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Higher Order Beam Equations

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Abstract

This paper considers wave propagation in circular cylindrical beams adopting a power series expansion method in the radial coordinate. Equations of motion together with consistent sets of end boundary conditions are derived in a systematic fashion up to arbitrary order using a generalized Hamilton's principle. These equations are believed to be asymptotically correct. Numerical examples for dispersion curves, eigenfrequencies, displacement and stress distributions are given for various sorts of finite beam structures. The results are presented for series expansion theories of different order and various classical theories, from which one may conclude that the present method generally models the beam accurately.

Keywords: circular beam, series expansion, recursion relations, asymptotic, eigenfrequency.

1 Introduction

There exist many models which describe the elastodynamic wave propagation in finite circular cylindrical beams. It has been treated at different levels; from a simple one-dimensional wave propagation problem to the complete three-dimensional theory of elastodynamics. The involved three-dimensional theory has been adopted in conjunction with various levels of approximations when studying dynamic beam problems for different standard end boundary conditions. Most such works consider eigenfrequency analyzes using fix frequency. There exists on one hand analytical solutions based on expansion in terms of Bessel functions [1, 2] where part of the boundary conditions are satisfied approximately, and on the other hand numerical solutions such as the Ritz method [3, 4, 5] or the finite element method [6].

However, the bulk of analysis has been on various approximate models due to the complexity of the exact theory. In these simplified theories, both the dynamic equations and the boundary conditions are often derived using various kinds of simplifying kinematic assumptions. The most used approximate theory is the simple Euler–Bernoulli equation, where shear and rotary inertia are neglected. This leads to a differential equation that has the undesired feature of being non-hyperbolic. However, if the beam radius is much smaller than the wavelengths this approximation is known to yield accurate results. The next level is to include shear and rotary inertia described by Timoshenko [7], resulting in a hyperbolic equation of motion. There are several other more advanced beam theories in use. Some of these concern only rectangular cross sections [8, 9, 10], while others are applicable for circular cross sections [11, 12, 13, 14].

Among the various beam theories, higher order power series expansion are used in [11, 12]. These work use approaches different from the present theory, such as the series expansion method, the use of recursion relations, the procedure when collecting terms or the truncation process as a whole. Besides isotropic beams, the present method has been used on rods, shells and plates [15, 16, 17, 18, 19].

The present paper aims at systematically develop higher order beam equations together with the end boundary conditions. These equations are supposed to be asymptotically correct, and may be derived to an (in principle) arbitrary order. To this end a generalized Hamilton's principle is used, where both the displacements and the stresses are varied independently. This results in traction and displacement boundary conditions, as well as the beam equation of motion. Besides presenting a hierarchy of beam equations with end boundary conditions, a more detailed comparison is performed between the lowest nontrivial theory and the Euler–Bernoulli and Timoshenko theories. The numerical results present the dispersion curves, the lowest eigenfrequencies for simply supported beams, together with corresponding displacement and stress distributions.

2 Hamilton's principle

Consider a cylindrical beam with length L and radius a. The beam is homogeneous, isotropic and linearly elastic with density ρ and Lamé constants λ and μ. Cylindrical coordinates are used with radial coordinate r, circumferential coordinate θ and axial coordinate z. The corresponding radial, circumferential and longitudinal displacement fields are denoted by u, v and w.

A generalized Hamilton's principle can be used to derive the differential equation describing the motion of the beam and the corresponding boundary conditions. Simultaneous and independent variations of displacements and stresses are adopted [19, 20]. The Hamilton's principle states that

$$\delta \int_{t_0}^{t_1} L \mathrm{d}t = 0, \qquad L = T - U + W, \tag{2.1}$$

where T is the kinetic energy, U is the potential energy and W is the work done by body forces and surface tractions. The energy densities $\overline{T}$ and $\overline{U}$ are defined as

$$\begin{aligned}
\overline{T} &= \rho/2\, \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} = \rho/2 \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right), \\
\overline{U} &= 1/2\, \boldsymbol{\sigma}{:}\boldsymbol{\epsilon} = 1/2 \left(\sigma_{rr}\epsilon_{rr} + \sigma_{\theta\theta}\epsilon_{\theta\theta} + \sigma_{zz}\epsilon_{zz}\right) + \sigma_{r\theta}\epsilon_{r\theta} + \sigma_{rz}\epsilon_{rz} + \sigma_{\theta z}\epsilon_{\theta z},
\end{aligned} \tag{2.2}$$

