Abstract

Galerkin-characteristics finite element method is a powerful numerical procedure for convection-diffusion problems even in the convection-dominated case. It is, however, reported that rough numerical integration for composite terms ruins the convergence property. In this paper we discuss two ways to avoid numerical quadrature referring to the recent results. One way is to use the lumping technique. A Galerkin-characteristics finite element scheme of lumped mass type is considered. The other way is to use a finite difference method derived from a Galerkin-characteristics finite element scheme. For these schemes the stability and convergence are discussed.

Keywords: convection-diffusion, characteristics, quadrature-free, finite element method, finite difference method, lumped mass, Péclet number, stability, convergence.

1 Introduction

Convection-diffusion problems appear and are solved in various fields of sciences and technologies. The convection-diffusion equation is linear, but to solve it is not always an easy task. When the Péclet number is high, that is, convection dominant cases, it is well-known that the Galerkin finite element scheme, or equivalently, the centered finite difference scheme, produces easily oscillation solutions. Hence, elaborate numerical schemes with new ideas have been developed to perform stable computation. Among them we focus on the method based on characteristics [1].

The procedure of the characteristic method is natural from the physical point of view since it approximates particle movements. It is also attractive from the computational point of view since it leads to a symmetric system of linear equations. Schemes derived from the characteristic method are recognized to be robust for high-Péclet numbers. Galerkin-characteristics method also has the advantage of the finite ele-
ment method, the geometrical flexibility and the extension to higher-order schemes. A unique disadvantage of this method is in the computation of composite function terms. Since the terms are not polynomials, some numerical quadrature is usually employed to compute them. It is, however, reported that much attention should be paid to the numerical quadrature, because a rough numerical integration formula may yield oscillating results caused by the non-smoothness of the composite function terms [2].

In this paper we discuss two ways to avoid numerical quadrature, referring to the recent results. One way is to use the lumping technique. A Galerkin-characteristics finite element scheme of lumped mass type [3] is considered. The other way is to use a finite difference method[4] derived from a Galerkin-characteristics finite element scheme[5]. Both schemes are free from numerical quadrature. For these schemes the stability and convergence are discussed.

Through the paper the symbol \( c \) with or without subscripts is used for a generic positive constant independent of the discretization parameters, which may take a different value at each occurrence. We often write \( C_j(X) \) in place of \( C_j([0,T];X) \) if there is no confusion.

2 Basic idea of the method of characteristics

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) and \( T \) be a positive constant. We consider an initial boundary value problem; find \( \phi : \Omega \times (0,T) \to \mathbb{R} \) such that

\[
\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi = f \quad \text{in } \Omega \times (0,T),
\]

\[
\phi = 0 \quad \text{on } \partial \Omega \times (0,T),
\]

\[
\phi(\cdot,0) = \phi^0 \quad \text{in } \Omega,
\]

where \( \nu \) is a positive constant, \( u \in C(\bar{\Omega} \times [0,T];\mathbb{R}^2) \) and \( f \in C(\bar{\Omega} \times [0,T];\mathbb{R}) \) are given functions. Physically \( u \) stands for the velocity of a fluid. We assume that it vanishes on the boundary.

**Hypothesis 1.** The velocity \( u \) satisfies

\[
u \in C \left( [0,T];W^{1,\infty}(\Omega) \right), \quad u = 0 \quad (x \in \partial \Omega).
\]

We denote the material derivative by

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla.
\]

Let \( \psi \) be any function in \( V \equiv H^1_0(\Omega) \). The weak formulation of (1) is

\[
\left( \frac{D\phi}{Dt}(t),\psi \right) + \nu(\nabla \phi(t), \nabla \psi) = (f(t),\psi), \quad (t \in (0,T)),
\]
where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$.

