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Paper 87

Adaptive Model Reduction for Thermoelastic Problems

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Abstract

In this paper we present a discrete *a posteriori* error estimate for a thermoelastic model problem approximated using the component mode synthesis (CMS) model reduction method. The problem is one-way coupled in the sense that heat transfer affects the elastic deformation, but not vice versa. The error estimate bounds the difference between the reduced and the standard finite element solution in terms of discrete residuals and corresponding dual weights. A feature of the estimate is that it automatically gives a quantitative measure of the propagation of error between the thermal and elastic solvers with respect to a certain computational goal. We accompany the analytical results by a numerical example.

Keywords: model reduction, component mode synthesis, adaptivity, a posteriori, error estimation.

1 Introduction

Many important problems in industry are so called multiphysics problems which involve several different types of physics. One such problem is thermoelastic stress analysis where the objective is to predict the elastic strain of a material caused by heat flow in order to prevent structural failure. A common technique for simulating thermoelasticity is to connect two finite element solvers, one for heat transfer and one for elastic deformation, into a network where each physics is solved for and data exchanged between the solvers.

Component mode synthesis (CMS) [1, 2] is classical model reduction method based on the idea of domain decomposition. Here, basis functions associated with the subdomains and the interface in a partition of the computational domain are used as to construct reduced finite element models. The basis functions associated with the subdomains are given by the solution of a constrained eigenvalue problem associated with each of the subdomains, and coupling of the response in the substructures is achieved through the inclusion of so called static modes defined as the structural response to prescribed displacements on the interface.

A difficulty in model reduction is choosing the model parameters such that satisfactory overall accuracy is guaranteed. In CMS this amounts to choosing the number of modes in each subspace, e.g. cf. [3, 4, 5].

In this paper we present a method to automatically control the reduction error in both the thermal and elastic solver for a one-way coupled thermoelastic problem where each of the physics is approximated using CMS. The method is based on a posteriori error estimation for multiphysics problems, e.g. cf. [6, 7, 8], and the error estimate measures the difference between the reduced and the full finite element solution in a given quantity of interest. The results presented herein extends the results in [5] by allowing temperature dependendent elastic parameters, leading to a linearized thermal dual problem.

The rest of this paper is organized as follows: In section 2 we present the thermoelastic model problem; in section 3 we describe the component mode synthesis method; in section 4 we present the a posteriori error estimate and formulate an adaptive algorithm; finally, in section 5 we present some numerical experiments.

2 The Equations of Thermoelasticity

2.1 The Heat Transfer Problem

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, be a polygonal or polyhedral domain with boundary $\partial\Omega$ and outward unit normal $\boldsymbol{n}$ occupied by a a homogeneous isotropic material. Assume that the temperature T within Ω is described by the stationary heat equation

$$-\nabla \cdot (c\nabla T) = s, \qquad \text{in } \Omega, \tag{1a}$$

$$T = 0, \qquad \text{on } \Gamma_D^T, \tag{1b}$$

$$\boldsymbol{n} \cdot (c\nabla T) = \kappa(T - T_\infty), \quad \text{on } \Gamma_N^T, \tag{1c}$$

where c is the thermal conductivity coefficient, and s is any external heat source. The temperature T is assumed to vanish on the boundary segment Γ_D^T, whereas the temperature flux $\boldsymbol{n} \cdot (c\nabla T)$ is assumed to be proportional to the difference between T and the temperature of the ambient media T_∞ on the boundary segment Γ_N^T. The constant of proportionality κ is the heat permeability on the boundary.

2.2 The Linear Elastic Problem with a Thermal Strain

Given the temperature T, assume that the displacements $\boldsymbol{u}$ and stress tensor $\boldsymbol{\sigma}$ of the body are described by the static equations of linear elasticity with a thermal strain load