where $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{\epsilon}$ is the strain and a dot denotes a time derivative. By considering displacement terms and force terms as independent, the variational expressions become

$$\int_{t_0}^{t_1} \left(\int_V \left(\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{f} - \rho \ddot{\boldsymbol{u}}\right) \cdot \delta \boldsymbol{u}\, \mathrm{d}V + \int_{S^t} \left(\hat{\boldsymbol{t}} - \boldsymbol{n} \cdot \boldsymbol{\sigma}\right) \cdot \delta \boldsymbol{u}\, \mathrm{d}S + \int_{S^u} \left(\hat{\boldsymbol{u}} - \boldsymbol{u}\right) \cdot \delta \boldsymbol{t}\, \mathrm{d}S \right) \mathrm{d}t = 0. \tag{2.3}$$

Since the virtual displacement components in $\delta\boldsymbol{u}$ and the virtual traction components $\delta\boldsymbol{t}$ are independent, equation (2.3) reduces to separate equations for each variational term. For clarity of sake, each equation is written below on component form. The equations of motion contained in the volume integrals are thus

$$\int_V \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho f_r - \rho \frac{\partial^2 u}{\partial t^2} \right) \delta u\, r \mathrm{d}r \mathrm{d}\theta \mathrm{d}z = 0, \tag{2.4}$$

$$\int_V \left(\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + 2\frac{\sigma_{r\theta}}{r} + \rho f_\theta - \rho \frac{\partial^2 v}{\partial t^2} \right) \delta v\, r \mathrm{d}r \mathrm{d}\theta \mathrm{d}z = 0, \tag{2.5}$$

$$\int_V \left(\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho f_z - \rho \frac{\partial^2 w}{\partial t^2} \right) \delta w\, r \mathrm{d}r \mathrm{d}\theta \mathrm{d}z = 0, \tag{2.6}$$

and the surface integrals follow directly.

3 Series expansion

The displacement components are now expanded in power series in the radial coordinate r

$$\begin{aligned}
u &= u_0(\theta, z, t) + r u_1(\theta, z, t) + r^2 u_2(\theta, z, t) + \ldots, \\
v &= v_0(\theta, z, t) + r v_1(\theta, z, t) + r^2 v_2(\theta, z, t) + \ldots, \\
w &= w_0(\theta, z, t) + r w_1(\theta, z, t) + r^2 w_2(\theta, z, t) + \ldots.
\end{aligned} \tag{3.1}$$

Using this ansatz in the stress–displacement relations this results in stress expressions on series form

$$\sigma_{ij} = r^{-1}\sigma_{ij,-1}(\theta, z, t) + \sigma_{ij,0}(\theta, z, t) + r\sigma_{ij,1}(\theta, z, t) + \ldots, \tag{3.2}$$

that are to be used in (2.4)–(2.6). Each term in the radial series ansatz (3.1) are expanded in Fourier series according to

$$u_k = \sum_{m=0}^{\infty} u_{k,m}(z, t) \cos m\theta, \qquad v_k = \sum_{m=0}^{\infty} v_{k,m}(z, t) \sin m\theta, \qquad w_k = \sum_{m=0}^{\infty} w_{k,m}(z, t) \cos m\theta.$$

(3.3)

Here the angle θ is measured from a vertical axis in a plane through the cross section of the cylinder with a horizontal z axis. Hereby, the case $m = 1$ correspond to the flexural motion in the vertical direction. The axisymmetric case $m = 0$ is for a rod with radial and longitudinal motion, treated in [15, 19].