Let $X: (0, T) \to \mathbb{R}^2$ be a solution of the ordinary differential equation
\[ \frac{dX}{dt} = u(X, t). \tag{3} \]

Then, we can write
\[ \frac{D\phi}{Dt}(X(t), t) = \frac{d}{dt}\phi(X(t), t). \tag{4} \]

Let $\Delta t$ be a time increment. We set $N_T = \lfloor T/\Delta t \rfloor$ and $t^n = n\Delta t$ for $n \in \mathbb{Z}$. Let $(x, t^n)$ be a given point in $\Omega \times (0, T)$. We denote by $X(t; t^n, x)$ the solution of (3) subject to an initial condition $X(t^n) = x$. By the backward Euler approximation the left-hand side of (4) is approximated by
\[ \frac{D\phi}{Dt}(x, t^n) \approx \frac{\phi(x, t^n) - \phi(X(t^{n-1}), t^{n-1})}{\Delta t}, \]
which is the basic idea of the method of characteristics used in the numerical analysis of flow problems. It approximates the material derivative term along the particle path $(X(t), t)$. Let $V_h$ be a finite element space in $V$. We denote by $\phi^n_h \in V_h$ an approximation to $\phi(\cdot, n\Delta t) \in V$. The Galerkin-characteristics finite element scheme for (1) is to find $\{\phi^n_h; n = 1, \cdots, N_T\} \subset V_h$ such that
\[ \left( \frac{\phi^n_h - \phi^{n-1}_h \circ X^n_1}{\Delta t}, \psi_h \right) + \nu(\nabla \phi^n_h, \nabla \psi_h) = (f^n, \psi_h) \quad (\forall \psi_h \in V_h) \tag{5} \]
subject to an initial condition $\phi^0_h \in V_h$, where
\[ X^n_1(x) = x - u^n(x)\Delta t \]
and $\circ$ means the composition of functions, $(\phi^{n-1}_h \circ X^n_1)(x) = \phi^{n-1}_h(X^n_1(x))$. Since $X^n_1(x)$ approximates $X(t^{n-1}, t^n, x)$, the first term of the left-hand side of (5) is an approximation of the first term of (2).

The Galerkin-characteristics finite element scheme has advantages as follows [1, 5]:

- geometrical flexibility since it is a finite element method;
- robustness for high Péclet number problems; it works even for pure convection problems;
- unconditional stability; there is no restriction on the choice of $\Delta t$;
- symmetricity of the matrix; the resultant matrix to be solved at each time step is symmetric and independent of the step number.
The scheme (5) is of the first order in $\Delta t$. The second order Galerkin-characteristics scheme in $\Delta t$ of Crank-Nicolson type is given by [5] as

$$
\left( \frac{\phi^n_h - \phi^{n-1}_h \circ X^n_2}{\Delta t}, \psi_h \right) + \frac{\nu}{2} \left( \nabla \phi^n_h + \nabla \phi^{n-1}_h \circ X^n_1, \nabla \psi_h \right)
$$

$$
+ \frac{\nu \Delta t}{2} \left\{ \left( J^n \nabla \phi^{n-1}_h, \nabla \psi_h \right) + \left( \nabla (\nabla \cdot u^n) \cdot \nabla \phi^{n-1}_h, \psi_h \right) \right\}
$$

$$
= \frac{1}{2} \left( f^n + f^{n-1} \circ X^n_1, \psi_h \right) \quad (\forall \psi_h \in V_h),
$$

(6)

where $J^n$ is the Jacobi matrix defined by

$$
J^n_{ij} = \frac{\partial u^n_i}{\partial x_j}
$$

and $X^n_2$ is the approximation of $X(t^{n-1}; t^n, x)$ obtained by the second order Runge-Kutta method or the Heun method,

$$
X^n_2(x) = x - u^{n-1/2} \left( x - u^n(x) \frac{\Delta t}{2} \right) \Delta t,
$$

$$
X^n_2(x) = x - \left( u^n(x) + u^{n-1}(x - u^n(x) \Delta t) \right) \frac{\Delta t}{2}.
$$

This scheme has also all the advantages mentioned above. For these schemes the convergence of the finite element solution $\phi_h$ to the exact one $\phi$ is proved to be of $O(\Delta t + h^k)$ and $O(\Delta t^2 + h^k)$, respectively, in a norm corresponding to $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ when $P_k$ finite element is employed [5].

