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Solving Homogeneous Trees of Sturm-Liouville Equations using an Infinite Order Determinant Method

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Abstract

Consideration is given to determining the eigenvalues of a graph with tree topology on which the second order Sturm Liouville operator is acting. This is directly analogous to the free vibration problem of a network of bars with identical topology. The eigenvalues are determined using the well-established Wittrick-Williams algorithm in conjunction with a dynamic stiffness approach. A plot of the determinant of the stiffness matrix versus trial eigenparameter has several poles and eigenvalues of the system can occur both when the determinant of the stiffness matrix is zero or infinite. A recent exact procedure has been established that transforms this plot into an equivalent one that has no poles and where all eigenvalues correspond to zero values of the determinant. The determinant of the transformed matrix is defined as the 'infinite order determinant'. This paper uses such a procedure and it is shown that the infinite order determinant plot tends to zero for large eigenvalues. However a small modification is seen to produce a plot that repeats at regular intervals and corresponds to the band gap structure of the spectrum for an infinitely long tree. In other published work it has been shown that the eigenvalues of a finite, n level tree are obtainable from a family of n stepped bars. Both techniques can be used to estimate the edges of the spectrum for the equivalent infinite system and can be extended to higher order Sturm Liouville problems. However in both cases the stepped bar technique is shown to be much superior computationally.

Keywords: exact dynamic stiffness matrix, Wittrick-Williams algorithm, infinite order determinant, graph theory, Sturm Liouville operator.

1 Introduction

Using the Wittrick–Williams algorithm [1] applied to the Sturm–Liouville problem on trees [2] it has been demonstrated that a numerical procedure can be used to provide insight on the spectral behaviour of the free Laplacian in one dimension acting on a metric tree [3]. The problem is directly analogous to the vibration analysis of a network of bars or beams connected together to form a tree topology [3]. The eigenvalues of the tree correspond to the square of the natural frequency of free vibration. In this paper the approach adopted is a mathematical one. The structural analogy is shown below, however Ref [4] discusses this analogy in more detail. In the case of the infinitely long tree, the eigenvalues form bands of continuous spectra with an eigenvalue of infinite multiplicity in the middle of the gaps. In this paper, as in previous work [4], only finite length trees are analysed.

Eigenvalues are found using the well-established Wittrick-Williams algorithm and a dynamic stiffness approach. The plot of the determinant of the stiffness matrix versus trial eigenparameter has several poles and eigenvalues of the system can occur both when the determinant of the stiffness matrix is zero or infinite.

The Wittrick-Williams algorithm locates with certainty, and up to computer accuracy, all the eigenvalues of the system. If the eigenvalue count on either side of a pole, in the plot of determinant versus trial eigenparameter, is identical then no eigenvalues are present. Typically poles occur at individual member eigenvalues that have clamped boundary conditions. In the case of the infinitely long tree the eigenvalue in the middle of the gap corresponds to the clamped/clamped member eigenvalue. For the finite length tree, if the Wittrick-Williams algorithm shows an eigenvalue is present as the determinant plot passes through a pole, the clamped/clamped member eigenvalue is also an eigenvalue of the system.

The structure of the spectrum, for the infinite tree, is a repeating band of continuous spectra with an eigenvalue in the middle of the gap. For the finite tree there are no bands of continuous spectra, however all eigenvalues fall within the lower and upper bounds of each band of continuous spectra. These bands of eigenvalues form repeating patterns. The determinant plot versus trial eigenparameter starts from a positive value at zero on the abscissa and after crossing the abscissa one or more times tends to infinity as the trial eigenparameter tends to the clamped/clamped member eigenvalue.

Williams *et al* [5] established a procedure to transform this plot so that it has no poles and all eigenvalues occur when the transformed stiffness matrix determinant becomes zero. The determinant of the transformed matrix is known as the ‘infinite order determinant’. This paper uses such a procedure and it is seen that the infinite order determinant plot tends to zero for large values of trial eigenparameter. However a small modification can produce a plot that repeats at regular intervals and corresponds to the band gap structure of the spectrum for an infinitely long tree. From the results of finite length trees it is then possible to estimate the edges of the spectrum for the infinite system.

Williams et al [6] show that the eigenvalues of the n level tree are obtainable from a family of n stepped bars. From the results of the stepped bar approach it is then possible to estimate the edges of the spectrum for the infinite system.

