# Limit Analysis of Highly-Undermatched Welded Joints with Cracks 

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#### Abstract

This paper deals with a general approach to build up kinematically admissible velocity fields for a class of highly undermatched welded joints. The approach is based on two principles. First, the known asymptotic singular behaviour of the real velocity field in the vicinity of velocity discontinuity surfaces is taken into account in the kinematically admissible velocity field. Second, a linear through-thickness distribution of one of the velocity components is assumed. The approach proposed is used to derive an upper bound limit load for tensile panels. It is shown that the difference between this solution and an accurate numerical solution is negligible.


Keywords: singular velocity field, upper bound method, limit load, highly undermatched welded joints.

## 1 Introduction

The limit load is an essential input parameter in many flaw assessment procedures [1]. A comprehensive overview of limit load solutions for structures with defects available at the time of writing can be found in [2]. An overview of limit load solutions for highly undermatched welded joints including joints containing cracks has been given in [3]. A distinguished feature of this class of welded joints is that the weld is much softer than the base material. In particular, plastic deformation is solely confined within the weld whereas the base material is elastic. Such structures are of practical interest [4]. The present paper concerns with a general method to build up kinematically admissible velocity fields for a class of highly undermatched welded joints. It is worthy of note that elastic properties have no effect on the limit load [5]. Therefore, the elastic material can be regarded as rigid. The approach proposed is based on two principles. First, it is known that the real velocity field is singular in the vicinity of maximum shear stress surfaces [6]. In particular, the equivalent strain rate involved in the formulation of the upper bound theorem of
plasticity (see, for example [7]) follows an inverse square root rule near such surfaces and, therefore, approaches infinity. In the case of highly undermatched welded joints surfaces of maximum shear stress coincide with the bi-material interface. It is advantageous to account for the singular behaviour of the real velocity fields in kinematically admissible velocity fields. Second, the thickness of the weld is usually much smaller than other geometric dimensions of specimens. It is therefore natural to assume a linear through-thickness distribution on the velocity component normal to the weld. The first principle has been adopted in [8] to propose a general method to evaluate the tensile strength of adhesive plastic layers of arbitrary simply connected contour. The method has been applied to estimate the effect of three-dimensional deformation on the limit load for a highly undermatched welded joint of rectangular cross-section [9]. The second principle has been ignored in [8]. On the other hand, the exact analytic solution for compression of a thin rigid/plastic layer between two rough parallel plates (it is known as Prandtl's problem and its solution is available in any monograph on the classical theory of continuum plasticity, for example [7]) predicts a linear through-thickness distribution of the velocity component normal to the layer. The width of the layer considered in this formulation is infinite. A very accurate numerical solution based on the slip-line technique for the layer of finite width has been obtained in [7]. It is shown in the present paper that the simple method to build up kinematically admissible velocity fields using the two aforementioned principles leads to a magnitude of the limit load which is practically the same as that from the numerical solution.

## 2 Conceptual approach

The class of structures under consideration is restricted to highly undermatched welded joints under plane strain conditions. The definition for highly undermatched welded joints assumes that plastic deformation at the instant of plastic collapse is localized within the weld whereas the base material is rigid. A consequence of such a flow pattern is that the yield stress of the base material has no effect of the limit load. Therefore, the mis-match factor, which is considered to be an important parameter of welded joints, is not involved in the present formulation. Numerous analytical and numerical limit load solutions for highly undermatched joints show that the velocity vector is discontinuous across a significant portion of the bimaterial interface. It is known [6] that the real velocity field is singular in the vicinity of velocity discontinuity surfaces (there are exceptions to this rule but those are not applicable to the problem under consideration). It is therefore advantageous to take into account this singular behaviour of the real velocity field in kinematically admissible velocity fields. In particular, the equivalent strain rate follows an inverse square root rule in the form