In these schemes we have to compute the term such as

$$
\int_K \phi^{n-1}_h \circ X^n_k w_{hi} \, dx \quad (k = 1, 2),
$$

where $K$ is an element and $w_{hi}$ is the base function associated with node $P_i$. Since the composite function $\phi^{n-1}_h \circ X^n_k$ is neither a polynomial nor smooth on $K$, some numerical integration is often employed. However, it is reported in [6] that rough numerical integration causes oscillation and that even the interruption of the computation is encountered by overflow in the worst case though the unconditional stability is proved. That is, quadrature error ruins the stability property of the schemes. This inconvenience is improved if we use the second order scheme (6) as shown in [2], but much attention should be paid in the treatment of these terms.

We focus on schemes based on the method of characteristics and they are free of numerical integration. One way of constructing such a scheme is the finite difference method and the other one is the lumping.

## 3 A finite difference scheme of the first order in time

Assume that $\Omega$ is a rectangle, $[0, L_1] \times [0, L_2]$. Let $h$ be a lattice size, and we assume that $N_1 = L_1/h$ and $N_2 = L_2/h$ be integers. We denote by $x_{i,j}$ the lattice point
\((ih, jh)\). We prepare sets of lattice points,
\[
\tilde{\Omega}_h = \{x_{i,j}; i = 0, \cdots, N_1, j = 0, \cdots, N_2\}, \\
\Omega_h = \{x_{i,j}; i = 1, \cdots, N_1 - 1, j = 1, \cdots, N_2 - 1\}, \\
\Gamma_h = \tilde{\Omega}_h \setminus \Omega_h.
\]

Let \(\phi_h\) be a lattice function defined on \(\tilde{\Omega}_h\). We set \(\phi_{h,i,j} = \phi_h(x_{i,j})\) for \(x_{i,j} \in \tilde{\Omega}_h\). We define the bilinear interpolation operator \(\Pi_h^{(1)}\) into \(C(\tilde{\Omega})\) by
\[
(\Pi_h^{(1)} \phi_h)(x) = \phi_{h,i,j} w_{i,j}(x) + \phi_{h,i+1,j} w_{i+1,j}(x) + \phi_{h,i,j+1} w_{i,j+1}(x) + \phi_{h,i,j+1} w_{i,j+1}(x) \quad (x \in [ih, ih + h) \times [jh, jh + h]),
\]
where
\[
i = \left\lfloor \frac{x_1}{h} \right\rfloor, \quad j = \left\lfloor \frac{x_2}{h} \right\rfloor,
\]
\[
w_{i,j}(x) = \left( i + 1 - \frac{x_1}{h} \right) \left( j + 1 - \frac{x_2}{h} \right), \quad w_{i+1,j}(x) = \left( \frac{x_1}{h} - i \right) \left( j + 1 - \frac{x_2}{h} \right),
\]
\[
w_{i,j+1}(x) = \left( i + 1 - \frac{x_1}{h} \right) \left( \frac{x_2}{h} - j \right), \quad w_{i+1,j+1}(x) = \left( \frac{x_1}{h} - i \right) \left( \frac{x_2}{h} - j \right).
\]

We denote by \(V_h\) a set of lattice functions defined by
\[
V_h = \{ v_h : \tilde{\Omega}_h \rightarrow \mathbb{R}; \ v_h = 0 \text{ on } \Gamma_h \}.
\]

A characteristics finite difference scheme for (1) is to find \(\{\phi^n_h; n = 1, \cdots, N_T\} \subset V_h\) such that
\[
\frac{\phi^n_h - (\Pi_h^{(1)} \phi_{h}^{n-1}) \circ X^n_h}{\Delta t} - \nu \Delta_h \phi^n_h = f^n \quad \text{on } \Omega_h
\]
subject to \(\phi_h^0 = \phi^0\) on \(\Omega_h\), where \(\Delta_h\) is the discrete Laplace operator defined by
\[
(\Delta_h \phi_h)(x_{i,j}) = \frac{1}{h^2} (\phi_{h,i+1,j} + \phi_{h,i-1,j} + \phi_{h,i,j+1} + \phi_{h,i,j-1} - 4 \phi_{h,i,j}).
\]