2 Theory

The theory presented in this section relates to the general form of the classical second-order Sturm-Liouville equation, which may be written as

$$-\frac{d}{dx}\left(p\frac{dy}{dx}\right)+qy=\lambda wy \tag{1}$$

where p, q and w are all real valued, positive constants and λ is the eigenparameter. By imposing the following boundary conditions

$$y=y_1 \quad \text{at} \quad x=0 \quad \text{and} \quad y=y_2 \quad \text{at} \quad x=L \tag{2}$$

leading to

$$\begin{bmatrix} -y_1' \\ y_2' \end{bmatrix} = k\begin{bmatrix} \cot kL & -\csc kL \\ -\csc kL & \cot kL \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3}$$

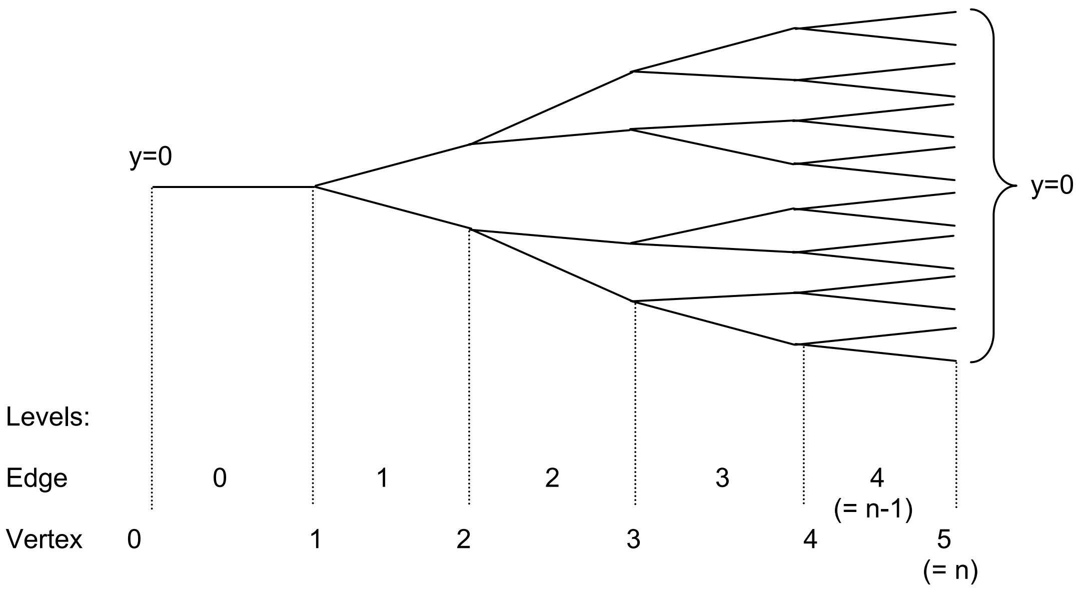
where $k=\sqrt{\frac{\lambda w-q}{p}}$.

This is the well known 'edge' equation with subscripts denoting the left and right end of each edge that is used to form the tree. Figure 1 typifies the tree topology, shows Dirichlet boundary conditions and defines the edge and vertex levels, together with n and b, the number of vertex levels and branching number, respectively. All trees are classified as repetitive or non-repetitive, depending upon whether or not the edge properties at all levels are identical, and such trees are sub-divided into uniform or non-uniform, depending upon whether or not L, p, q and w are all constant. Hence, a repetitive uniform tree is a homogeneous one, whereas a repetitive non-uniform tree is not. This paper deals only with homogeneous trees with p=w=L=1 and q=0. For the analysis of an n level tree the required eigenvalues are obtainable from n stepped bars each containing r members, where r =1,2,…,n. The stepped bar representations for a tree with n=3 and b=3 are shown in Figure 2. All edges at the same level are part of the same member in this representation.