Adopting (3.3) in (2.4)–(2.6) using the stress–displacement relations gives the recursion relations for each Fourier mode m as

$$
\begin{aligned}
&(k+1)(k+3)\left[(k+1)(k+3)+4m(k+2)+4m^2\right]\mu(\lambda+2\mu)\,u_{m+k+2,m} = \\
&\left[(m+k+1)(m+k+3)\mu - m^2(\lambda+2\mu)\right]\left(\rho\ddot{u}_{m+k,m} - \mu u''_{m+k,m}\right) \\
&-\left[m(m+k+1)(\lambda+\mu) - 2m\mu\right]\left(\rho\ddot{v}_{m+k,m} - \mu v''_{m+k,m}\right) \\
&-(k+1)(m+k+3)(2m+k+1)\mu(\lambda+\mu)w'_{m+k+1,m}, \qquad k = 1, 3, \ldots,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&(k+1)(k+3)\left[(k+1)(k+3)+4m(k+2)+4m^2\right]\mu(\lambda+2\mu)\,v_{m+k+2,m} = \\
&\left[(m+k+1)(m+k+3)(\lambda+2\mu) - m^2\mu\right]\left(\rho\ddot{v}_{m+k,m} - \mu v''_{m+k,m}\right) \\
&+\left[m(m+k+3)(\lambda+\mu) + 2m\mu\right]\left(\rho\ddot{u}_{m+k,m} - \mu u''_{m+k,m}\right) \\
&+m(k+1)(2m+k+1)\mu(\lambda+\mu)w'_{m+k+1,m}, \qquad k = 1, 3, \ldots,
\end{aligned}
\tag{3.5}
$$

$$
\begin{aligned}
&\left[(m+k+2)^2 - m^2\right]\mu\,w_{m+k+2,m} = \rho\ddot{w}_{m+k,m} - (\lambda+2\mu)w''_{m+k,m} \\
&-(m+k+2)(\lambda+\mu)u'_{m+k+1,m} - m(\lambda+\mu)v'_{m+k+1,m}, \qquad k = 0, 2, \ldots.
\end{aligned}
\tag{3.6}
$$

Here a prime denotes a z-derivative. Two further equations may also be obtained by combining the recursion relations for negative k values, resulting in

$$u_{m-1,m} + v_{m-1,m} = 0, \tag{3.7}$$

$$
\begin{aligned}
&\left[m(m+2)(\lambda+2\mu) - m^2\mu\right]u_{m+1,m} + \left[m^2(\lambda+\mu) - 2m\mu\right]v_{m+1,m} \\
&= \rho\ddot{u}_{m-1,m} - \mu u''_{m-1,m} - m(\lambda+\mu)w'_{m,m}.
\end{aligned}
\tag{3.8}
$$

By inspection, these equations reveal that the terms in (3.3) are such that

$$u_{k,m} = v_{k,m} = w_{k,m-1} \equiv 0, \qquad k < m-1. \tag{3.9}$$

Moreover, $u_{k,m}$ and $v_{k,m}$ are zero when k and m are either both even or both odd, respectively. The opposite situation holds for $w_{k,m}$. The recursion formulas (3.4)–(3.6) together with (3.7) and (3.8) allow for expressing higher order index terms in the mutually independent lowest order index terms. This is to be used in the derivation process for obtaining a hierarchy of beam equations with pertinent boundary conditions.

The expressions for the stresses follow directly from (3.1) and the stress–displacement relations. Hereby $\{\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}\}$ are expanded in $\cos m\theta$ and $\{\sigma_{r\theta}, \sigma_{\theta z}\}$ are expanded in $\sin m\theta$. The stresses may be written

$$\sigma_{ij} = \sum_{m=0}^{\infty} \tilde{\sigma}_{ij,m}(r, z, t)\{\cos m\theta; \sin m\theta\}, \tag{3.10}$$

using either $\cos m\theta$ or $\sin m\theta$ according to above. The Fourier modes are

$$\begin{aligned} \tilde{\sigma}_{ab,m} &= r^{m-2}\sigma_{ab,\{m-2,m\}} + r^{m}\sigma_{ab,\{m,m\}} + r^{m+2}\sigma_{ab,\{m+2,m\}} + \dots, \\ \tilde{\sigma}_{cd,m} &= r^{m-1}\sigma_{cd,\{m-1,m\}} + r^{m+1}\sigma_{cd,\{m+1,m\}} + r^{m+3}\sigma_{cd,\{m+3,m\}} + \dots, \end{aligned} \tag{3.11}$$

where ab is for $\{rr, \theta\theta, zz, r\theta\}$ and cd is for $\{rz, \theta z\}$. Each stress term is expressed as

