Paper 11

©Civil-Comp Press, 2012
Proceedings of the Eighth International Conference
on Engineering Computational Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

The Analysis of Porous Media using the Mixed Finite Element Method and the FETI Method

K. Lee, M. Tak and T. Park
Department of Civil and Environmental Engineering
Hanyang University, Seoul, South Korea

Abstract

Porous media analysis is very difficult to accomplish using general solid mechanics theory, because saturated porous media is composed of different materials. Therefore, it is necessary to use porous theory considering solid mechanics and the fluid flow theory. Recently, in order to resolve this problem, the staggered method was proposed by Park and Tak [14]. However, the numerical efficiency is decreased because the iteration step in the staggered method is performed more than one time. Thus, in order to improve the numerical efficiency, a combined method is proposed by combining the finite element tearing and interconnecting (FETI) method with the staggered method for the efficient analysis of saturated porous media.

Keywords: porous media, staggered method, finite element method, parallel analysis, domain decomposition, FETI.

1 Introduction

The saturated porous media are composed of solid and fluid phases. Solid and fluid have different properties, so it is difficult to analyse by the continuum theory, therefore, it is necessary to use porous theory considering solid mechanics and fluid flow. Theoretical formulation for the porous media was introduced by Biot in 1941. Biot proposed three dimensional governing equations by Poroelasticity that applied continuum mechanics to Soil Consolidation Theory by Terzaghi [1]. In order to numerical analysis model using Biot's governing equations to be considered behavior of solid and fluid, the finite element method using hybrid element has been studied by many researchers. The mixed formulations based upon a variational theorem were firstly introduced by Hermann [2] to solve a plate bending. In addition, the mixed formulation was extended by Ghaboussi and Wilson [3], Zienkiewicz et al. [4], Borja [5], Voyiadjis and Abu-Farsakh [6], Park et al. [7, 8, 9] in the soil scope for the instantaneous or consolidation analysis. The governing equations are represented into a mixed form in terms of displacement and pore pressure formulation. However, the mixed finite element analysis cannot be solved easily by direct solution because the continuity requirements for the shape functions for each constituent are different. Especially, if solid and fluid are nearly incompressible and the fluid is nearly impermeable, then solving for the equations becomes a difficult task due to element locking. Indeed, finite element should be satisfied Baduska-Brezzi Condition [10, 11, 12] or pass to the patch test in which different high order interpolation is required for an element. In order to solve this problem, staggered method is proposed by Park and Tak [13, 14]. The nearly incompressible and impermeable saturated porous media using equal order element and staggered method. This result was much close to one dimensional and two dimensional consolidation theory of value. But Numerical efficiency is reduced because staggered method performs iteration more than one time. Thus, we, in order to improve the numerical efficiency, propose the method through graft FETI method onto staggered method for saturated porous media.

The FETI method is an effective domain decomposition method for parallel analysis that was introduced by C. Farhat [15]. FETI method is requires fewer interprocessor communications than traditional domain decomposition algorithms, while it still offer the same amount of parallelism. Typically, the spatial domain is decomposed onto a set of subdomains and each of these is assigned to an individual processor. Moreover, subdomains analyses are progressed by direct and iterative method. Continuities of subdomains are defined by Lagrange multiplier of among the subdomains. Here is how to obtain Lagrange multiplier using PCPG (Preconditioned Conjugate Projected Gradient) algorithm. In this paper, we present through combine staggered method with FETI method implemented analysis of porous media and then, the proposed method is verified numerical efficiency by point to point MPI library.

2 Governing Equation

The governing equations derived by Biot [16], the momentum balance equation for displacement and the mass balance equation for pore pressure, that are introduced here under the linear flow and the linear elastic behavior. In particular, the relative velocity concept and Biot's constant are used here in the macroscopic sense.