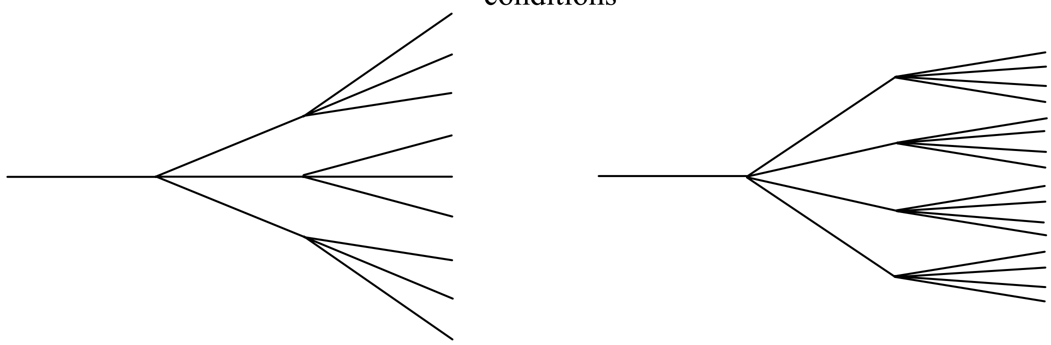
It should now be noted that Equation (1) is exactly analogous to the axial vibration equation of a non-uniform bar on a uniform elastic foundation, as follows

$$-\frac{d}{dx}\left(EA\frac{dy}{dx}\right)+\kappa y=\omega^2 my \tag{4}$$

where EA is the extensional rigidity of the bar, κ is the stiffness per unit length of the elastic foundation, m is the mass per unit length of the bar and ω is the natural radian frequency.

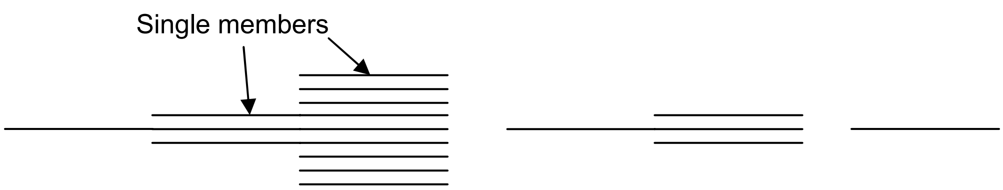


(a) A five level, two branching tree *i.e.* $n = 5$, $b = 2$ with Dirichlet boundary conditions



(b) $n = 3$, $b = 3$ (c) $n = 3$, $b = 4$

Figure 1: The topology of typical trees showing edge and vertex levels together with their branching number



(a) 3 level, stepped bar (b) 2 level stepped bar (c) 1 level stepped bar

Figure 2: The stepped bar representations for a tree with n=3 and b=3. All edges at the same level are part of the same member in this representation so that the three level stepped bar has three members not thirteen.

2.1 Dynamic stiffness matrix

The necessary eigenvalue relationship for a single edge, evaluated at a trial eigenparameter, *k*, is given by Equation (3) as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A & -B \\ -B & A \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{5}$$

where $A = k \cot kL$ and $B = k \csc kL$. The edge equation of equation (3) is used to assemble the global stiffness matrix, *K(k)*, for the tree. For a tree with b=3, n=3 and Dirichlet boundary conditions, as shown Figure 1(b), the global stiffness matrix is given as

$$K(k) = \begin{bmatrix} 4A & 0 & 0 & -B \\ 0 & 4A & 0 & -B \\ 0 & 0 & 4A & -B \\ -B & -B & -B & 4A \end{bmatrix} \tag{6}$$

Using Gauss elimination this matrix can be converted to upper triangular form to give $K^{\Delta}(k)$

$$K^{\Delta}(k) = \begin{bmatrix} 4A & 0 & 0 & -B \\ 0 & 4A & 0 & -B \\ 0 & 0 & 4A & -B \\ 0 & 0 & 0 & \dfrac{16A^2 - 3B^2}{4A} \end{bmatrix} \tag{7}$$

The determinant of $K(k)$ is then

$$\begin{aligned} |K(k)| &= 16A^2(16A^2 - 3B^2) \\ &= 16k^4 \cot^2 k(16 \cot^2 k - 3 \csc^2 k) \end{aligned} \tag{8}$$

It can be seen that the value of the determinant becomes large as the trial eigenparameter *k* becomes large. (At an eigenvalue the determinant will be zero). If the value of $|K(k)|/k^4$ versus *k* is plotted instead, then this leads to a repeating plot that still has poles.