$$\sigma_{rr,\{k,m\}}(z,t) = \left[(k+2)(\lambda+2\mu) - 2\mu\right] u_{k+1,m} + m\lambda v_{k+1,m} + \lambda w'_{k,m}, \tag{3.12}$$

$$\sigma_{\theta\theta,\{k,m\}}(z,t) = \left[(k+2)\lambda + 2\mu\right] u_{k+1,m} + m(\lambda+2\mu) v_{k+1,m} + \lambda w'_{k,m}, \tag{3.13}$$

$$\sigma_{zz,\{k,m\}}(z,t) = (k+2)\lambda u_{k+1,m} + m\lambda v_{k+1,m} + (\lambda+2\mu) w'_{k,m}, \tag{3.14}$$

$$\sigma_{r\theta,\{k,m\}}(z,t) = \mu\left[k v_{k+1,m} - m u_{k+1,m}\right], \tag{3.15}$$

$$\sigma_{rz,\{k,m\}}(z,t) = \mu\left[u'_{k,m} + (k+1) w_{k+1,m}\right], \tag{3.16}$$

$$\sigma_{\theta z,\{k,m\}}(z,t) = \mu\left[v'_{k,m} - m w_{k+1,m}\right]. \tag{3.17}$$

4 Equations of motion

The lateral boundary conditions at $r = a$ constitute the beam equations of motion. Considering the standard case of only prescribed tractions, the set of three lateral boundary conditions thus becomes from (3.11), (3.12), (3.15) and (3.16)

$$\begin{aligned} &a^{m-2}\left[(m(\lambda+2\mu) - 2\mu)\, u_{m-1,m} + m\lambda v_{m-1,m}\right] + \\ &a^{m}\left[((m+2)(\lambda+2\mu) - 2\mu)\, u_{m+1,m} + m\lambda v_{m+1,m} + \lambda w'_{m,m}\right] + \dots = \hat{t}_{r,m}, \end{aligned} \tag{4.1}$$

$$a^{m-2}\mu\left[(m-2)v_{m-1,m} - m u_{m-1,m}\right] + a^{m}\mu\left[m v_{m+1,m} - m u_{m+1,m}\right] + \dots = \hat{t}_{\theta,m} \tag{4.2}$$

$$a^{m-1}\mu\left[u'_{m-1,m} + m w_{m,m}\right] + a^{m+1}\mu\left[u'_{m+1,m} + (m+2) w_{m+2,m}\right] + \dots = \hat{t}_{z,m}. \tag{4.3}$$

Adopting the recursion relations (3.4)–(3.6) together with (3.7) and (3.8), a hierarchy of beam equations is obtained expressed in terms of the mutually independent lowest order index terms. These beam displacement terms are $u_{m-1,m}$, $v_{m+1,m}$ and $w_{m,m}$ for $m > 0$, and $u_{1,0}$, $v_{1,0}$, and $w_{0,0}$ for $m = 0$. The axisymmetric case using $u_{1,0}$ and $w_{0,0}$ in (4.1) and (4.3) is described in [15, 19].

The differential orders of the beam equations depend on the number of terms used in (4.1)–(4.3). Consider now the case when $m > 0$. The resulting hyperbolic beam equations using n_r terms in (4.1), n_θ terms in (4.2) and n_z terms in (4.3) (including zero terms) are of total differential order $2(n_r + n_\theta + n_z) - 8$ in both space and time. This is readily seen by eliminating within the set of equations, so as to obtain one equation in one of the fields, say $u_{m-1,m}$. Due to the same differential terms appearing in both (4.1) and (4.2), it is natural to set $n_\theta = n_r$. As for (4.3), the appearance of higher order derivatives implies that one should choose $n_z = n_r$ or $n_z = n_r - 1$. For nontrivial solutions, (4.1)–(4.3) are solved using $n_i > 1$. However, since for $m = 1$ the first a^{-1} terms in (4.1) and (4.2) are zero, one has here that $n_r > 2$ and $n_\theta > 2$.