2.1 Mixed Finite Element Formulation for Porous Media

Mass balance equation for porous media is represented fluid flow in the porous media using Darcy's law and material time derivative. Momentum balance equation is expressed as displacement in the solid phase by the relative equation of Cauchy stress tensor and pore pressure represented deformation of porous media through constitutive equation. For the finite element analysis, initial condition and boundary condition for displacement and fluid flow are applied to the mass balance equation and the momentum balance equation, respectively. Then transformation weak form is represented as follows;

$$\mathbf{C}_2 \frac{\partial \tilde{\mathrm{u}}^s}{\partial t} + \mathbf{S}\frac{\partial \tilde{\mathrm{p}}^f}{\partial t} + \mathbf{D}\tilde{p}^f = \mathbf{f}^f \tag{1}$$

$$\mathbf{K}_{\mathrm{t}}\tilde{\mathrm{u}}^s - \mathbf{C}_1\tilde{\mathrm{p}}^f = \mathbf{f}^s \tag{2}$$

where, $\mathbf{K_t}$ is stiffness matrix, $\mathbf{D}$ is drainable matrix, $\mathbf{S}$ is compressible matrix and $\mathbf{C}$ is coupled matrix.

Then, above equations are applied to the Backward Difference Method for time analysis. The displacement and pore pressure are given by;

$$\mathbf{u}^s{}_{n+1}(t_1) = \mathbf{K}^{-1}{}_{T,n+1}\{\mathbf{f}^s{}_{n+1} + \mathbf{C}_{1,\mathrm{n+1}}\mathrm{p}^f{}_{n+1}(t_2)\} \tag{3}$$

$$\begin{aligned}\mathrm{p}^f{}_{n+1}(t_2) = &\left[\mathbf{D}_{\mathrm{n+1}} + \frac{1}{\Delta t_{2,n+1}}\mathbf{S}_{\mathrm{n+1}}\right]^{-1}\left[\mathbf{f}^f{}_{n+1} + \frac{1}{\Delta t_{2,n1+}}\mathbf{S}_{\mathrm{n+1}}\mathrm{p}^f{}_n\right. \\ &\left. + \frac{1}{\Delta t_{1,n+1}}\left(\mathbf{C}_{2,\mathrm{n+1}}(\mathbf{u}^s{}_n - \mathbf{u}^s{}_{n+1})\right)\right]\end{aligned} \tag{4}$$

2.2 Staggered Method for Porous Media

Saturated porous media analysis is considered to be of different properties of solid and fluid phases. Thus, for the analysis equation (1) and (2) should be used iterative method or direct method. But the shape function for continuity problems of solid and fluid should be resolved in the coupling matrices $\mathbf{C_1}, \mathbf{C_2}$. To solve the above problems, element order must be satisfied with the Babuska-Brezzi conditions or patch test.

It is difficult to secure convergences in state variables such as displacements and pore pressures for the transient analysis of porous media that is nearly incompressible and fluid flow that is very slow which is termed nearly impermeable. In order word, when the compressible matrix $\mathbf{S}$ and the drainable matrix $\mathbf{D}$ have very small values, a standard implicit process using the Gaussian elimination procedure cannot be solved by the simple time step analysis. Therefore, partitioned analyses such as explicit-explicit and explicit-implicit are used. However, the restrictions on the determination of the tome step are quite severe. In order to remedy these problems, Park and Tak [13, 14] proposed Staggered method with multi time step method, remeshing scheme and sub-iteration. At the multi time step and the sub-iteration step, compatibility and the convergence problems are solved with equal order shape functions. Namely, multi time step, remeshing and sub- iteration is applied to Eq. (4) that is expressed as follows;

$$\begin{aligned}\tilde{\mathrm{p}}^f{}_{n+(i+2)}(t_2) = &\left[\mathbf{D}_{n+(i+2)}\frac{1}{\Delta t_{2,n+(i+2)}}\mathbf{S}_{n+(i+2)}\right]^{-1}\left[\mathbf{f}^f{}_{n+(i+2)}\right. \\ &+ \frac{1}{\Delta t_{1,n+(i+2)}}\left(\mathbf{C}_{2,n+(i+2)}(\tilde{\mathrm{u}}^s{}_n(t_1) - \tilde{\mathrm{u}}^s{}_{n+(i+1)}(t_1)\right. \\ &\left. + \tilde{\mathrm{u}}^s{}_n(t_1) - \tilde{\mathrm{u}}^s{}_{n+(i+2)}(t_1))\right) \\ &\left. + \frac{1}{\Delta t_{2,n+(i+2)}}\mathbf{S}_{n+(i+2)}\tilde{\mathbf{p}}^f{}_{n+(i+1)}(t_2)\right]\end{aligned} \tag{5}$$

where, subscript i is defined iteration number.