$$
\begin{equation*}
\xi_{e q}=\frac{D}{\sqrt{s}}+o\left(\frac{1}{\sqrt{s}}\right), \quad s \rightarrow 0 \tag{1}
\end{equation*}
$$

where $s$ is the normal distance to the velocity discontinuity surface and $D$ is a function of in-surface coordinates. It is evident from Equation (1) that $\xi_{e q} \rightarrow \infty$ as $s \rightarrow 0$. Under plane strain conditions, the equivalent strain rate is defined by

$$
\begin{equation*}
\xi_{e q}=\frac{2}{\sqrt{3}} \sqrt{\xi_{\alpha \alpha}^{2}+\xi_{\alpha \beta}^{2}}=\frac{2}{\sqrt{3}} \sqrt{\xi_{\beta \beta}^{2}+\xi_{\alpha \beta}^{2}} \tag{2}
\end{equation*}
$$

where $\xi_{\alpha \alpha}, \xi_{\beta \beta}$ and $\xi_{\alpha \beta}$ are the components of the strain rate tensor in an arbitrary orthogonal coordinate system ( $\alpha, \beta$ ). The equation of incompressibility

$$
\begin{equation*}
\xi_{\alpha \alpha}+\xi_{\beta \beta}=0 \tag{3}
\end{equation*}
$$

has been taken into account to arrive at Equation (2). Since the normal strain rate components $\xi_{\alpha \alpha}$ and $\xi_{\beta \beta}$ are bounded, the condition (1) is equivalent to

$$
\begin{equation*}
\left|\xi_{\alpha \beta}\right|=\frac{\sqrt{3}}{2} \frac{D}{\sqrt{s}}+o(s), \quad s \rightarrow 0 \tag{4}
\end{equation*}
$$

Even though Equation (1) is valid for quite an arbitrary pressure-independent yield criterion, the present study is restricted to Mises yield criterion. According to this criterion $\sigma_{e q}=\sigma_{0}$ where $\sigma_{e q}$ is the equivalent Mises stress and $\sigma_{0}$ is the yield stress in tension, a material constant. In this case the upper bound theorem reads [7]

$$
\begin{equation*}
\iint_{S_{v}}\left(t_{i} v_{i}\right) d S \leq \sigma_{0} \iiint_{\Omega} \xi_{e q} d \Omega+\frac{\sigma_{0}}{\sqrt{3}} \iint_{S_{d}}\left|\left[u_{\tau}\right]\right| d S \tag{5}
\end{equation*}
$$

where $\Omega$ is the volume of material loaded by prescribed velocities $v_{i}$ over a part $S_{v}$ of its surface and $\left|\left[u_{\tau}\right]\right|$ is the amount of velocity jump across the velocity discontinuity surface $S_{d}$. The equivalent strain rate $\xi_{e q}$ and $\left[u_{\tau}\right]$ involved in Equation (5) should be found using any kinematically admissible velocity field $u_{i}$. Since the normal velocity must be continuous across any velocity discontinuity surface, the velocity component $u_{\tau}$ is tangent to this surface. Equation (5) enables the stresses $t_{i}$ applied over $S_{v}$ to be evaluated. If the only unknown load is a tensile force $F$ then Equation (5) can be transformed to

$$
\begin{equation*}
F_{u} V=\sigma_{0} \iint_{\Omega} \xi_{e q} d \Omega+\frac{\sigma_{0}}{\sqrt{3}} \iint_{S_{d}}\left|\left[u_{\tau}\right]\right| d S \tag{6}
\end{equation*}
$$

where $F_{u}$ is the upper bound on the magnitude of $F$ at plastic collapse and $V$ is the velocity of the point at which the force is applied. It has been assumed here that the
vectors $\mathbf{F}$ and $\mathbf{V}$ are collinear. It follows from Equation (1) that the volume integral in Equations (5) and (6) is improper. However, it is easy to show convergence.

