the beam cross section can be performed either in a global form or considering a discretization of the cross section into rectilinear elements, being the displacement approximation performed over each element. The later procedure has the advantage of having a more efficient implementation as well as enhancing *h* refinement for the displacement approximation. The approximation of the axial displacements for the cross section element considering quadratic functions is represented in figure (1).

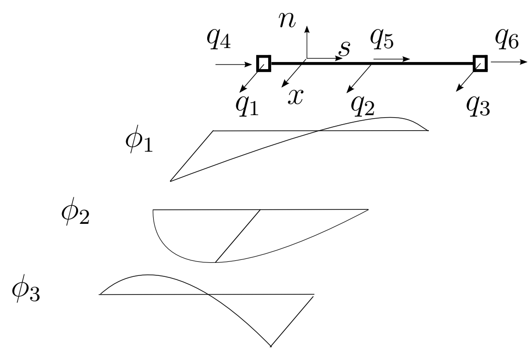

Figure 1: Approximation functions for a cross section element.

The beam governing equations are obtained from the assembly of the element equations coefficient matrices by taking in account the compatibility between cross section elements. However, the tangential deformability of each cross section element is considered by this form. Since the cross section in-plane deformability is not considered in this work, the cross section transverse displacements are referred to the displacements of a generic point of the cross section through a proper change of coordinates or alternatively a matrix of restrictions regarding the tangential relative displacement for each wall can be built.

The classic and warping modes are obtained for the beam governing equations through the procedures described in 3.2 and in 3.3. A change of coordinates for the beam equations is considered by a set of orthogonal displacement modes obtained from the generalised eigenvectors and eigenvectors, allowing to establish a set of "uncoupled" equilibrium equations, which can be written in terms of internal forces allowing to recover the beam classic equilibrium equations and establish a similar set of equations in terms of generalised internal forces associated with warping modes.

A solution for the axial beam displacement is derived taking in account the eigenvalues obtained through equation (40). The beam axial displacement can then be obtained as follows,

$$\boldsymbol{\delta_x}(x) \;=\; \alpha_1 \tilde{\mathbf{u}}_{a,1}\, e^{\mu_1\, x} \;+ \alpha_2 \tilde{\mathbf{u}}_{a,2}\, e^{\mu_2\, x} \;\cdots\; \alpha_m \tilde{\mathbf{u}}_{a,m}\, e^{\mu_m\, x} \tag{41}$$

where $m \leq n$ because the null eigenvalues have an algebraic multiplicity. The constants α_i are obtained taking in account the boundary conditions. However, a finite element procedure is adopted to solve the differential equations along the beam axis. Therefore, instead of considering the constants α_i together with the associated exponential form $e^{\mu_i\, x}$, a new variable dependent on the beam axis coordinate $\tilde{\alpha}_i(x)$ is introduced rewriting the beam displacements as follows,

$$\boldsymbol{\delta_x}(x) \;=\; \tilde{\alpha}_1(x)\mathbf{q}_{x1} \;+ \tilde{\alpha}_2(x)\mathbf{q}_{x2} \;\cdots\; \tilde{\alpha}_m(x)\mathbf{q}_{xm} \tag{42}$$

where $\mathbf{q}_{x1}$ to $\mathbf{q}_{xm}$ correspond to the amplitudes of the interpolation functions along the beam axis for the warping deformation modes.

5 Examples of application

Some example are presented considering the developed beam model. Firstly, the definition of the warping modes of a cross section with different profile geometry is illustrated through the proposed uncoupling procedure. The local influence of the higher order warping modes can be verified through the results obtained for the structural analysis of the beam in terms of stresses distributions along the beam axis, being identified a decaying pattern.

5.1 Cross section modes identification

A thin-walled rectangular hollow section made of concrete is considered. The concrete behaviour is admitted to be linear, elastic and isotropic. The elastic modulus is $E = 21\, GPa$. In terms of the cross section discretization, three solutions were analysed: a) the cross section divided into 6 elements (two elements for the flange and 1 element for the web) considering the axial displacements to be approximated through Lagrange quadratic functions and b) the cross section divided into 12 elements (6 elements for each flange and two elements per web) adopting linear functions for the approximation of the axial displacement. The tangential displacement were approximated by linear function for all the three solutions.