3 Finite Element Tearing and Interconnecting (FETI) Method

FETI Method is proposed by C. Farhat [15] that is domain decomposition method for a parallel finite element computational method for the solution of static problems. FETI method requires fewer interprocessor communications than traditional domain decomposition algorithms, while it still offers the same amount of parallelism. Traditional domain decomposition is realized directly about continuity in subdomain boundary. But FETI is solved using Lagrange multiplier.

The FETI method derivation is the subdomain energy J^{s};

$$J^s = \mathbf{u}^{\mathrm{s}^{\mathrm{T}}}\mathbf{f}^{\mathrm{s}} - \frac{1}{2}\mathbf{u}^{\mathrm{s}^{\mathrm{T}}}\mathbf{K}^{\mathrm{s}}\mathbf{u}^{\mathrm{s}} \tag{6}$$

where, superscript s denotes the number of subdomain, $\mathbf{u}^{\mathrm{s}}$ and $\mathbf{f}^{\mathbf{s}}$ are the displacements and force vectors of subdomain and $\mathbf{K}^{\mathrm{s}}$ is stiffness matrix.

In order for the subdomain displacement to yield the required global displacement, $\mathbf{u}^{\mathbf{s}}$ must satisfy the interface condition;

$$\sum_{s=1}^{n_s} \mathbf{B}^s\mathbf{u}^s = 0 \tag{7}$$

where, $\mathbf{B}^{\mathrm{s}}$ connectivity matrix, the signed matrix with entries -1, 0, 1 describing the subdomain interconnectivity. The energy expression for the global domain is just the sum of subdomain level energy with the constraint condition Eq.(7) augment via Lagrange multiplier as shown below;

$$J_{total} = \sum_{s=1}^{n_s} J^s - \lambda^T{}_b \sum_{s=1}^{n_s} \mathbf{B}^s\mathbf{u}^s \tag{8}$$

It is observed that the constraint equation (7) produces no work. The stationary of (8) yield the following subdomain level governing equation:

$$\mathbf{K}^{\mathrm{s}}\mathbf{u}^{\mathrm{s}} = \mathbf{f}^{\mathrm{s}} - \mathbf{B}^{\mathrm{s}^{\mathrm{T}}}\lambda(s = 1, \ldots., n) \tag{9}$$

Observe that, Eq. (9) is strictly local for each subdomain provided λ is known, the constraint equation (7) extends over several subdomains that are connected. Nevertheless, the subdomain displacement vector $\mathbf{u}^{\mathrm{s}}$ is of local nature.

The differentially partitioned FETI method begins with the solution of $\mathbf{u}^{\mathrm{s}}$ from Eq. (9)

$$\mathbf{u}^{s} = \mathbf{K}^{s+}\left(\mathbf{f}^{s} - \mathbf{B}^{s^{T}}\lambda\right) - \mathbf{R}^{s}\alpha^{s} \tag{10}$$

where, $\mathbf{K}^{s+}$ is an inverse of K^{s} and R^{s} is the null space matrix. Null space matrix $\mathbf{R}^{s}$ is satisfying as below;

$$\mathbf{R}^{s^{T}}\mathbf{K}^{s}\mathbf{u}^{s} = 0 \tag{11}$$

α^{s} is the complementary displacement vector that accounts for the rigid body motions for floating subdomain. Thus, the solution of $\mathbf{u}^{s}$ by the FETI method is reduced to the solution of the interface force λ and the complementary displacement vector α^{s}.

In order to obtain the appropriate equation for the λ and α^{s}, substitute Eq. (9) into Eq. (11) and Eq. (10) into (7) to arrive at

$$\begin{bmatrix} \mathbf{F} & -\mathbf{G} \\ -\mathbf{G}^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} \lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} \mathrm{d} \\ -\mathrm{e} \end{bmatrix} \tag{12}$$

where,

$$\mathbf{F} = \sum_{s=1}^{n_s} \mathbf{B}^{s}\mathbf{K}^{-1}\mathbf{B}^{s^{T}}$$
$$\mathbf{G} = [\mathbf{B}^{1}\mathbf{R}^{1} \dots\dots \mathbf{B}^{n_s}\mathbf{R}^{n_s}]$$
$$\alpha = \left[\alpha^{1^{T}} \dots\dots \alpha^{n_s^{T}}\right]$$
$$\mathrm{d} = \sum_{s=1}^{n_s} \mathbf{B}^{s}\mathbf{K}^{-1}\mathbf{f}^{s}$$
$$\mathrm{e} = \left[\mathbf{f}^{1^{T}}\mathbf{R}^{1} \dots\dots \mathbf{f}^{n_s^{T}}\mathbf{R}^{n_s}\right]^{T}$$