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{f}, && \text{in } \Omega, && \text{(2a)}\\
\boldsymbol{\sigma} &= 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + (\lambda\nabla\cdot\boldsymbol{u} - \beta(T)(T-T_0))\boldsymbol{I}, && \text{in } \Omega, && \text{(2b)}\\
\boldsymbol{u} &= \boldsymbol{0}, && \text{on } \Gamma_D^u, && \text{(2c)}\\
\boldsymbol{n}\cdot\boldsymbol{\sigma} &= \boldsymbol{0}, && \text{on } \Gamma_N^u, && \text{(2d)}
\end{aligned}
$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ is the linear strain tensor, $\boldsymbol{f}$ is any body force, $\boldsymbol{I}$ is the $d \times d$ identity matrix, and λ and μ are the Lamé parameters, which are given by $\lambda = E\nu/((1+\nu)(1-2\nu))$ and $\mu = E/(2(1+\nu))$, with $E = E(T)$ Young's elastic modulus and $\nu = \nu(T)$ Poisson's ratio. Further, T_0 is the temperature of the body in its stress free state, and $\beta(T) = (3\lambda + 2\mu)\alpha$ with $\alpha = \alpha(T)$ the thermal expansion coefficient.

2.3 Finite Element Approximation

Let $V^h \subset V = \{H^1(\Omega) : v|_{\Gamma_D^T} = 0\}$ be the standard finite element space of all continuous piecewise linear polynomials on a partition $\mathcal{K} = \{K\}$ of Ω into shape regular triangles or tetrahedra depending on the dimension d. Let further $\boldsymbol{V}^h \subset \boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^3 : \boldsymbol{v}|_{\Gamma_D^u} = \boldsymbol{0}\}$ be the space of all continuous piecewise linear d-dimensional vector polynomials.

The finite element approximation of (1) takes the form: find $T^h \in V^h$ such that

$$a_T(T^h, v) = l_T(v), \quad \forall v \in V^h, \tag{3}$$

where the linear forms $a_T(\cdot,\cdot)$ and $l_T(\cdot)$ are defined by

$$a_T(v, w) = (c\nabla v, \nabla w) + (\kappa v, w)_{\partial\Omega}, \tag{4}$$
$$l_T(v) = (s, v) + (\kappa T_\infty, v)_{\partial\Omega}, \tag{5}$$

respectively.

Similarly, the finite element approximation of (2) takes the form: find $\boldsymbol{U}^h \in \boldsymbol{V}^h$ such that

$$a_u(\boldsymbol{U}^h, \boldsymbol{v}) = l_u(T^h; \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}^h. \tag{6}$$

Here, the linear forms $a_u(\cdot,\cdot)$ and $l_u(T^h;\cdot)$ are defined by

$$a_u(\boldsymbol{w}, \boldsymbol{v}) = 2(\mu\boldsymbol{\varepsilon}(\boldsymbol{w}) : \boldsymbol{\varepsilon}(\boldsymbol{v})) + (\lambda\nabla\cdot\boldsymbol{w}, \nabla\cdot\boldsymbol{v}), \tag{7}$$
$$l_u(T^h; \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + (\beta(T^h)(T^h - T_0^h), \nabla\cdot\boldsymbol{v}), \tag{8}$$

where we have written $l_u(T^h;\cdot)$ to emphasize the dependence on the temperature T^h.

3 Component Mode Synthesis

Let $\mathcal{S} = \{\Omega_i\}_{i=1}^N$ be a partition of Ω into N subdomains Ω_i joined at the interface Γ, such that each $\Omega_i = \cup_{K\in\mathcal{K}_i} K$, for some subset $\mathcal{K}_i \subset \mathcal{K}$. An a_T-orthogonal decomposition

$$V^h = \bigoplus_{i=1}^N V_i^h, \tag{9}$$

of V^h may then be constructed by letting $V_i^h = \{v \in V^h : v|_{\Omega\setminus\Omega_i} = 0\}$, $i = 1,\dots,N$, and by letting

$$V_0^h = \{\mathcal{E}\nu \in V^h : \nu \in V^h|_\Gamma\}, \tag{10}$$

where $\mathcal{E}\nu \in V^h$ denotes the harmonic extension of a function $\nu \in V^h|_\Gamma$ to Ω, and $V^h|_\Gamma$ denotes the trace space of V^h associated with Γ.