2.2 Infinite Order Determinant Method

The procedure using the infinite order determinant method is described in detail elsewhere [5]. The essence of using this approach is to be able to plot a determinant that does not have poles at any trial value of the eigenparameter. The procedure involves a number of steps. The first of these steps is to evaluate the determinant at zero eigenparameter i.e. k=0. Using the matrix K derived above for the b=3 and n=3 tree we obtain the following

$$|K(0)| = 16\times(16-3) = 208 \tag{9}$$

The second step is to examine each individual member to establish the normalised member stiffness determinant. For the repetitive tree all members are identical hence only one member needs to be analysed. From Ref. [5] this determinant is given as

$$\overline{\Delta}_m = \frac{\sin k}{k} \tag{10}$$

For all members

$$\overline{\Delta} = \prod_m \overline{\Delta}_m \tag{11}$$

The infinite order stiffness determinant is then given as

$$|K_\infty(k)| = \overline{\Delta}(k)\frac{|K(k)|}{|K(0)|} \tag{12}$$

For the tree considered b=3, n=3 and m=13 (number of members). Hence

$$\begin{aligned}|K_\infty(k)| &= \left(\frac{\sin k}{k}\right)^{13}\frac{16k^4\cot^2 k(16\cot^2 k - 3\csc^2 k)}{208}\\ &= \frac{\cos^2 k\sin^9 k(16\cos^2 k - 3)}{13k^9}\end{aligned} \tag{13}$$

The infinite order determinant tends to a small number for large values of k if the determinant is non-zero. If this determinant is multiplied by k^9 then a repeating plot is produced and is shown in Figure 3. Figure 3 shows three plots which are:

(i) $|K(k)|$ v k;

(ii) $|K_\infty(k)|$ v k; and

(iii) $|K_\infty(k)|k^9$ v k. (This plot is amplified by a factor of 100).

There are some points to note about Figure 3. The plot of $|K(k)|$ v k has a non-zero value at k=0. As seen above $|K(0)| = 208$. Because this plot has an amplification factor of .001 the ordinate value is 0.208. The plot of $|K_\infty(k)|$ tends to zero very rapidly. This is due to the presence of k^9 in the denominator. This plot then does not appear to give any useful information. The plot of $|K_\infty(k)|k^9$ v k only has transcendental terms and is therefore cyclical. Herein this determinant is referred to as the normalised infinite order determinant because it repeats at fixed intervals and does not have any poles. This plot has been amplified by a factor of 100.

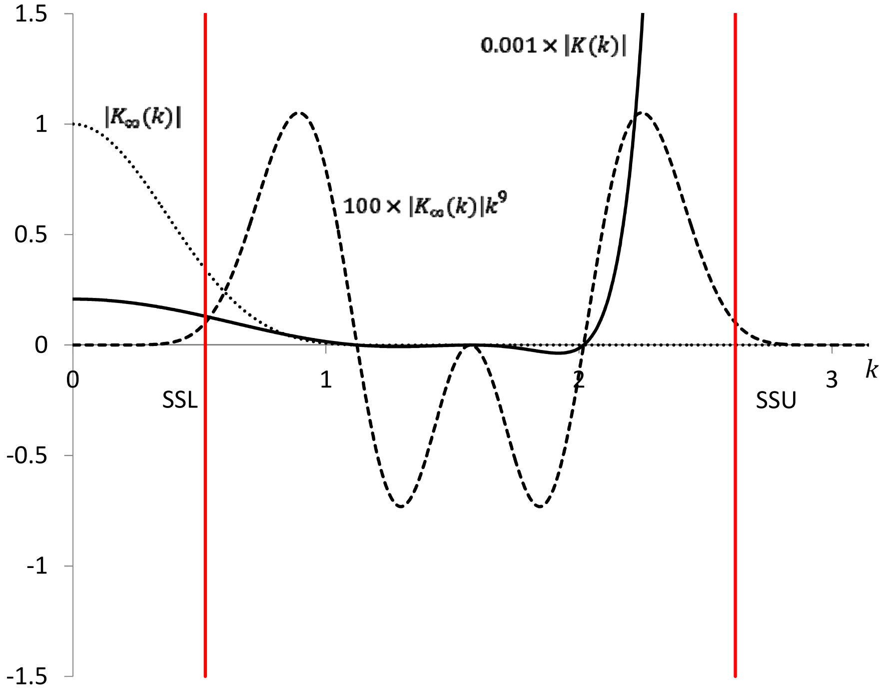


Figure 3: Determinant plots for the *n*=3, *b*=3 tree. The three plots are the ordinary determinant; infinite order determinant and normalised infinite order determinant plots. The theoretical upper and lower bounds given by Sobolev and Solomyak [2] are shown as solid vertical lines and labelled as SSL and SSU for the lower and upper bound, respectively.