**Remark 1.** In general, \(X^n_h(x_{i,j}) = x_{i,j} - u^n(x_{i,j})\Delta t\) is not a lattice point. Therefore, the interpolation operator \(\Pi_h^{(1)}\) is necessary in (7). The equation (7) can be written equivalently as
\[
\begin{align*}
(1 + \frac{4\nu \Delta t}{h^2} \phi_{h,i,j} - \nu \Delta t \frac{\phi_{h,i+1,j}}{h^2} - \nu \Delta t \frac{\phi_{h,i-1,j}}{h^2} - \nu \Delta t \frac{\phi_{h,i,j+1}}{h^2} - \nu \Delta t \frac{\phi_{h,i,j-1}}{h^2})
&= w_{k,\ell}(y_{i,j}) \phi_{h,k,\ell}^{n-1} + w_{k+1,\ell}(y_{i,j}) \phi_{h,k+1,\ell}^{n-1} + w_{k,\ell+1}(y_{i,j}) \phi_{h,k,\ell+1}^{n-1} + w_{k+1,\ell+1}(y_{i,j}) \phi_{h,k+1,\ell+1}^{n-1} + \Delta t f^n_{i,j},
\end{align*}
\]
where
\[
y_{i,j} = x_{i,j} - u^n(x_{i,j})\Delta t, \quad k = \left\lfloor \frac{(y_{i,j})_1}{h} \right\rfloor, \quad \ell = \left\lfloor \frac{(y_{i,j})_2}{h} \right\rfloor.
\]
Since (7) is a finite difference scheme, no quadrature is required. We show the stability and convergence results. We use the following norms,

\[ \| \phi_h \|_{\ell^\infty(\ell^\infty)} = \max_{0 \leq n \leq N_T} \| \phi_h^n \|_{\ell^\infty}, \quad \| \phi_h \|_{\ell^1(\ell^\infty)} = \Delta t \sum_{n=0}^{N_T} \| \phi_h^n \|_{\ell^\infty}, \]

\[ \| \phi_h^n \|_{\ell^\infty} = \max\{ |\phi_h^n(x_{i,j})| ; \ x_{i,j} \in \bar{\Omega}_h \}. \]

**Hypothesis 2.**

\[ \Delta t < \| u \|_{C(W^1,\infty(\Omega))}. \] (9)

**Lemma 1.** Under Hypotheses 1 and 2, scheme (7) is $L^\infty$-stable, i.e., it holds

\[ \| \phi_h \|_{\ell^\infty(\ell^\infty)} \leq \| \phi_h^0 \|_{\ell^\infty} + \| f \|_{\ell^1(\ell^\infty)}. \] (10)

**Proof.** Condition (9) implies that $X^n_i(x) \in \bar{\Omega}$ as shown in [5, Proposition 1]. In general, it holds that for any $y \in \bar{\Omega}$

\[ w_{k,\ell}(y), w_{k+1,\ell}(y), w_{k+1,\ell+1}(y), w_{k,\ell+1}(y) \geq 0, \]

\[ w_{k,\ell}(y) + w_{k+1,\ell}(y) = w_{k+1,\ell+1}(y) + w_{k,\ell+1}(y) = 1 \quad (k = \lfloor \frac{y_1}{h} \rfloor, \ell = \lfloor \frac{y_2}{h} \rfloor). \]

Hence we can derive the result from (8) by a maximum principle. \( \square \)

We introduce the function space

\[ Z_C^m = \{ \phi \in C^j(C^{m-j}(\bar{\Omega})); \ j = 0, \cdots, m, \| \phi \|_{Z_C^m} < +\infty \}, \]

\[ \| \phi \|_{Z_C^m} = \max\{ \| \phi \|_{C^j(C^{m-j}(\bar{\Omega}))}; \ j = 0, \cdots, m \}. \]

**Theorem 1.** Assume hypotheses 1 and 2. Let $\phi$ and $\phi_h$ be the solution of (1) and (7), respectively. If $\phi$ and $u$ satisfy

\[ \phi \in Z_C^2 \cap C^0([0,T]; C^3(\bar{\Omega})), \quad u \in Z_C^1, \] (11)

then we have

\[ \| \phi_h - \phi \|_{\ell^\infty(\ell^\infty)} \leq c(\| u \|_{Z_C^1}) \| \phi \|_{Z_C^2 \cap C(C^3)(h + \Delta t)}. \]