Each subspace V_i^h, $i = 0,\dots,N$, is spanned by a basis of d_i, $i = 0,\dots,N$, eigenmodes obtained from the discrete eigenvalue problems: find $(\Lambda_i, Z_i) \in \mathbb{R} \times V_i^h$ for $i = 0,\dots,N$, such that

$$a_T(Z_i, v) = \Lambda_i(Z_i, v), \quad \forall v \in V_i^h, \quad i = 0,\dots,N. \tag{11}$$

To reduce V^h we let $V^{h,\boldsymbol{m}} \subset V^h$ be defined by

$$V^{h,\boldsymbol{m}} = \bigoplus_{i=0}^N V_i^{h,m_i}, \tag{12}$$

where $\boldsymbol{m} = (m_i)_{i=0}^N$, $1 \le m_i \le d_i$, is a multi-index, and

$$V_i^{h,m_i} = \mathrm{span}\{Z_{i,j}\}_{j=1}^{m_i}, \quad i = 0,\dots,N. \tag{13}$$

An a_u-orthogonal decomposition of $\boldsymbol{V}^h$, and a reduced space $\boldsymbol{V}^{h,\boldsymbol{n}} \subset \boldsymbol{V}^h$, where $\boldsymbol{n} = (n_i)_{i=0}^n$, can be similarly constructed.

3.1 The Reduced Problem

Introducing the spaces $V^{h,\boldsymbol{m}}$ and $\boldsymbol{V}^{h,\boldsymbol{n}}$ in the thermoelastic model we get the following reduced problem: find $T^{\boldsymbol{m}} \in V^{h,\boldsymbol{m}}$ such that

$$a_T(T^{\boldsymbol{m}}, v) = l_T(v), \quad \forall v \in V^{h,\boldsymbol{m}}, \tag{14}$$

and find $\boldsymbol{U}^{\boldsymbol{n}} \in \boldsymbol{V}^{h,\boldsymbol{n}}$ such that

$$a_u(\boldsymbol{U}^{\boldsymbol{n}}, \boldsymbol{v}) = l_u(T^{\boldsymbol{m}}; \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}^{h,\boldsymbol{n}}. \tag{15}$$

We wish to estimate the discrete error $\boldsymbol{E} = \boldsymbol{U}^h - \boldsymbol{U}^{\boldsymbol{n}}$ in the reduced problem (15). In doing so, we are lead to a posteriori error estimates, which account for the reduction error in both the thermal and elastic models.

4 An A Posteriori Error Estimate

Let the operators $\mathcal{R}_i : V^h \to V_i^h$, $i = 0, \ldots, n$, denote Ritz projectors defined by $a_T(w - \mathcal{R}_i w, v) = 0$, $\forall v \in V_i^h$, the operators $\mathcal{P}_i^{m_i} : V^h \to V_i^{h,m_i}$, $i = 0, \ldots, n$, denote Fourier expansion in the space V_i^{h,m_i}, i.e. $\mathcal{P}_i^{m_i} w = \sum_{j=1}^{m_i} (w, Z_{i,j}) Z_{i,j}$. The same conventions also apply for operators on subspaces in $\boldsymbol{V}^h$. We further define the discrete subspace residuals $R_i^h(w) \in V_i^h$ and $\boldsymbol{R}_i^h(\boldsymbol{w}) \in \boldsymbol{V}_i^h$ respectively, for $w \in V^h$ by

$$(R_i^h(w), v) = l_T(v) - a_T(w, v), \quad \forall v \in V_i^h, \tag{16}$$

and for $\boldsymbol{w} \in \boldsymbol{V}^h$ by

$$(\boldsymbol{R}_i^h(\boldsymbol{w}), \boldsymbol{v}) = l_u(\boldsymbol{v}) - a_u(\boldsymbol{w}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_i^h. \tag{17}$$

The following error representation formula holds

$$\begin{aligned} m_u(\boldsymbol{U}^h - \boldsymbol{U}^{\boldsymbol{n}}) = l_u(T^{\boldsymbol{m}}; \boldsymbol{\phi}_u - \mathcal{P}^{\boldsymbol{n}} \boldsymbol{\phi}_u) - a_u(\boldsymbol{U}^{\boldsymbol{n}}, \boldsymbol{\phi}_u - \mathcal{P}^{\boldsymbol{n}} \boldsymbol{\phi}_u) \\ + m_T(T^h) - m_T(T^{\boldsymbol{m}}), \end{aligned} \tag{18}$$