3 Higher order trees

The stiffness matrix for an n=4, *b* branching tree is shown below. It shows how the stiffness matrix is constructed for higher order trees. The stiffness matrix is square and the order of the matrix is equal to the number of vertices in the tree. The number of vertices, *N*, can be equated as follows:

$$N = \frac{b^n - 1}{b - 1} \tag{14}$$

For each new additional level in tree length the order of the matrix increases by a factor of b+1. Performing analytical methods on large trees gets difficult very quickly due to this exponential growth in number of vertices and edges.

$$
\begin{array}{c}
\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}^{b^2}\overbrace{\hphantom{xxxxxxxxxxxxxxxxxx}}^{b}\overbrace{\hphantom{xxxxxxxx}}^{1}\\
\left[\begin{array}{cccccccc|ccc|c}
(b+1)A & & & & & & & & -B & & & \\
 & \cdots & & & & & & & \cdots & & & \\
 & & \cdots & & & & & & \cdots & & & \\
 & & & \cdots & & & & & & -B & & \\
 & & & & \cdots & & & & & \cdots & & \\
 & & & & & \cdots & & & & \cdots & & \\
 & & & & & & \cdots & & & & -B & \\
 & & & & & & & \cdots & & & \cdots & \\
 & & & & & & & \cdots & & & \cdots & \\
\hline
-B & \cdots & \cdots & & & & & & (b+1)A & & & -B \\
 & & & -B & \cdots & \cdots & & & & \cdots & & \cdots \\
 & & & & & & -B & \cdots\ \cdots & & & \cdots & \cdots \\
\hline
 & & & & & & & & -B & \cdots & \cdots & (b+1)A
\end{array}\right]
\end{array}
\qquad (15)
$$

When this matrix is converted to upper triangular form the leading diagonal is made up of:

(i) b^2 diagonal elements containing (b+1)A

(ii) b diagonal elements containing $(b+1)A - \dfrac{bB^2}{(b+1)A}$; and

(ii) 1 element containing $(b+1)A - \dfrac{bB^2}{(b+1)A - \dfrac{bB^2}{(b+1)A}}$;

As trees get larger it can be see that for each new level the above elements grow by a factor of b+1 and the new eigenvalues are given by the last diagonal element of the matrix. Note that the locations of the individual terms, in the matrix, are as a consequence of the numbering technique employed.

4 Stepped bar model

The analysis of the stepped bar representation of the tree is different to that of the tree. The principal difference lies in the construction of the stiffness matrix. The members at each level are not identical. However the terms *p* and *w* in Equation (1) vary down the length of the stepped bar. This needs to be accounted for as follows.

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = b^i \begin{bmatrix} A & -B \\ -B & A \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{16}$$

where b^i is the factor for each member at level i as defined in the tree model of Figure 1. The construction of the stiffness matrix then becomes for a three member stepped bar with two internal vertices as follows:

$$K_{SB} = \begin{bmatrix} (b+1)A & -bB \\ -bB & b(b+1)A \end{bmatrix} \tag{17}$$

Performing Gauss elimination we obtain

$$K_{SB}^{\Delta} = \begin{bmatrix} (b+1)A & -bB \\ 0 & b(b+1)A - \dfrac{b^2B^2}{(b+1)A} \end{bmatrix} \tag{18}$$

Hence

$$\begin{aligned} |K_{SB}| &= (b+1)A \times \left((b+1)A - \frac{b^2B^2}{(b+1)A} \right) \\ &= (b+1)A \times \left(\frac{b(b+1)^2 A^2}{(b+1)A} - \frac{b^2B^2}{(b+1)A} \right) \\ &= b(b+1)^2 A^2 - b^2B^2 \end{aligned} \tag{19}$$