The set of deformation modes were obtained from the uncoupling procedure described in 3.2 and 3.3 through the generalised eigenvalue problem. The eigenvalues obtained are real numbers and occur in symmetric pairs. A null eigenvalue with an algebraic multiplicity of 12 was obtained, which since the corresponding geometric multiplicity was different implies the computation of a Jordan chain of matrices. The Jordan matrices obtained allowed to identify the corresponding classic deformation modes as well as a warping associated with the shear deformation.

The higher order warping modes for the rectangular hollow section are represented in figures (2) and (3). The modes obtained from two approximations are presented: a linear approximation that considers the cross section divided into 12 elements and a quadratic approximation with 6 elements. The modes are presented hierarchically according to the respective decay length, the warping modes of figure (2) have slower decay than the modes in (3). The higher order modes represented in figure (2) represent shear-lag modes, being associated with the corresponding flexure modes. The modes in figure (3) represent a shear-lag mode associated with the extension (on the left side) and a higher order mode related to shear deformation of both flanges.

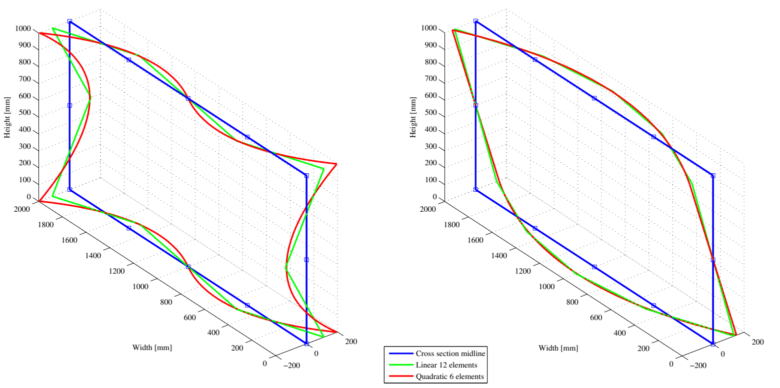

Figure 2: Rectangular hollow section, 1^{st} and 2^{nd} warping modes

A twin hollow cross section obtained by introducing a sept into the previous rectangular hollow section was considered. For the approximation of the displacement field both quadratic and linear functions were considered, namely a discretization into 7 quadratic elements (one per wall of each cell) and into 7 and 14 linear elements (one and two elements per wall of each cell, respectively). The discretization into 7 linear elements is represented in figure 4, being the corresponding degrees of freedom iden-

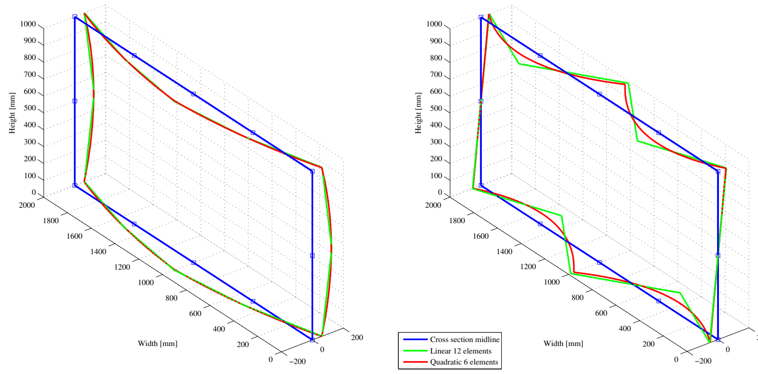

Figure 3: Rectangular hollow section, 3^{rd} and 4^{th} warping modes

tified; the cross section is considered rigid by restricting the tangential displacement along each element, e. g. $q_{11} = q_{12}$.