The weld is idealised by a narrow layer of constant thickness. A typical weld configuration is illustrated in Figure 1. The thickness of the weld is denoted by 2 H and its width by $2 L$. There are two axes of symmetry coinciding with the axes of Cartesian coordinates $(x, y)$. In what follows, it is assumed that $x \equiv \alpha$ and $y \equiv \beta$. For a sake of simplicity, it is assumed that the boundary value problem is symmetric relative to these axes. It is therefore sufficient to get the solution in the domain $x \geq 0$ and $y \geq 0$. There should a rigid zone in the vicinity of the $x$-axis. This zone sticks to the base material whose motion is prescribed. Therefore, the motion of the rigid zone is prescribed as well. There are two velocity discontinuity surfaces (curves in the plane of flow), $0 b$ and $b c$ (Figure 2). The shape of the velocity discontinuity surface $0 b$ should be found from the solution.


Figure 1: Idealised weld configuration and Cartesian coordinate system.


Figure 2: Flow pattern including rigid zone, plastic zone and velocity discontinuity surface.

Let $u_{x}$ and $u_{y}$ be the velocity components in the Cartesian coordinate system. Because of symmetry, one of the velocity boundary conditions is

$$
\begin{equation*}
u_{x}=0 \tag{7}
\end{equation*}
$$

at $x=0$. In the case under consideration $s=H-x$. Therefore, it follows from Equation (4) that

$$
\begin{equation*}
u_{y}=U_{0}+\frac{U_{1}}{\sqrt{H-x}}+o\left(\frac{1}{\sqrt{H-x}}\right), \quad x \rightarrow H \tag{8}
\end{equation*}
$$

where $U_{0}$ and $U_{1}$ may depend on $y$. By assumption, $u_{x}$ is a linear function of $x$. Taking into account the boundary condition (7) it is possible to get

$$
\begin{equation*}
u_{x}=\beta(y) \frac{x}{H} . \tag{9}
\end{equation*}
$$

## 3 Tensile panel: general analysis

Assume that the specimen is a tensile panel. The line of action of the tensile force is parallel to the $x$-axis. The rigid base material moves along the $x$-axis with a velocity $V$. Then, the motion of the rigid zone (Figure 2) is a translation along the $x$-axis with the same velocity. It is evident that the magnitude of $V$ is immaterial. The velocity component $u_{x}$ should satisfy the boundary scondition

$$
\begin{equation*}
u_{x}=V \tag{10}
\end{equation*}
$$

at $x=H$. Comparing Equations (9) and (10) yields $\beta(y)=V$. Therefore,

$$
\begin{equation*}
u_{x}=\frac{V x}{H} . \tag{11}
\end{equation*}
$$

The equation of incompressibility (3) becomes

$$
\begin{equation*}
\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}=0 \tag{12}
\end{equation*}
$$

Substituting Equation (11) into Equation (12) and integrating yield

$$
\begin{equation*}
\frac{u_{y}}{V}=-\frac{y}{H}+g\left(\frac{x}{H}\right) \tag{13}
\end{equation*}
$$

where $g(x / H)$ is an arbitrary function of its argument. In addition to the boundary conditions (7) and (10), any kinematically admissible velocity field must satisfy the condition that the normal velocity is continuous across velocity discontinuity surfaces. This condition can be written as

$$
\begin{equation*}
\mathbf{u} \cdot \mathbf{n}=\mathbf{v} \cdot \mathbf{n} \tag{14}
\end{equation*}
$$

on the velocity discontinuity surface. Here $\mathbf{n}$ is the unit normal vector to the surface, $\mathbf{u}$ is the vector velocity field in the plastic zone and $\mathbf{v}$ is the vector velocity field in the rigid zone. Let $\varphi$ be the orientation of the tangent to the velocity discontinuity surface $0 b$ relative to the $x$-axis (Figure 3). Then, the unit normal vector to $0 b$ can be represented in the form

$$
\begin{equation*}
\mathbf{n}=-\sin \varphi \mathbf{i}+\cos \varphi \mathbf{j} \tag{15}
\end{equation*}
$$

where $\mathbf{i}$ and $\mathbf{j}$ are the base vectors of the Cartesian coordinate system (Figure 3). The vector velocity field in the rigid zone is

$$
\begin{equation*}
\frac{\mathbf{v}}{V}=\mathbf{i} . \tag{16}
\end{equation*}
$$



Figure 3: Velocity discontinuity surface (curve in the plane of flow).