where $m_u(\cdot)$ is a linear functional representing the goal of the computation, $\boldsymbol{\phi}_u \in \boldsymbol{V}^h$ is the solution to the associated dual problem

$$m_u(\boldsymbol{v}) = a_u(\boldsymbol{v}, \boldsymbol{\phi}_u), \quad \forall v \in \boldsymbol{V}^h, \tag{19}$$

and $m_T(T^h) - m_T(T^{\boldsymbol{m}})$ is the so-called modeling error, which accounts for the effect of the discrete error in the temperature on the elastic equations. The modeling error takes the explicit form

$$\begin{aligned} m_T(T^h) - m_T(T^{\boldsymbol{m}}) &= l_u(T^h; \boldsymbol{\phi}_u) - l_u(T^{\boldsymbol{m}}; \boldsymbol{\phi}_u) && (20) \\ &= (\beta(T^h) T^h - \beta(T^{\boldsymbol{m}}) T^{\boldsymbol{m}}, \nabla \cdot \boldsymbol{\phi}_u). && (21) \end{aligned}$$

Making a Taylor expansion of $\beta(T)$ around $T = T^{\boldsymbol{m}}$, assuming $T^h \approx T^{\boldsymbol{m}}$, yields

$$\begin{aligned} \beta(T^h) T^h - \beta(T^{\boldsymbol{m}}) T^{\boldsymbol{m}} &= \beta(T^{\boldsymbol{m}}) T^h + \beta'(T^{\boldsymbol{m}})(T^h - T^{\boldsymbol{m}}) T^h && (22) \\ &\quad + \mathcal{O}(|T^h - T^{\boldsymbol{m}}|^2) - \beta(T^{\boldsymbol{m}}) T^{\boldsymbol{m}} \\ &= \beta(T^{\boldsymbol{m}})(T^h - T^{\boldsymbol{m}}) + \beta'(T^{\boldsymbol{m}}) T^{\boldsymbol{m}} (T^h - T^{\boldsymbol{m}}) && (23) \\ &\quad + \mathcal{O}(|T^h - T^{\boldsymbol{m}}|^2) \end{aligned}$$

Introducing the linearized modeling error

$$\bar{m}_T(T^h - T^{\boldsymbol{m}}) = ((\beta(T^{\boldsymbol{m}}) + \beta'(T^{\boldsymbol{m}}) T^{\boldsymbol{m}})(T^h - T^{\boldsymbol{m}}), \nabla \cdot \boldsymbol{\phi}_u) \tag{24}$$

and the linearized thermal dual problem: find $\phi_T \in V^h$, such that

$$\bar{m}_T(v) = a_T(v, \phi_T), \quad \forall v \in V^h, \tag{25}$$

where

$$\bar{m}_T(v) = ((\beta(T^{\boldsymbol{m}}) + \beta'(T^{\boldsymbol{m}})T^{\boldsymbol{m}})v, \nabla \cdot \boldsymbol{\phi}_u), \tag{26}$$

it can be shown that the following a posteriori error estimate holds

$$\begin{aligned} |m_u(\boldsymbol{U} - \boldsymbol{U}^{\boldsymbol{n}})| \leq \sum_{i=0}^{N} & \left|(R_i^h(T^{\boldsymbol{m}}), \mathcal{R}_i\phi_T - \mathcal{P}_i^{m_i}\mathcal{R}_i\phi_T)\right| \\ & + \left|(\boldsymbol{R}_i^h(\boldsymbol{U}^{\boldsymbol{n}}), \mathcal{R}_i\boldsymbol{\phi}_u - \mathcal{P}_i^{n_i}\mathcal{R}_i\boldsymbol{\phi}_u)\right|. \end{aligned} \tag{27}$$

4.1 Adaptive Algorithm

An adaptive algorithm that automatically refines the subspaces V_i^{h,m_i}, and $\boldsymbol{V}_i^{h,n_i}$ to increase the accuracy with respect to the goal functional $m_u(\cdot)$ is outlined below:

1. Begin with a starting guess of the subspace dimensions $\boldsymbol{m}$ and $\boldsymbol{n}$.

2. Solve equations (14) and (15) for $T^{\boldsymbol{m}}$ and $\boldsymbol{U}^{\boldsymbol{n}}$.

3. Compute the subspace indicators

   $$\eta_{T,i} = \left|(R_i^h(T^{\boldsymbol{m}}), \mathcal{R}_i\phi_T - \mathcal{P}_i^{m_i}\mathcal{R}_i\phi_T)\right| \tag{28}$$

   and

   $$\eta_{u,i} = \left|(\boldsymbol{R}_i^h(\boldsymbol{U}^{\boldsymbol{n}}), \mathcal{R}_i\boldsymbol{\phi}_u - \mathcal{P}_i^{n_i}\mathcal{R}_i\boldsymbol{\phi}_u)\right|. \tag{29}$$

4. Increase the number of modes in the corresponding subspaces according to a predetermined refinement strategy.

5. Repeat until satisfactory results are obtained.

5 Numerical Experiment

As our numerical example we study the static deformation of a micro electro mechanical system (MEMS) called a microspring thermal actuator [9], see Figure 1. This device consists of several chevron structures linked together to form a spring like system. The actuator is connected to an electrical source through two contact pads. Joule heating is produced in the material when voltage is applied to the contact pads, and thermal stresses makes the actuator expand. Insulating beams with low thermal expansion coefficient are included to constrain the motion to one dimension.

This problem may be modeled by including an equation for the electric potential ϕ_E in the structure, in addition to the heat and elastic equations. We consider the electrostatic case where the potential ϕ_E is given by

$$-\nabla \cdot (\sigma \nabla \phi_E) = 0, \quad x \in \Omega, \tag{30}$$

$$\phi_E = g_E, \quad x \in \Gamma_D^{\phi}, \tag{31}$$

$$\boldsymbol{n} \cdot \nabla \phi_E = 0, \quad x \in \Gamma_N^{\phi}, \tag{32}$$

where σ is the electric conductivity, and g is a prescribed voltage on the boundary Γ_D^{ϕ}. The potential is coupled to the heat equation through a heat source s in (1) of the form $s = \sigma|\nabla \phi_E|^2$.

Voltage of $g_E = \pm 1$ is applied to the boundary of the contact pads which is assumed to be fixed, $\boldsymbol{u} = \boldsymbol{0}$, to the surrounding substrate. It is also assumed that the fixed boundary has temperature $T = 0$, and that the rest of the boundary is thermally insulated, $\kappa = 0$, and stress free $\boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{0}$. We further assume that the insulating beams have electric conductivity $\sigma = 10^{-12}$ and that $\sigma = 1$ in the rest of the structure. Finally, the elastic properties of the structure are defined by $E = 1$, and $\nu = 0.3$. Further, $\alpha = 10^{-6}$ for the insulating beams, and $\alpha(T) = \arctan(20) + \arctan(T - 20)$ in the rest of the structure, see Figure 2. We remark that the properties of the structure are arbitrarily chosen and do not reflect the actual properties of a microspring thermal actuator.

The simplified computational domain, partitioned into subdomains, can be seen in Figure 3. We begin by obtaining a discrete electrostatic potential Φ_E by solving the full associated finite element problem. Since we are using the full finite element model, the discrete modeling error associated with the potential equation is zero. Using the heat source $S = \sigma|\nabla \Phi_E|^2$ as in-data to the heat equation we use the adaptive algorithm to obtain reduced solutions $T^{\boldsymbol{m}}$ and $\boldsymbol{U}^{\boldsymbol{n}}$. With the goal of computing the displacements in the x_2 direction accurately near the point $(0, -2.85)$, the goal functional is chosen as $m_u(\boldsymbol{v}) = (\boldsymbol{g}, \boldsymbol{v})$, where $\boldsymbol{g} = [0, -\exp(-50r^2)]^T$, with $r = \sqrt{x_1^2 + (-2.85 - x_2)^2}$.

Iterating five times using the refinement rule: add 10 modes to the basis in each subspace V_i^{h,m_i} and $\boldsymbol{V}_i^{h,n_i}$, where the subspace indicators $\eta_{T,i}$ and $\eta_{u,i}$ satisfies

$$\eta_{T,i} > 0.5 \max_i \eta_{T,i}, \tag{33}$$

and

$$\eta_{u,i} > 0.5 \max_i \eta_{u,i}, \tag{34}$$

where $\eta_{T,i}$ and $\eta_{u,i}$ where defined in (28) and (29), respectively, we obtain the primal temperature and displacement seen in Figure 5 and Figure 6. For simplicity we have solved the dual equations using the full finite element model, in practice a richer reduced model may be used to approximate the dual problems. The obtained dual solutions may be seen in figures 7 and 8.