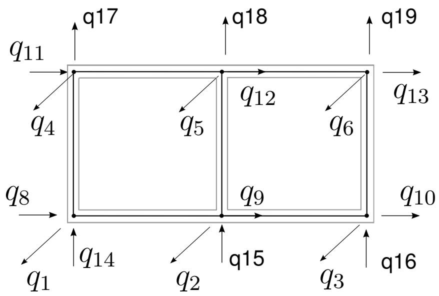

Figure 4: Twin box beam discretization.

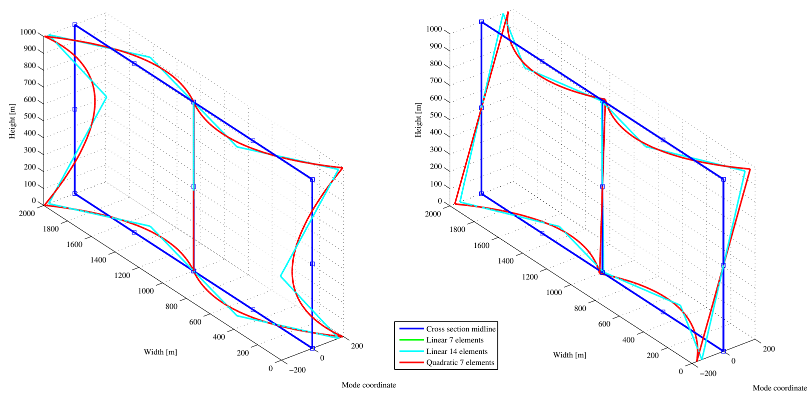

Figure 5: Shear lag warping modes.

A set of shear-lag warping modes are represented in figures 5 for the quadratic element approximation and for the linear approximation with 14 elements. The eigenvalues obtained for these modes through the two approximations solutions are similar. The discretization with 7 linear elements does not allow to capture such modes, being the corresponding eigenvalues higher and closer to the warping modes represented in figure 6. The modes depicted in 6 represent higher warping modes associated with torsion (on the left side) and with the extension (on the right side).

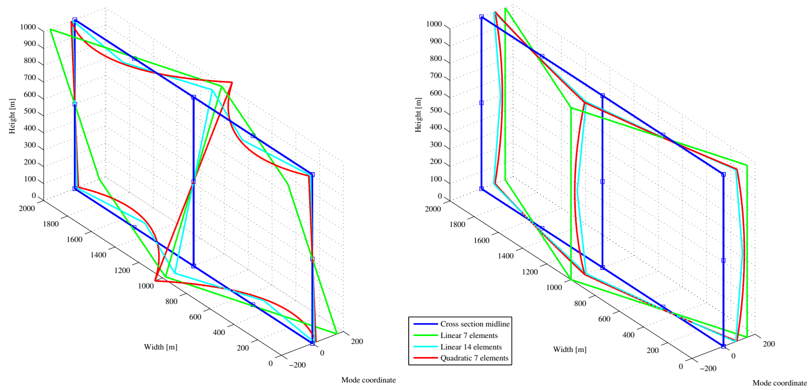

Figure 6: Torsion and extension higher order warping modes.

5.2 Beam analysis

A simple supported thin-walled structure with a twin cell hollow section is considered in order to verify the use of the warping modes derived within the context of a higher order beam model. The structure represents a concrete girder spanning a length of $L = 15\,m$. The beam cross section is a twin cell box shaped with an overall width of $2\,b = 2\,m$ and an height of $h = 1\,m$, having a constant thickness of $t = 0.15\,m$. For the material an elastic *moduli* of $E = 21\,GPa$ with a Poisson coefficient of $\nu = 0.2$ was considered. The results are therefore compared with a shell finite element implemented in [45]. This example was chosen since it was already performed in [8], and in [15], allowing to establish a comparison between results.

The linear approximation with 7 elements represented in figure 4 was considered for the warping modes definition. A concentrated load of $P = 10\,kN$ is applied at mid span in one of the outer webs of the cross section, as represented in figure 7, submitting the beam to a restrained torsion.