**Proof.** Let $e_h^n = \phi_h^n - \phi^n$ be the function defined on $\bar{\Omega}_h$. Then $e_h = \{ e_h^n; n = 1, \cdots, N_T \}$ satisfies (7) with the initial condition $e_h^0 = 0$ and the right-hand side $R^n$ defined by

\[ R^n \equiv R^n_1 + R^n_2 + R^n_3 + R^n_4, \]

\[ R^n_1 \equiv \frac{D\phi^n}{\Delta t} \phi^n - \phi^{n-1} \circ X^{n-1}, \]

\[ R^n_2 \equiv \frac{\phi^{n-1} \circ X^n - \phi^{n-1} \circ X^{n-1}}{\Delta t}, \]

\[ R^n_3 \equiv \frac{\Pi^{(1)} \phi^{n-1} \circ X^n - \phi^{n-1} \circ X^n}{\Delta t}, \]

\[ R^n_4 \equiv \nu \Delta h \phi^n - \nu \Delta \phi^n. \]
Let \( g(t) = \phi(X(t; t^n, x), t) \). Then, we have

\[
\left| R^n_1 \right| = \left| \frac{d g}{d t}(t^n) - \frac{g(t^n) - g(t^{n-1})}{\Delta t} \right| \\
\leq \frac{1}{2} \left\| \frac{d^2 g}{d t^2} \right\|_{L^\infty(t^{n-1}, t^n)} \Delta t \leq c(\|u\|_{Z^1_h}) \|\phi\|_{Z^2_h} \Delta t.
\]

The second term is evaluated as

\[
\left| R^n_2 \right| = \left| \frac{1}{\Delta t} \nabla \phi^{n-1}(X^n_1 - X^{n-1}) \right| \\
\leq c(\|u\|_{Z^1_h}) \|\phi\|_{C(C^1)} \Delta t.
\]

Noting that

\[
(\Pi^{(1)} - I)\phi^{n-1} = 0 \quad (\forall x_{i,j} \in \Omega_h),
\]

we evaluate the third term as

\[
\left| R^n_3 \right| = \left| \frac{1}{\Delta t} (\Pi^{(1)} - I)\phi^{n-1} \circ X^n_1 \right| \\
= \left| \frac{1}{\Delta t} \{ (\Pi^{(1)} - I)\phi^{n-1} \circ X^n_1 - (\Pi^{(1)} - I)\phi^{n-1} \circ I \} \right| \\
= \frac{1}{\Delta t} \left| \nabla \{ (\Pi^{(1)} - I)\phi^{n-1} \} \right| \left| X^n_1 - x \right| \\
\leq c(\|u\|_{C}) \left| \nabla \{ (\Pi^{(1)} - I)\phi^{n-1} \} \right| \\
\leq c(\|u\|_{C}) \|\phi\|_{C(C^2)} \Delta t.
\]

The fourth term is evaluated as

\[
\left| R^n_4 \right| \leq c\nu \|\phi\|_{C(C^4)} \Delta t.
\]

Gathering these estimates, we obtain

\[
\left| R^n \right| \leq c(\|u\|_{Z^1_h}) \|\phi\|_{Z^2 \cap C(C^3)} (h + \Delta t),
\]

which implies the result by virtue of Lemma 1.

\[\square\]

4 A Galerkin-characteristics finite element scheme of lumped mass type

The result of the previous section can be extended to a finite element method of lumped mass type [3]. Lumped mass technique requires no quadrature.