The absolute value of the discrete error in the goal functional $|m_u(\boldsymbol{U}^h - \boldsymbol{U}^{\boldsymbol{n}})|$ together with the estimated error as the adaptation proceeds can be seen in Figure 4.

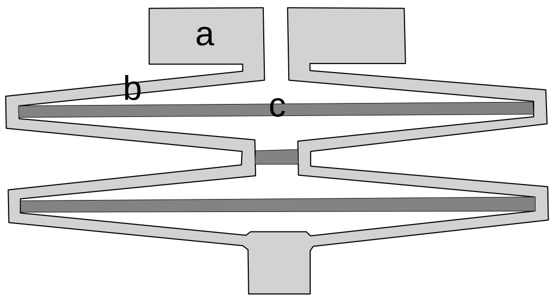

Figure 1: Microspring thermal actuator with (a) contact pads, (b) chevron structures, and (c) insulating beams.

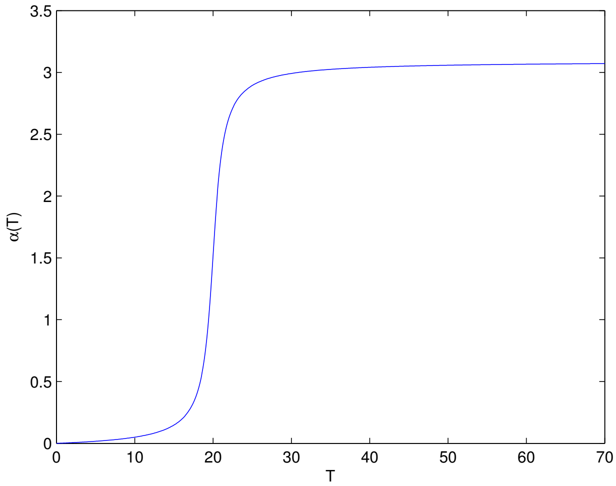

Figure 2: The thermal expansion coefficient $\alpha(T) = \arctan(20) + \arctan(T - 20)$.

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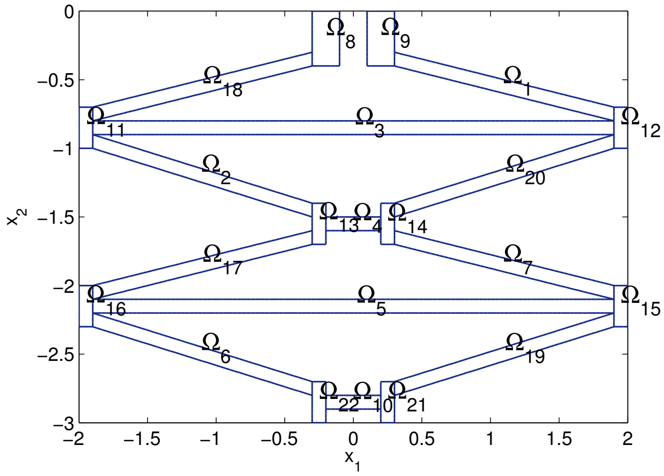

Figure 3: The computational domain Ω partitioned into non-overlapping subdomains Ω_i interfacing a Γ.

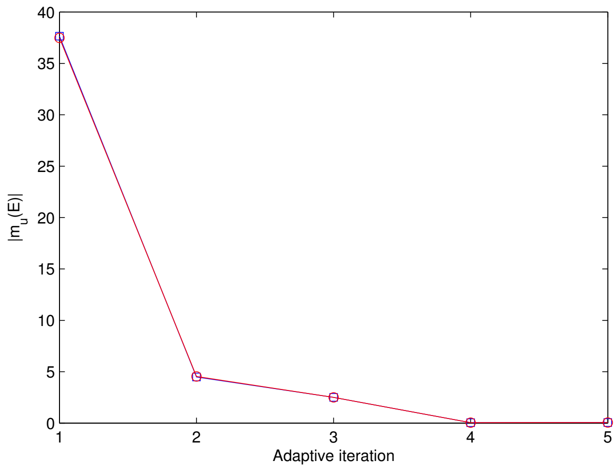

Figure 4: Absolute value of the error in the goal functional and the estimate as the adaptation proceeds. Legend: circle, estimate; square, error.