The velocity vector field in the plastic zone is determined from Equations (11) and (13) as

$$
\begin{equation*}
\frac{\mathbf{u}}{V}=\frac{x}{H} \mathbf{i}+\left[-\frac{y}{H}+g\left(\frac{x}{H}\right)\right] \mathbf{j} . \tag{17}
\end{equation*}
$$

Substituting Equations (15) - (17) into Equation (14) gives

$$
\begin{equation*}
\left(1-\frac{x}{H}\right) \sin \varphi=\left[\frac{y}{H}-g\left(\frac{x}{H}\right)\right] \cos \varphi . \tag{18}
\end{equation*}
$$

It is evident (see Figure 3) that $\tan \varphi=d y / d x$. Therefore, Equation (18) becomes

$$
\begin{equation*}
\left(1-\frac{x}{H}\right) \frac{d y}{d x}=\frac{y}{H}-g\left(\frac{x}{H}\right) \tag{19}
\end{equation*}
$$

It is convenient to introduce the following dimensionless coordinates

$$
\begin{equation*}
\zeta=\frac{x}{H}, \quad \eta=\frac{y}{H} . \tag{20}
\end{equation*}
$$

Then, Equation (19) transforms to

$$
\begin{equation*}
\frac{d \eta}{d \zeta}=\frac{\eta-g(\zeta)}{1-\zeta} \tag{21}
\end{equation*}
$$

This is an ordinary linear differential equation of first order with respect to $\eta$. The general solution to Equation (21) is

$$
\begin{equation*}
\eta=\frac{1}{(1-\zeta)}\left(\eta_{0}-\int_{1}^{\zeta} g(\chi) d \chi\right) \tag{22}
\end{equation*}
$$

where $\eta_{0}$ is a constant of integration and $\chi$ is a dummy variable of integration. It follows from the structure of the solution (22) that the velocity discontinuity surface $0 b$ may have a common point with the line $x=H$ (or $\zeta=1$ ) if and only if $\eta_{0}=0$. Then, Equation (22) becomes

$$
\begin{equation*}
\eta=\eta_{0 b}(\zeta)=-\frac{1}{(1-\zeta)} \int_{1}^{\zeta} g(\chi) d \chi \tag{23}
\end{equation*}
$$

Applying l'Hospital rule it is possible to find from this equation that

$$
\begin{equation*}
\eta_{b}=g(1) \tag{24}
\end{equation*}
$$

where $\eta_{b}$ is the value of $\eta$ at $\zeta=1$. Thus, it follows from Equation (20) that the $y$ coordinate of point $b$ (Figure 2) is $y_{b}=H g(1)$. It is seen from Figure 2 that the solution under consideration is valid if and only if $y_{b} \leq L$. Therefore, a restriction on the function $g(\zeta)$ is

$$
\begin{equation*}
g(1) \leq \frac{L}{H} . \tag{25}
\end{equation*}
$$

Equation (23) determines the shape of the velocity discontinuity surface $0 b$. It follows from Equations (11) and (13) that

$$
\begin{equation*}
\xi_{x x}=-\xi_{y y}=\frac{V}{H}, \quad \xi_{x y}=\frac{V}{2 H} g^{\prime}(\zeta) \tag{26}
\end{equation*}
$$

where $g^{\prime}(\zeta) \equiv d g / d \zeta$. Substituting Equation (26) into Equation (2) leads to

$$
\begin{equation*}
\xi_{e q}=\frac{V}{\sqrt{3} H} \sqrt{4+\left[g^{\prime}(\zeta)\right]^{2}} . \tag{27}
\end{equation*}
$$