Let \( T_h \equiv \{ K \} \) be a partition of \( \Omega \) by triangles. Let \( X_h(\subset H^1(\Omega)) \) be the P1-finite element space, and \( V_h \) be \( X_h \cap H^1_0(\Omega) \). A Galerkin-characteristics finite element scheme of lumped mass is to find \( \{ \phi^n_h; n = 1, \cdots, N_T \} \subset V_h \) such that

\[
\left( \frac{\phi^n_h - \bar{I}_h(\phi^{n-1}_h \circ X^n_1)}{\Delta t}, \bar{\psi}_h \right) + \nu(\nabla \phi^n_h, \nabla \psi_h) = \left( \bar{I}_h f^n, \bar{\psi}_h \right), \quad \forall \psi_h \in V_h \quad (12)
\]
subject to the initial condition $\phi_h^0 = I_h \phi^0$. Here, $I_h : C(\Omega) \to X_h$ is the interpolation operator defined by

$$(I_h \psi)(P) = \psi(P), \quad (\forall \text{node } P \in \bar{\Omega}),$$

$\bar{\cdot} : V_h \to L^2(\Omega)$ is the lumping operator defined by

$$\bar{\psi}_h(x) = \psi_h(P), \quad (x \in D_P)$$

and $D_P$ is the barycentric domain [7] associated with node $P$ shown in Fig. 1.

![Diagram](image.png)

**Figure 1:** The barycentric domain $D_P$ associated with $P$.

Let $N$ be the number of interior nodes, and $w_{hi}, i = 1, \cdots, N$, be the base function associated with node $P_i \in \Omega$,

$$w_{hi} \in V_h, \quad w_{hi}(P_j) = \delta_{ij}, \quad i, j = 1, \cdots, N.$$  

Let $A = \{a_{ij}\}$ be the stiffness matrix with

$$a_{ij} = (\nabla w_{hj}, \nabla w_{hi}), \quad i, j = 1, \cdots, N.$$  

**Hypothesis 3.** $a_{ij} \leq 0$ for $i \neq j, \quad i, j = 1, \cdots, N.$

**Remark 2.** A sufficient condition for Hypothesis 3 is that for any edge $E$ it holds that

$$\alpha_1^E + \alpha_2^E \leq \pi,$$

where $\alpha_i^E, \quad i = 1, 2$, are two angles (of two elements sharing $E$) opposite to $E$ [8]. A little stronger but more familiar condition is every angle of all triangles is less than or equal to $\pi/2$ [9].
Let $m_i$ be

$$m_i = \text{meas } D_{P_i}.$$ 

Setting $\psi_h = w_{hi}$ in (12) and dividing the $i$-th equation by $m_i$, we obtain an equivalent equation to (12), for $i = 1, \ldots, N$,

$$\frac{1}{\Delta t} \left( \phi_h^n(P_i) - (\phi_h^{n-1} \circ X_i^n)(P_i) \right) + \nu \frac{1}{m_i} \sum_{j=1}^{N} a_{ij} \phi_h^n(P_j) = f^n(P_i).$$ (13)

For a set of functions $\{\phi^n\}_{n=0}^{N_T} \subset L^\infty(\Omega)$ we define norms,

$$\|\phi\|_{L^\infty(\Omega)} = \max_{0 \leq n \leq N_T} \|\phi^n\|_{L^\infty(\Omega)}, \quad \|\phi\|_{C(\Omega)} = \Delta t \sum_{n=0}^{N_T} \|\phi^n\|_{L^\infty(\Omega)}.$$ (14)

**Lemma 2.** Under Hypotheses 1, 2 and 3, scheme (12) is $L^\infty$-stable, i.e., it holds

$$\|\phi_h\|_{L^\infty(\Omega)} \leq \|\phi^0\|_{L^\infty} + \|I_h f\|_{C(\Omega)}.$$ (15)

The proof is similar to that of Theorem 1.

**Theorem 2.** Assume hypotheses 1, 2 and 3. Let $\phi$ and $\phi_h$ be the solution of (1) and (12), respectively. If $\phi$ and $u$ satisfy (11), then for any $\epsilon \in (0, 1)$ we have

$$\|\phi_h - I_h \phi\|_{L^\infty(\Omega)} \leq C \epsilon \|u\|_{Z^1_C} \|\phi\|_{Z^2_C \cap C(\Omega^3)} \left( h + \Delta t + \frac{h^{2-\epsilon}}{\Delta t} \right).$$ (16)

By taking $\Delta t = O(h)$ we have

$$\|\phi_h - I_h \phi\|_{L^\infty(\Omega)} \leq C \epsilon \|u\|_{Z^1_C} \|\phi\|_{Z^2_C \cap C(\Omega^3)} h^{1-\epsilon}.$$ (17)

We omit the proof, which is found in [3].