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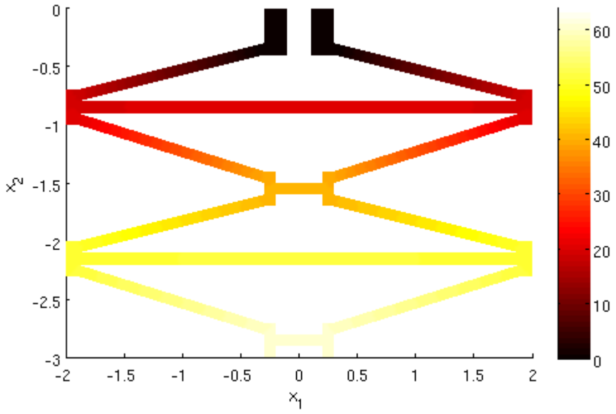

Figure 5: Primal temperature

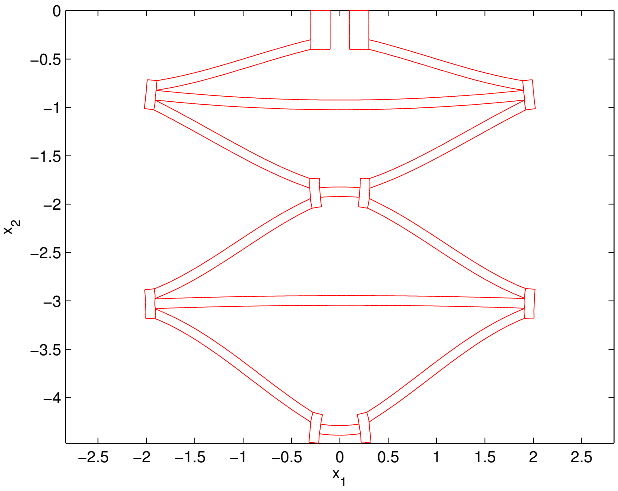

Figure 6: Primal displacements

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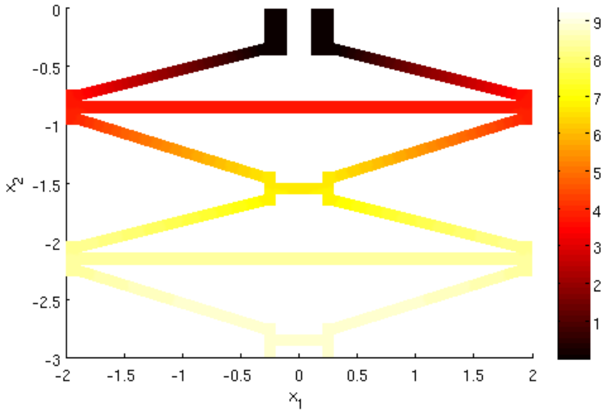

Figure 7: Dual temperature

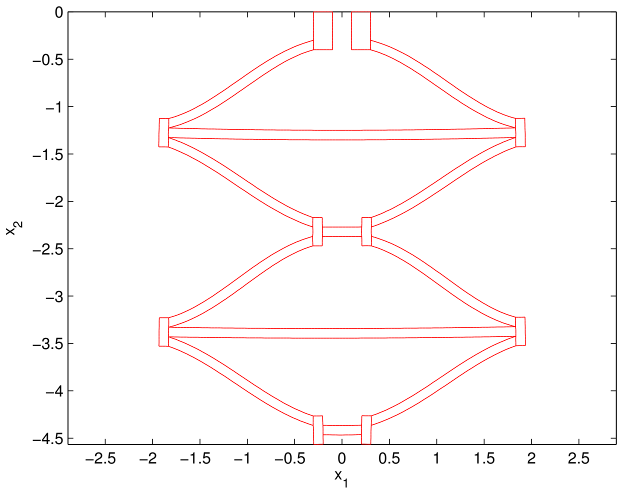

Figure 8: Dual displacements