The [45] model implemented considers quadrilateral shell elements (S4), adopting a mesh of 28 elements along the beam cross section and considering the span divided into 60 intervals, with a total of 1680 elements, which correspond to 9882 degrees of freedom. The membrane axial stress distribution along the beam longitudinal axis obtained through the higher order beam mode herein developed and a finite element model implemented in [45] is presented in figure 8. Nodal line A is relative to the top of the beam right web, whereas the nodal B represent the top of the beam left web.

A stress color map obtained using the developed model is represented in figures 9 and 10, where the effect of the local load can be observed. A comparison of the axial stresses distribution at the mid span cross section obtained from the current model (green pentagrams), the [8] model (blue pentagrams), the [15] model (red pentagrams) and results obtained from the [45] model (red circles) is presented in figure 11. A fairly good agreement of results exist, being the distribution of Sedlacek's the one which deviates more from the others.

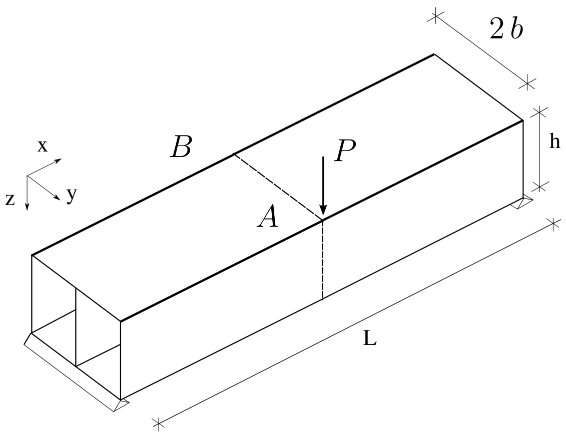

Figure 7: Twin box beam loading.

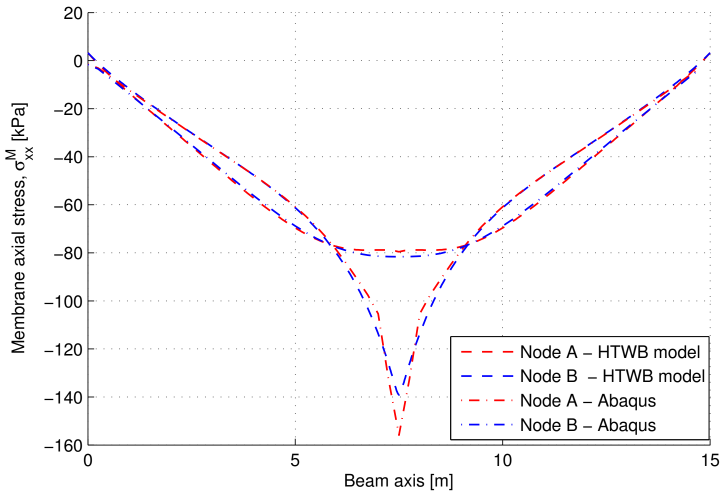

Figure 8: Axial stress longitudinal distribution.

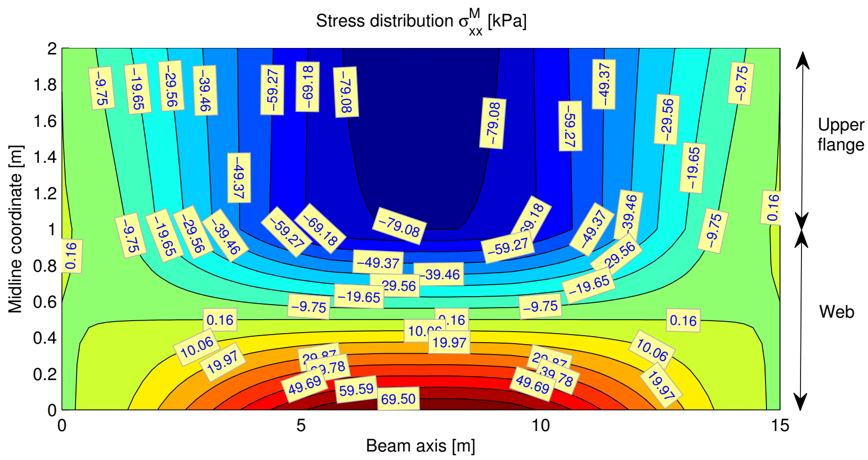

Figure 9: Membrane axial stress [kPa], right web-top flange.