Substituting for *A* and *B*

$$\begin{aligned} |K_{SB}| &= b(b+1)^2 k^2 \cot^2 k - b^2 k^2 \csc^2 k \\ &= k^2 \left(b(b+1)^2 \cot^2 k - b^2 \csc^2 k \right) \end{aligned} \tag{20}$$

Using the same techniques as used for the tree to obtain the infinite order determinant plot we obtain the following:

$$|K_{SB}(0)| = b(b+1)^2 - b^2 = 48 - 9 = 39 \tag{21}$$

It can be shown that for all members

$$\overline{\Delta}_m = \frac{\sin k}{k} \tag{22}$$

For all members

$$\overline{\Delta} = \prod_m \overline{\Delta}_m \tag{23}$$

The Infinite Order Stiffness Determinant is then given as

$$|K_{SB\infty}(k)| = \overline{\Delta}(k)\frac{|K_{SB}(k)|}{|K_{SB}(0)|} \tag{24}$$

For the stepped with bar b=3 and n=3, there are 3 members hence

$$|K_{SB\infty}(k)| = \left(\frac{\sin k}{k}\right)^3 \frac{k^2\left(b(b+1)^2\cot^2 k - b^2\csc^2 k\right)}{39} = \frac{\sin k\left(16\cos^2 k - 3\right)}{13k} \tag{25}$$

The normalised plots for n=2; n=4; n=8 and n=16 are shown in Figure 4.

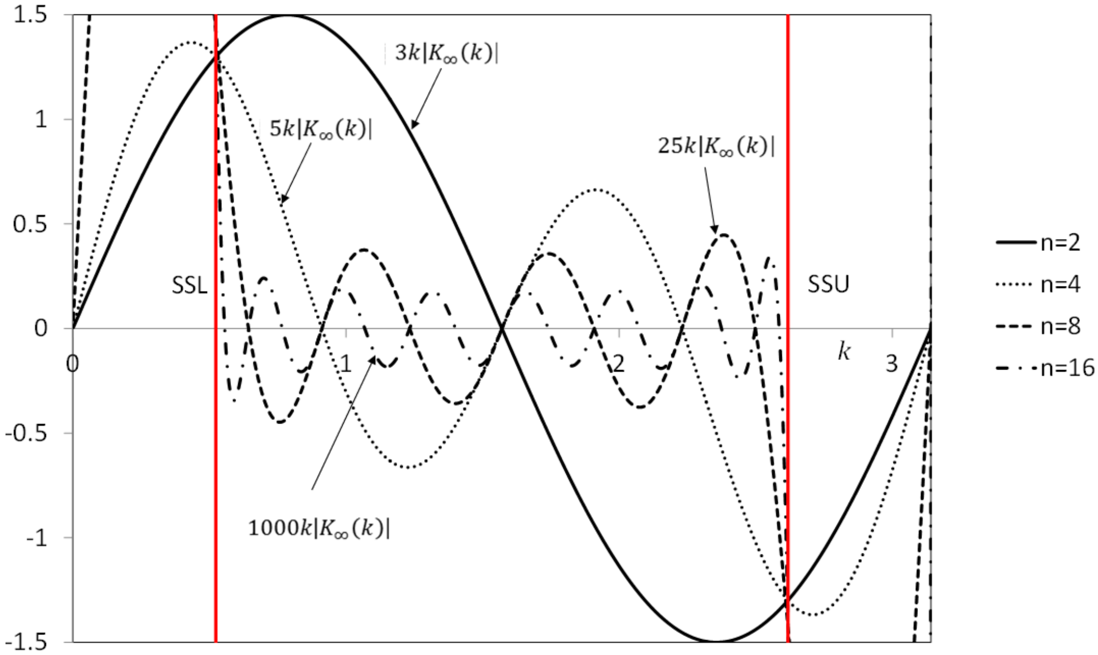


Figure 4: Normalised infinite order determinant plots for n=2, n=4, n=8 and n=16 stepped bar. The theoretical upper and lower bounds given by Sobolev and Solomyak are shown as solid vertical lines and labelled as SSL and SSU for the lower and upper bound respectively.