The lowest order set of beam equations that incorporates flexural motion is for the present theory when $n_r = n_\theta = 3$ and $n_z = 2$ in (4.1)–(4.3). Written out explicitly, the truncated system may be expressed as

$$(3\lambda + 4\mu)\, u_{2,1} + \lambda v_{2,1} + \lambda w'_{1,1} + a^2 \left[(5\lambda + 8\mu)\, u_{4,1} + \lambda v_{4,1} + \lambda w'_{3,1}\right] = 0, \tag{4.4}$$

$$v_{2,1} - u_{2,1} + a^2 \left[3v_{4,1} - u_{4,1}\right] = 0, \tag{4.5}$$

$$u'_{0,1} + w_{1,1} + a^2 \left[u'_{2,1} + 3w_{3,1}\right] = 0. \tag{4.6}$$

Using the recursion relations, this is seen to be a hyperbolic 8:th order system.

5 Numerical results

The behavior using the present beam theory is to be compared to exact and classical beam theories. These comparisons comprise dispersion relation curves for an infinite beam, eigenfrequencies for a finite beam, as well as mode shapes and stress distributions. These dynamical problems are for laterally free beams.

5.1 Dispersion curves

In order to illustrate the effects from the number of terms adopted in (4.1)–(4.3), dispersion relations are calculated for $n_r = n_\theta = n$ and $n_z = n - 1$, where $n = 3, 4, 5, 6$. A normalized frequency $\Omega = \omega a / c_E$ is introduced with $c_E^2 = E/\rho$. Considering the flexural case $m = 1$, Figure 1 shows the three lowest modes using both the series expansion theories (4.1)–(4.3) and the exact theory. It is clear that higher accuracy is obtained as more terms are used. Among the results, the lowest curve is accurately captured in the lower frequency range for all theories. Note that the $n = 5$ curve for the second mode virtually coincides with the exact curve, which is also the case for the first mode over most of presented frequencies. Here the case $n = 6$ is not plotted as these three curves are indistinguishable from the exact curves in the presented range.

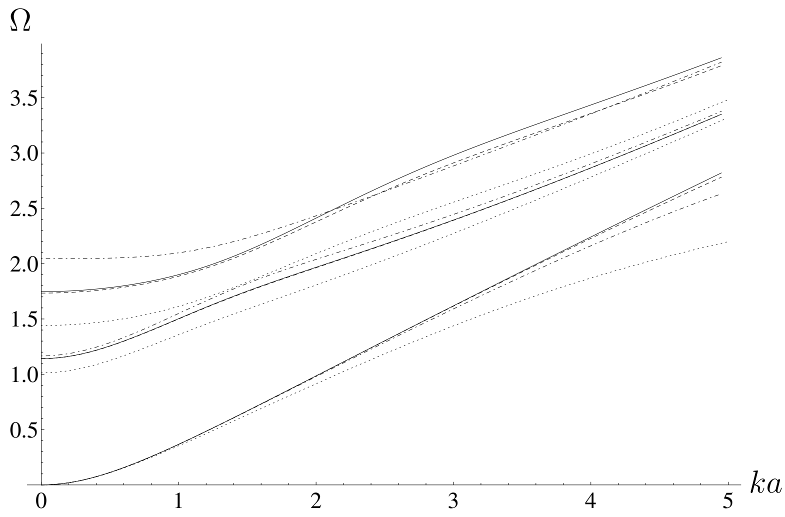


Figure 1: Dispersion curves for $m = 1$: —— Exact, $\cdots$ $n = 3$, - $\cdot$ - $n = 4$,- - - $n = 5$

5.2 Eigenfrequencies

In this section, the eigenfrequencies for the flexural series expansions theories $m = 1$ are compared with one another using different truncation orders. These expansions are also compared to other classical theories as well as the exact theory for simply supported ends. As for the dispersion relations $n_r = n_\theta = n$ and $n_z = n - 1$ are chosen. This results in $3n - 5$ BCs at each end. The three lowest eigenfrequencies for $L/a = 10$ and $L/a = 2$ are presented for the bending dominant mode in Table 1. It is clear from the table that the series expansion results converge to the exact results as the power series orders are increased. These series results seem to converge monotonically from below to the exact values. It is clear from Table 1 that more accurate results are obtained for lower frequencies and slender beams. The Timoshenko theory is astonishingly accurate. This behavior relates to the choice of shear correction factor, here chosen as $\kappa = 0.932$. The results for the Euler-Bernoulli theory confirm the well known fact that this theory renders reasonably accurate results for slender beams in the low frequency spectra.