4 FETI method within Staggered method

In order to improve for the numerical efficiency of Staggered method, we performed Staggered method using FETI method applied to both solid and fluid phase, in this paper. FETI method is applied to Staggered method, vector **d** and **e** is represented in the FETI interface problem as below;

$$\mathbf{d} = \sum_{s=1}^{n_s} \mathbf{B}^{s}\mathbf{K}^{-1}(\mathbf{f}^{s} + \mathrm{C}\mathrm{p}^{f}) \tag{14}$$

$$\mathbf{e} = \left[\left(\mathbf{f}^{1^{T}} + \mathrm{C}^{1}\mathrm{p}^{f^{1}}\right)\mathbf{R}^{1} \dots \left(\mathbf{f}^{n_s^{T}} + \mathrm{C}^{1}\mathrm{p}^{f^{1}}\right)\mathbf{R}^{n_s}\right]^{T} \tag{15}$$

Also, vector **d** and **e** is expressed in the FETI interface problem for the fluid phase as follows;

$$d = \sum_{s=1}^{n_s} \mathbf{B}^s \left[\mathbf{D} + \frac{1}{\Delta t}\mathbf{S}\right]^{-1} \left(\mathbf{f}^s + \frac{1}{\Delta t}\mathbf{S}p^f + \frac{1}{\Delta t}\left(\mathbf{C}_2(u^s{}_n - u^s{}_{n+1})\right)\right) \tag{16}$$

$$e = \left[\left(\mathbf{f}^{1^T} + C^1 \frac{1}{\Delta t}\mathbf{S}p^f + \frac{1}{\Delta t}\left(\mathbf{C}_2(u^s{}_n - u^s{}_{n+1})\right)^{f^1}\right)\mathbf{R}^1 \ldots \left(\mathbf{f}^{n_s{}^T} + C^1 \frac{1}{\Delta t}\mathbf{S}p^f + \frac{1}{\Delta t}\left(\mathbf{C}_2(u^s{}_n - u^s{}_{n+1})\right)^{f^1}\right)\mathbf{R}^{n_s}\right]^T \tag{17}$$

FETI method applied to Staggered method for all phase is used parallel analysis using MPI library

5 Numerical Verification

In order to improve the numerical efficiency, FETI method applied to Staggered Method. For the verified numerical efficiency of the above method, we are used to finite element analysis program and decomposed 2-dimensional porous media (Figure 1) and porous media has conditions of the permeable and compressible. Also in order to the verified accuracy, we are compared with pore pressure of Staggered method and pore pressure of applied FETI method. In this paper, for the numerical verification of parallel analysis for porous media, it is used for Linux cluster of the Nehalem CPU (2.4GHz, 8Core) of 54 nodes (432 cores) and QRD transmission speed network. Also, MPI communication for parallel analysis is used library that is offered MPICH2 [17] and GCC complier.

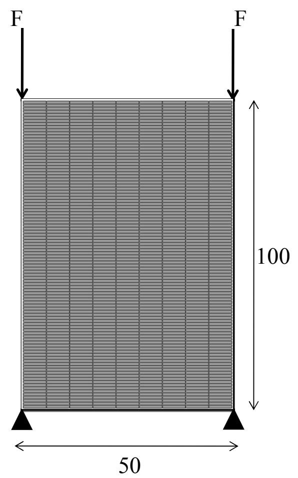


Figure 1. Porous Media for 2-Dimensional

Material Properties	
E	3.0×10^6 Pa
$\boldsymbol{\nu}$	0.25
$\boldsymbol{\rho}$	1000 Kg/m^3
n	0.3
$\boldsymbol{\alpha}$	1
F	100kN

Table 2. Material Properties

In this numerical verification, used to porous media for width of 50cm, height of 100cm. Porous media is assumed to fully saturated and 4 node linear behaviour.

In the numerical test, two dimensional equal order elements of 100, 300, 500, 700, 900 are used to porous media analysis. Also, elements are divided to one of subdomain from 10 of subdomains. In order to express numerical efficiency, solid and fluid phases running time in porous media are represented by each element. Fig. 2 is represented Staggered method running time for number of subdomain. As the number of subdomain is increasing, numerical efficiency is improve. But after the 4 subdomain, numerical efficiency is slight difference.