Table 1 shows the first eigenvalue for the four stepped bars of Figure 4 along with the lower bound of Sobolev and Solomyak (SSL). A numerical convergence technique is used to estimate the lower bound based upon the finite values of *n* and compares reasonably well with the analytical value.

n	k_1	Estimate of SSL	SSL	% difference
2	1.5707963			
4	0.9117383	0.6920523	0.5235988	32.2
8	0.6433292	0.5538595	0.5235988	5.78
16	0.5559775	0.5268603	0.5235988	0.623

Table 1: First eigenvalue, k_1, for stepped bar of length *n* and estimate of Sobolev Solomyak lower bound SSL. The % difference compares Estimate of SSL to SSL.

Figure 4 has several interesting features. The n=16 stepped bar has the same eigenvalues as all the smaller stepped bars plus additional ones. This is a consequence of choosing the stepped bar lengths. All the zeros of the four determinants plotted are eigenvalues except for the origin where all four plots start. The eigenvalues shown in the figure are not the complete set for the *n*=16 tree. To obtain a complete set one would need to include the same plots for all stepped bars in the range 2 to 16 that are not already shown. As *n* tends to get larger it can be seen the amplitude of the plots is reducing inside the SSL and SSU range. In fact the plots have been amplified by factors 3, 5, 25 and 1000 for each of *n*=2, 4, 8 and 16 respectively. As *n* tends to infinity the determinant plot would be a flat line from the value SSL to SSU representing a set of solutions which form a continuous band of spectra [2]. The infinitely long stepped bar would thus have all the eigenvalues of the tree although their multiplicity would not be correct. Although not shown, all the four plots repeat every interval of π and it can be seen that the number of eigenvalues given by an *n* level stepped bar is exactly *n*-1 in the interval of SSL to SSU.

6 Summary and Conclusions

The analysis presented shows that large trees can be analysed very easily with use of the infinite order determinant method and normalising the numerical data to produce plots that repeat at intervals of π. To achieve this, the stepped bar representation was used, which reduced the order of the problem substantially. The length of the stepped bar and order of its stiffness matrix varies linearly, so reasonably long stepped bars e.g. n=16 can be modelled. The order of the tree stiffness matrix is a function of the branching number and length of the tree, hence the order of the stiffness matrix increases exponentially with each new level. For the stepped bar, the order of the stiffness matrix is a linear function of the length of the bar only and is independent of the branching number *b*.

Each stepped bar does not contain the information of the eigenvalues of the whole tree and so a series of stepped bars must be considered in order to obtain this information. However the first eigenvalue of the stepped bar of length n has the same first eigenvalue of the tree and so only one shorter stepped bar is required to carry out a numerical convergence analysis to converge on the value for the edge of the spectrum. It was shown that for $n=8$ and n=16, the eigenvalues obtained provided a very good estimate on the edge of the spectrum. This was validated by comparing with the theoretical result.

References

[1] W.H. Wittrick & F.W. Williams, F. W. "A general algorithm for computing natural frequencies of elastic structures", Q. J. Mech. Appl. Math, 24, 263–284, 1971.

[2] A.V. Sobolev, M. Solomyak, "Schrödinger operators on homogeneous metric trees spectrum in gaps", Reviews in Mathematical Physics 14(5), 421-467, 2002.

[3] F.W. Williams, W.P. Howson, A. Watson, "Application of the Wittrick-Williams algorithm to the Sturm-Liouville problem on homogeneous trees: A structural mechanics analogy", Proceedings of the Royal Society series A: Mathematical, Physical and Engineering Sciences, 460(2045), 1243-1268, 2004.

[4] W.P. Howson & A. Watson, "Homogeneous Trees of Second Order Sturm-Liouville Equations: A General Theory and Program", Computers & Structures, 104-105C, 13-20, 2012

[5] F.W. Williams, D. Kennedy, M.S. Djoudi, "The member stiffness determinant and its uses for the transcendental eigenproblems of structural engineering and other disciplines", Proceedings of The Royal Society: Mathematical, Physical and Engineering Sciences , 459(2045), 1001-1019, 2003.

[6] F.W. Williams, A. Watson, W.P. Howson, A.J. Jones. "Exact solutions for Sturm-Liouville problems on trees via novel substitute systems and the Wittrick-Williams algorithm", Proceedings of the Royal Society series A: Mathematical, Physical and Engineering Sciences, 463(2088), 3195-3224 2007.