Since the normal velocity is continuous, the amount of velocity jump across the velocity discontinuity surface $0 b$ is determined by the equation $\left|\left[u_{\tau}\right]\right|_{0 b}=|\mathbf{v}-\mathbf{u}|$ where the vectors $\mathbf{v}$ and $\mathbf{u}$ are given in Equations (16) and (17). Then, using Equations (20) and (23)

$$
\begin{equation*}
\left.\left[u_{\tau}\right]\right]_{0 b}=V \sqrt{(1-\zeta)^{2}+\left[\eta_{0 b}(\zeta)-g(\zeta)\right]^{2}} \tag{28}
\end{equation*}
$$

An infinitesimal surface element of the velocity discontinuity surface $0 b$ is given by

$$
\begin{equation*}
d S_{0 b}=2 H B \sqrt{1+\left(\frac{d \eta_{0 b}}{d \zeta}\right)^{2}} d \zeta \tag{29}
\end{equation*}
$$

where $2 B$ is the thickness of the specimen. The derivative $d \eta_{0 b} / d \zeta$ is determined from Equation (21) in which $\eta$ on its right hand side should be replaced with $\eta_{0 b}(\zeta)$ from Equation (23). Then, Equation (29) becomes

$$
\begin{equation*}
d S_{0 b}=\frac{2 H B}{(1-\zeta)} \sqrt{(1-\zeta)^{2}+\left[\eta_{0 b}(\zeta)-g(\zeta)\right]^{2}} d \zeta \tag{30}
\end{equation*}
$$

An infinitesimal volume element is $d \Omega=2 B d x d y$. Taking into account Equation (20) it is possible to get $d \Omega=2 B H^{2} d \zeta d \eta$. Then, using Equation (27) and integrating with respect to the thickness direction and $\eta$ the rate of work dissipation in the plastic zone is determined as

$$
\begin{equation*}
\sigma_{0} \iiint_{\Omega} \xi_{e q} d \Omega=\frac{2 V B H \sigma_{0}}{\sqrt{3}} \int_{0}^{1}\left[\frac{L}{H}-\eta_{0 b}(\zeta)\right] \sqrt{4+\left[g^{\prime}(\zeta)\right]^{2}} d \zeta \tag{31}
\end{equation*}
$$

Here $\eta_{0 b}(\zeta)$ should be replaced with a function of $\zeta$ by means of Equation (23). Then, in general, the integral in Equation (31) can be evaluated. Using Equations (28) and (30) the rate of work dissipation at the velocity discontinuity surface $0 b$ is given by

$$
\begin{equation*}
\frac{\sigma_{0}}{\sqrt{3}} \iint_{S_{d}}\left|\left[u_{\tau}\right]\right|_{0 b} d S=\frac{2 V B H \sigma_{0}}{\sqrt{3}} \int_{0}^{1} \frac{\left\{(1-\zeta)^{2}+\left[\eta_{0 b}(\zeta)-g(\zeta)\right]^{2}\right\}}{(1-\zeta)} d \zeta \tag{32}
\end{equation*}
$$

The amount of velocity jump across the velocity discontinuity surface $b c$ (Figure 2) is simply equal to $\left|u_{y}\right|$ at $x=H$ (or $\zeta=1$ ). Therefore, it follows from Equations (13), (20) and (24) that

$$
\begin{equation*}
\mid\left[u_{\tau}\right] \|_{b c}=V\left(\eta-\eta_{b}\right) . \tag{33}
\end{equation*}
$$