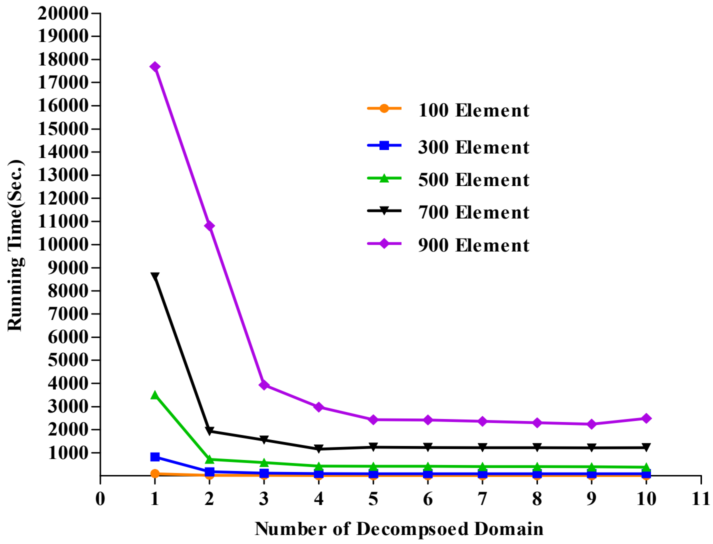


Figure 2. Total Running Time of Subdomain

In order to obtain the parallel analysis, we are compared with Speedup and Efficiency. Speedup is value of comparison with parallel algorithm and serial algorithm. Generally, Speedup is expressed to computation algorithm using one of CPU more than faster computation algorithm using number of CPUs. Speedup is defined as follows;

$$S_p = \frac{T_1}{T_p} \tag{18}$$

where, p is number of processes, T_1 is running time using one process and T_p is running time using pth processes. Ideal Speedup is S_p and p is equal that is defined Linear Speedup. Numerical analysis result in this section is presented about Speedup as follows (Fig. 3);

Efficiency is defined the waste time in the communication and synchronization. Generally Efficiency value is represented to 0 from 1 and represented as follows (Fig 3);

$$E_p = \frac{T_1}{pT_p} \tag{19}$$

Efficiency of Linear Speedup and algorithm using one CPU is obtained 1. And increasing CPU numbers, Efficiency is represented like $\frac{1}{\log p}$.

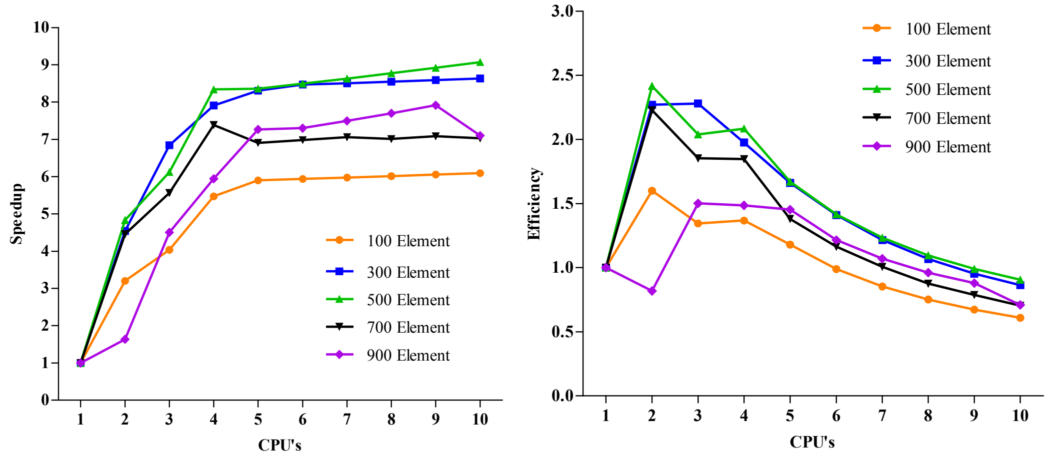
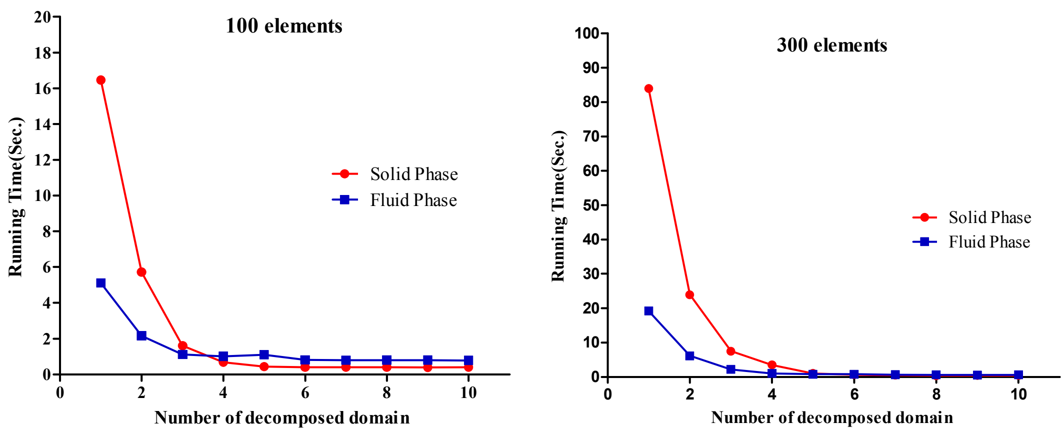