**Remark 3.** The $\epsilon$ of the power in $h$ is the consequence of the error estimate for the Poisson equation by the $P1$ finite element [9]. It disappears for three dimensional problems.

## 5 A characteristics finite difference scheme of second order in time

We present a characteristics finite difference scheme of second order in time [4], which corresponds to (6). Since it is a finite difference scheme, no quadrature is required.
Here we consider again functions $\phi_h$ defined on the lattice set $\Omega_h$. A finite difference scheme corresponding to (6) is

$$
\frac{\phi^n_h - (\Pi_h^{(1)} \phi_h^{n-1}) \circ X^n_h}{\Delta t} - \nu \left( \Delta_h \phi_h^n + \tilde{\Delta}_h (\phi_h^{n-1}) \right)
- \nu \Delta t \left\{ \frac{\partial u^n_h}{\partial x_1} \nabla h_1 \nabla h_1^\top + \frac{\partial u^n_h}{\partial x_2} \nabla h_2 \nabla h_2^\top + \left( \frac{\partial u^n_h}{\partial x_1} + \frac{\partial u^n_h}{\partial x_2} \right) \nabla (2h_1) \nabla (2h_2) \right\} \phi_h^{n-1}
= \frac{1}{2} \left( f^n + f^{n-1} \circ X^n_h \right) \quad \text{on } \Omega_h,
$$

(18)

where

$$
\tilde{\Delta}_h (\phi) \equiv \nabla h_1 \tilde{\nabla} (\nabla h_1) + \nabla h_2 \tilde{\nabla} (\nabla h_2),
$$

$$
(\nabla h_k v_h) (x) \equiv \frac{1}{h} \left\{ v_h (x + \frac{h}{2} e_k) - v_h (x - \frac{h}{2} e_k) \right\},
$$

$$
\tilde{\nabla} (\nabla h_k) \equiv \left\{ \Pi_h^{(2)} e_k, (\nabla h_k) \right\} \circ X^n_h,
$$

$$
(\nabla (2h_k) v_h) (x) \equiv \frac{1}{2h} \left\{ v_h (x + h e_k) - v_h (x - h e_k) \right\} \quad (k = 1, 2),
$$

$e_k$ is the unit vector to $x_k$ direction, and $\Pi_h^{(2)} e_k, (\nabla h_k)$ is the bilinear interpolation operators using the values at $x_{i,j} \pm \frac{1}{2} h e_k$ for $i, j \in \mathbb{Z}$.

**Remark 4.** We suppose that $\Pi_h^{(2), (1)} v_h$ is defined in the whole $\bar{\Omega}$. For that purpose we extend the values of $v_h$ by defining that

$$
v_h (x_{-1/2,j}) = 2v_h (x_{1/2,j}) - v_h (x_{3/2,j}),
$$

$$
v_h (x_{N_1+1/2,j}) = 2v_h (x_{N_1-1/2,j}) - v_h (x_{N_1-3/2,j}) \quad (j = 0, \cdots, N_2).
$$

Similar extension is done for functions operated by the bilinear interpolation $\Pi_h^{(0, \frac{1}{2}), (1)}$.

For this scheme we have the $L^2$ stability and convergence result. We prepare the following norms and seminorms. For a lattice function $\psi_h$ defined on $\Omega_h$ or a set of the functions $\psi_h = \{ \psi^n_h \}_{n=0}^{N_T}$,

$$
\| \psi_h \|_{L^2 (\Omega_h)} \equiv \left\{ h^2 \sum_{x_{i,j} \in \Omega_h} \psi_h (x_{i,j})^2 \right\}^{1/2},
$$

$$
\| \psi_h \|_{L^\infty (\Omega_h)} \equiv \max_{0 \leq n \leq N_T} \| \psi^n_h \|_{L^2 (\Omega_h)},
$$