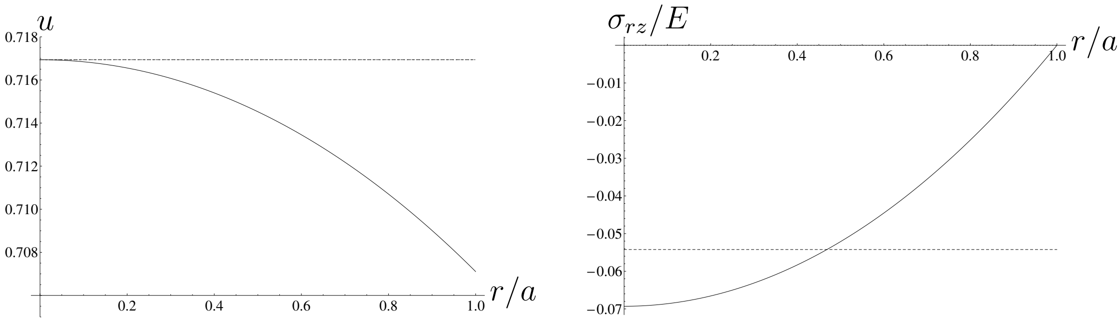
5.3 Mode shapes and stress distributions

In order to illustrate the differences between the current beam theory, classical theories and the exact theory when $m = 1$, various plots on mode shapes and stress distributions are compared for the fundamental frequency in case of the lowest bending mode for a simply supported beam when $L/a = 10$. The presented numerical results are such that the curves using the lowest series expansion theory $n = 3$ are very close to the exact curves. Hereby, separate exact curves virtually on top of the $n = 3$ curves are not plotted. Moreover, the eigenmodes are normalized so that the maximum radial displacement u at $r = a$ is equal to unity.

L/a	Ω	Exact	EB	T	$n=3$	$n=4$	$n=5$	$n=6$
	Ω_1	0.047215	0.049348	0.047214	0.046910	0.047214	0.047215	0.047215
10	Ω_2	0.16955	0.19739	0.16952	0.16619	0.16950	0.16955	0.16955
	Ω_3	0.33422	0.44413	0.33404	0.32308	0.33388	0.33422	0.33422
	Ω_1	0.71391	1.2337	0.71310	0.67335	0.71115	0.71381	0.71391
2	Ω_2	1.7079	4.9348	1.7074	1.5053	1.6748	1.7040	1.7076
	Ω_3	2.6765	11.103	2.6864	2.1134	2.5227	2.6465	2.6717

Table 1: The eigenfrequencies for $L/a = 10$ and $L/a = 2$ using exact, Euler-Bernoulli (EB), Timoshenko (T) and the series expansion theories of orders $n = 3, 4, 5, 6$ for the bending dominant mode in a beam simply supported at $z = 0, L$

Figure 2a illustrates the radial displacement u as a function of the radius for $z = 3L/4$. For the exact and the present beam theories, the radial displacement varies, while both the Euler-Bernoulli and the Timoshenko theories describe a constant displacement field lying on top of each other. Still the differences between theories are quite small, which partly stem from the normalization process. For the stress distribution, Figure 2b presents the shear stress σ_{rz}. It is seen that the Timoshenko theory generates a constant shear stress, while the stress magnitudes from the Euler-Bernoulli theory are zero. Note that the shear stress using the present theory is zero at the lateral boundary, as expected.



(a) Radial displacement u at $z = 3L/4$.

(b) Shear stress σ_{rz} at $z = 3L/4$.

Figure 2: Cross section distribution for the lowest eigenfrequency. —— Exact, · · · Euler–Bernoulli, - - - Timoshenko

6 Conclusions

This paper presents the beam equation and corresponding boundary conditions to arbitrary order according to the power series expansion theory. The method used is a generalized Hamilton's principle resulting in variationally consistent equations that seem to be asymptotically correct. Numerical results are presented for different beam theories. Here, all theories are fairly adequate for calculating the eigenfrequencies, but the distribution of stresses varies considerably between theories. One application of this theory is to implement it in finite element codes. Hereby one benefits from the accurate results using the present theory, and at the same time the number of elements can be considerably reduced compared to using three-dimensional elements.

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