Figure 3. Speedup and Efficiency by Number of Subdomain

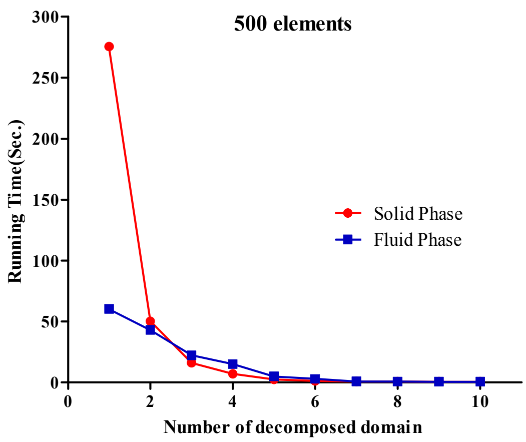
In order to verify numerical efficiency, numerical analysis is performed about 100, 300, 500, 700, 900 element, respectively. Displacement and pore pressure running time is represented as follows;

As shown in Fig. 4, increasing number of subdomains, solid and fluid phases running times are decreased. Especially running time of pore pressure is lower than running time of displacement, thus pore pressure running time is almost uniformed with increasing subdomains.

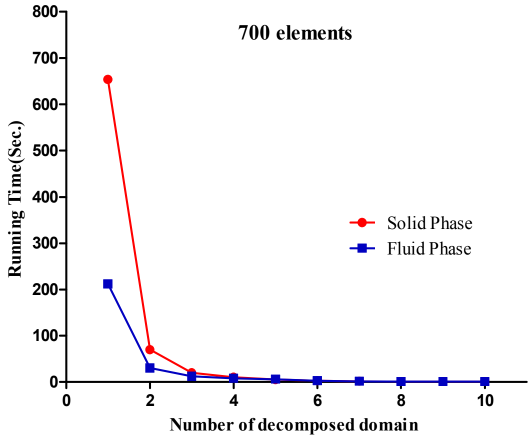


(a) 100 Elements (b) 300 Element

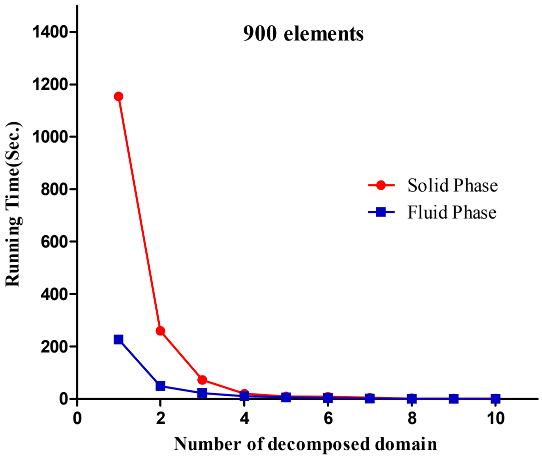
Figure 4. Running Time by the Number of Decomposed Domain