$$
\| \psi_h \|_{H^s (\Omega_h)} \equiv \left\{ \Delta t \sum_{n=0}^{N_T} \left\| \partial_t \psi_h \right\|_{L^2 (\Omega_h)} \right\}^{1/2},
$$

$$
\| \psi_h \|_{\dot{H}^s (\Omega_h)} \equiv \left\{ \Delta t \sum_{n=1}^{N_T} \left\| \nabla_h \psi^n_h + \tilde{\nabla} (\nabla h_k) \psi_h^{n-1} \right\|_{L^2 (\Omega_h)} \right\}^{1/2},
$$

$$
\| \psi_h \|_{L^2 (\bar{\Omega}_h)} \equiv \left\{ \Delta t \sum_{n=1}^{N_T} \left\| \nabla_h \psi^n_h + \tilde{\nabla} (\nabla h_k) \psi_h^{n-1} \right\|_{L^2 (\bar{\Omega}_h)} \right\}^{1/2},
$$

$$
\| \psi_h \|_{L^2 (\bar{\Omega}_h)} \equiv \left\{ \Delta t \sum_{n=1}^{N_T} \left\| \nabla_h \psi^n_h + \tilde{\nabla} (\nabla h_k) \psi_h^{n-1} \right\|_{L^2 (\bar{\Omega}_h)} \right\}^{1/2},
$$

$$
\| \psi_h \|_{L^2 (\bar{\Omega}_h)} \equiv \left\{ \Delta t \sum_{n=1}^{N_T} \left\| \nabla_h \psi^n_h + \tilde{\nabla} (\nabla h_k) \psi_h^{n-1} \right\|_{L^2 (\bar{\Omega}_h)} \right\}^{1/2}.
where $\ell^2(\Omega_{h}^\frac{1}{2}e_k), k = 1, 2$, is a norm for functions $\psi_h$ defined on the lattice $\Omega_{h} \pm \frac{1}{2} e_k$,

$$\|\psi_h\|_{\ell^2(\Omega_{h}^\frac{1}{2}e_k)} \equiv \left[ h^2 \sum \{\psi_h(y_{i,j})^2; y_{i,j} = x_{i,j} + \frac{1}{2} e_k \in \bar{\Omega}, x_{i,j} \in \Omega_{h}\} \right]^{1/2}.$$  

**Lemma 3.** Under Hypotheses 1 and 2, scheme (18) is $L^2$-stable, i.e., it holds

$$\|\phi_h\|_{l^\infty(l^2)} + \sqrt{\nu} \|\phi_h\|_{l^2(h^1u)} \leq c \left( \|\phi^0\|_{l^2(\Omega_{h})} + \sqrt{\nu} \Delta t \|\phi^0\|_{l^1(h)(\Omega_{h})} + \|f\|_{l^2(l^2)} \right).$$

**Theorem 3.** Assume hypotheses 1 and 2. Let $\phi$ and $\phi_h$ be the solution of (1) and (18), respectively. If $\phi$, $u$ and $f$ satisfy

$$\phi \in Z^3_C, \ u \in Z^2_C, \ f \in Z^2_C$$

then we have

$$\|\phi - \phi_h\|_{l^\infty(l^2)} + \sqrt{\nu} \|\phi - \phi_h\|_{l^2(h^1u)} \leq c(\|u\|_{Z^2_C})(\|\phi\|_{Z^3_C} + \|f\|_{Z^2_C})(h + \Delta t^2).$$

For the proofs of these results we refer to [4], where a scheme having a better convergence order in space is also presented.

## 6 Conclusion

We have discussed two ways to avoid quadrature in treating characteristic methods for convection diffusion problems from the recent results. One is to use finite difference methods, which does not need quadrature in nature with the loss of geometrical flexibility. On rectangle domains we have shown two characteristics finite difference schemes of first and second orders in time increment. In the latter case discrete $L^2$ norm is employed, which will be applicable to the Navier-Stokes equations. The other way is to use the lumping technique for the Galerkin-characteristics finite element method. It keeps the geometrical flexibility of the finite element method, but the convergence order is of first order. To develop numerical schemes having both advantages of these methods is the future work.

## References