It has been taken into account here that $\eta \geq \eta_{b}$ between points $b$ and $c$ (Figure 2). Also, using Equation (20) an infinitesimal surface element of the velocity discontinuity surface $b c$ can be represented as $d S_{b c}=2 B d y=2 B H d \eta$. Therefore, using Equation (33) and integrating the rate of work dissipation at the velocity discontinuity surface $b c$ can be found as

$$
\begin{equation*}
\frac{\sigma_{0}}{\sqrt{3}} \iint_{S_{d}}\left|\left[u_{\tau}\right]\right|_{b c} d S=\frac{2 V B H \sigma_{0}}{\sqrt{3}} \int_{\eta_{b}}^{L / H}\left(\eta-\eta_{b}\right) d \eta=\frac{V B H \sigma_{0}}{\sqrt{3}}\left(\frac{L}{H}-\eta_{b}\right)^{2} . \tag{34}
\end{equation*}
$$

In the case under consideration Equation (6) is valid. Therefore, using Equations (31), (32) and (34) the upper bound on the limit load is represented in the following form

$$
\begin{align*}
& \frac{F_{u}}{2}=\frac{2 B H \sigma_{0}}{\sqrt{3}} \int_{0}^{1}\left[\frac{L}{H}-\eta_{0 b}(\zeta)\right] \sqrt{4+\left[g^{\prime}(\zeta)\right]^{2}} d \zeta+ \\
& +\frac{2 B H \sigma_{0}}{\sqrt{3}} \int_{0}^{1} \frac{\left\{(1-\zeta)^{2}+\left[\eta_{0 b}(\zeta)-g(\zeta)\right]^{2}\right\}}{(1-\zeta)} d \zeta+\frac{V B H \sigma_{0}}{\sqrt{3}}\left(\frac{L}{H}-\eta_{b}\right)^{2} \tag{35}
\end{align*}
$$

The factor $1 / 2$ on the left hand side of this equation has appeared because two identical forces act and one quarter of the specimen has been considered to derive Equations (31), (32) and (34). It is convenient to rewrite Equation (35) in nondimensional form. Then, denoting $h=H / L$

$$
\begin{align*}
& f_{u}=\frac{F_{u}}{4 B L \sigma_{0}}=\frac{h}{\sqrt{3}} \int_{0}^{1}\left[\frac{1}{h}-\eta_{0 b}(\zeta)\right] \sqrt{4+\left[g^{\prime}(\zeta)\right]^{2}} d \zeta+ \\
& +\frac{h}{\sqrt{3}} \int_{0}^{1} \frac{\left\{(1-\zeta)^{2}+\left[\eta_{0 b}(\zeta)-g(\zeta)\right]^{2}\right\}}{(1-\zeta)} d \zeta+\frac{h}{2 \sqrt{3}}\left(\frac{1}{h}-\eta_{b}\right)^{2} \tag{36}
\end{align*}
$$

## 4 Tensile panel: numerical example

An accurate numerical solution for compression of a thin layer of plastic material between two parallel plates has been proposed in [7]. It has been assumed that the maximum friction law occurs at the interface between the plate and plastic material. Mathematically, this boundary condition is equivalent to the conditions at velocity discontinuity surfaces. Therefore, the problem solved in [7] is equivalent to the problem formulated in the previous section. The difference in sign is immaterial for pressure-independent materials. The solution given in [7] can be approximated by

$$
\begin{equation*}
f_{0}=\frac{F_{0}}{4 B L \sigma_{0}}=\frac{1}{2 \sqrt{3}}\left(3+\frac{1}{h}\right) . \tag{37}
\end{equation*}
$$

It is of interest to compare this solution and the solution given in Equation (36). Symmetry demands that the function $g(\zeta)$ involved in Equation (13) is an even function of its argument. One of the simplest functions satisfying this condition and Equation (8) is

$$
\begin{equation*}
g(\zeta)=\beta_{0}+\beta_{1} \sqrt{1-\zeta^{2}} \tag{38}
\end{equation*}
$$