(c) 500 Elements



(d) 700 Elements



(e) 900 Elements

Figure 4. (Continued) Running Time by the Number of Decomposed Domain

6 Summary and Conclusion

Saturated porous media is consisting of solid and fluid. Thus porous media analysis is very difficult because solid and fluid properties are different. In order to overcome these problems, the staggered method is proposed for saturated porous media analysis. The staggered method for porous media is numerically accurate, but the numerical efficiency is low because iteration uses remeshing and sub-iteration. Thus, in this paper, a combination of the staggered method and the FETI method is performed for the purpose to increasing the numerical efficiency. First, the FETI method is applied to the solid phase and the fluid phase in porous media. In this paper, combined method obtained a satisfactory result. The proposed method is expected to be efficient for large and complex models. Also, for the optimization of the analysis, further research will be undertaken about the load balancing. If load balancing is applied to the proposed method, then this method will provide a very effective analysis without wasting computational time.

Acknowledgement

This work was financially supported by the National R&D project of the "Development of Energy utilization technology with Deep Ocean Water" supported by the Korean Ministry of Land, Transport and Maritime Affairs.

References

[1] K. Terzaghi, “Principles of Soil Mechanics”, Engineering News-Record. 1925.

[2] L. R. Herrmann, “Finite element bending analysis for plates”, Journal of The Engineering Mechanics Division, ASCE, 93(1), 13~26, 1967.

[3] J. Ghaboussi, and E. L. Wilson, “Flow of compressible fluid in porous elastic media, International Journal for Numerical Methods in Engineering, 5(3), 419~442, 1973.

[4] O.C. Zienkiewicz, C.T. Chang, and P. Bettess, “Drained, Undrained, Consolidating and Dynamic Behavior Assumptions in Soils”, Geothchnique, 30(4), 385~395, 1980.

[5] M. Borja, “Finite Element Formulation for Transient Pore Pressure Dissipation: A Variational Approach”, International Journal of Solid and Structures, 22(11), 1201~1211, 1986.

[6] G.Z. Voyiadjis, and M.Y. Abu-Farsakh, “Coupled Theory of Mixtures for Clayey Soils”, Journal of Computers and Geotechnics, 20(3/4), 195~222, 1997.

[7] T. Park, M. Tak, and H. Kim, “Analysis of Saturated Porous Media Using Arbitrary Lagrangian Eulerian Method: I. Theoretical Formulation”, KSCE Journal of Civil Engineering, KSCE, 9(3), pp.233~239, 2005.

[8] T. Park, M. Tak, and H. Kim, “Analysis of Saturated Porous Media Using Arbitrary Lagrangian Eulerian Method: II. Finite Element Formulation”, KSCE Journal of Civil Engineering, KSCE, 9(3), 241~246, 2005.

[9] T. Park, M. Tak, and H. Kim, “Analysis of Saturated Porous Media Using Arbitrary Lagrangian Eulerian Method: III. Numerical Examples”, KSCE Journal of Civil Engineering, KSCE, 9(3), pp.247~254, 2005.

[10] I. Babuska, “The Finite Element Method with Lagrange Multipliers”, Numerische Mathematik, 20(3), 179~192, 1973.

[11] F. Brezzi, “On the Existence, Uniqueness and Approximation of Saddle-Point Problems Arising from Lagrangian Multipliers”, RAIRD, 8(2), 129~151, 1974.

[12] O.C. Zienkiewicz, R.L. Taylor, and J.Z. Zhu, “The Finite Element Method Sixth edition”, Elsevier Butterworth-Heinemann, 334, 2000.

[13] T. Park, and M. Tak, "A New Coupled Analysis for Nearly Incompressible and Impermeable Saturated Porous Media on Mixed Finite Element Method: I. Proposed Method", KSCE, 14(1), 7~16, 2010.
[14] M. Tak, and T. Park, "A New Coupled Analysis for Nearly Incompressible and Impermeable Saturated Porous Media on Mixed Finite Element Method: II. Verifications", KSCE, 14(1), 17~24, 2010.
[15] C. Farhat., "A Method of Finite Element Tearing and Interconnecting and Its Parallel Solution Algorithm", International Journal for Numerical Methods in Engineering, 32, 1205~1277, 1991.
[16] M. Biot, "General Theory of Three-Dimensional Consolidation", Journal of Applied Physics, 12(1), 155~164, 1941.
[17] W. Gropp, E. Lusk, and R. Thakur, "Using MPI-2: Advanced Features of the Message-Passing Interface", MIT Press, 382, 1999.