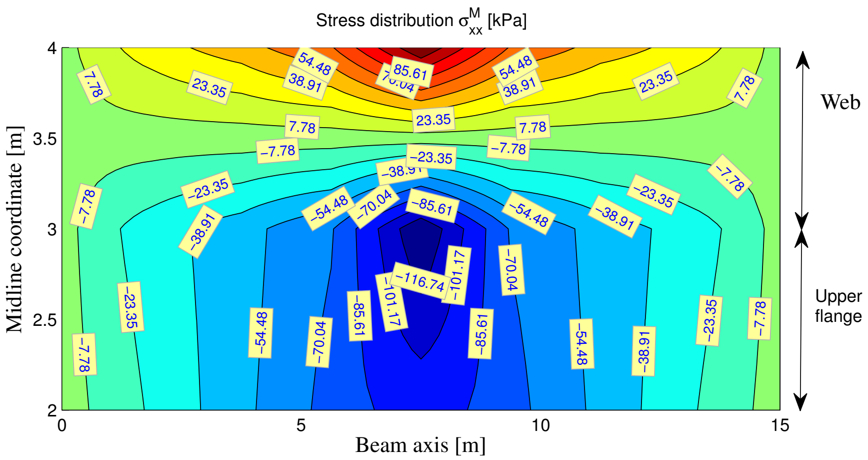

Figure 10: Membrane axial stress [kPa], left web-top flange.

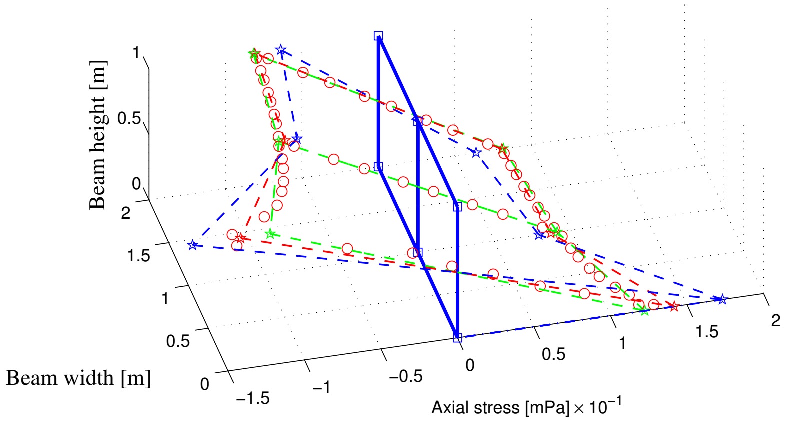

Figure 11: Axial stress transverse distribution.

6 Conclusions

A beam model for the analysis of thin-walled beams was presented. The beam model relies on the approximation over the cross section using a set of linear independent basis functions in order to properly capture the three-dimensional structural behaviour. The cross section in-plane is deformation is neglected, allowing to write the beam governing equations in terms of the axial components of the displacement field and establish a procedure for the uncoupling of the beam deformation modes through the corresponding generalised eigenvalue problem. A set of warping modes for some thin-walled cross sections was hierarchically obtained by sorting the modes according with the respective eigenvalue. A twin-cell concrete girder simple supported was analysed considering the warping modes obtained, being the results successfully compared with a) the results obtained through a shell finite element model implemented in [45] (where through kinematical constraints the cross section was considered rigid), b) the results obtained by Sedlacek in [8] and c) the results obtained from the generalised beam theory formulation of Möller in [15].

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