Substituting Equation (38) into Equation (23) leads to

$$
\begin{equation*}
\eta_{0 b}(\zeta)=\beta_{0}+\frac{\beta_{1}}{2(\zeta-1)}\left[\zeta \sqrt{1-\zeta^{2}}+\arcsin \zeta-\frac{\pi}{2}\right] \tag{39}
\end{equation*}
$$

Also, it follows from Equations (24) and (38) that

$$
\begin{equation*}
\eta_{b}=\beta_{0} . \tag{40}
\end{equation*}
$$

The curve $\eta_{0 b}(\zeta)$ must contains the origin of the coordinate system (otherwise, the rigid zones in the domains $x \geq 0$ and $x \leq 0$ cannot move in the opposite directions). Therefore, it follows from Equation (39) that

$$
\begin{equation*}
\beta_{1}=-\frac{4 \beta_{0}}{\pi} . \tag{41}
\end{equation*}
$$

The restriction (25) transforms to

$$
\begin{equation*}
\beta_{0} \leq \frac{1}{h} . \tag{42}
\end{equation*}
$$

Substituting Equations (39) and (40) into Equation (36) and using Equation (41) it is possible to obtain $f_{u}$ as a function of one variable, $\beta_{0}$. Minimizing this function with respect to $\beta_{0}$ gives the best upper bound limit load based on the kinematically admissible velocity field chosen. This minimization has been carried out numerically. The condition (42) has been checked in course of numerical minimization. It has been found that this condition is not satisfied for $h>0.54$. As a result of numerical minimization, the dependence of $f_{u}$ on $h$ has been found. This dependence is illustrated in Figure 4 by the broken line. The solid line corresponds to Equation (37). It is seen from this figure that the limit load found from Equation (36) with the use of the singular velocity field is just slightly higher than that from the accurate numerical solution.


Figure 4: Comparison on numerical and semi-analytical solutions.
It is straightforward to account for a crack located within the weld. General methods have been derived in [10, 11]. These methods can be used in conjunction with the solution (36) to determine the upper bound limit load for tensile panels with a crack.

## 5 Conclusions

The main finding of this paper is that a unified and simple approach to build up kinematically admissible velocity fields for a class of highly undermatched welded joints leads to an upper bound limit load whose magnitude is very close to that found by means of the slip-line technique (Figure 4). The approach proposed can be extended to bent specimens. To this end, it is necessary to assume that the rigid zone (Figure 2) rotates around the origin of the coordinate system with an angular velocity $\omega$. In this case the function $\beta(y)$ in Equation (9) should be chosen in the
form $\beta(y)=\omega y$. Then, the analysis similar to that presented in Section 3 should be carried out. A further generalization can be obtained assuming a general motion of the rigid zone in an $x y$-plane. As a result, the solution for simultaneous bending and tension can be found.

It is also possible to consider more general geometry of the weld. In particular, the shape shown in Figure 1 can be replaced with a curvilinear rectangle. Then, the coordinate system $(\alpha, \beta)$ can be chosen such that its long sides of the rectangular are determined by the equations $\alpha= \pm 1$ and the axis of symmetry $x=0$ is determined by the equation $\alpha=0$. The subsequent general analysis is similar to that presented in Section 3.

It is straightforward to account for plastic anisotropy of the weld. This material property has no effect of the general method to build up kinematically admissible velocity fields. In order to get the solution for anisotropic materials, it is just necessary to replace the right hand side of Equation (5) with appropriate functionals. Examples of anisotropic solutions for highly undermatched welded joints accounting for the singular behaviour of real velocity fields in the vicinity of maximum shear stress surfaces are given in $[12,13]$.

Finally, three-dimensional deformational deformation within the weld can be considered assuming that the velocity component normal to the $x y$-plane does not vanish. Examples are provided in [10, 14]. Indeed, the general analysis to be carried out is more complicated in this case as compared with plane strain deformation. However, the final result of analytical derivation is still feasible for rather a simple numerical treatment.

## Acknowledgment

The research described in this paper has been supported by grant RFBR-12-0891704.

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