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Free vibration Analysis of Laminated Plates using Wavelet Collocation and a Unified Formulation

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Abstract

A study of free vibrations of shear flexible isotropic and laminated composite plates with the Carrera's unified formulation is presented. The analysis is based on collocation with a Deslaurier Dubuc interpolating basis to produce highly accurate results. The high order collocation method presented in this paper proved to be very accurate for this type of problems and the numerical efficiency is as good as other numerical schemes, such as finite element solutions.

Keywords: collocation, wavelets, vibrations, composites, plates.

1 Introduction

This paper deals with the free vibration analysis of composite plates by a wavelet collocation method [1, 2]. The unified formulation by Carrera [3, 4, 5, 6] is used to model the kinematics of the laminated plate deformations.

The analysis of static deformations and free vibration of shear-flexible plates by numerical techniques, was performed by [20, 10, 22], using the differential quadrature method. In [9, 31, 27]) the finite element method was used with success. More recently the analysis of isotropic and laminated plates by Kansa's non-symmetric radial basis function collocation method was performed by Ferreira [13, 19, 76, 78, 77, 87, 14, 16].

The method employed for the numerical solution is a collocation method based on Deslaurier-Dubuc interpolating basis in hierarchical form [35].

2 Interpolating Wavelets

The Deslaurier-Dubuc fundamental function [36] of order $N = 2L + 1$ is defined as the autocorrelation of Daubechies scaling functions, ϕ_L [37], as follows:

$$\vartheta(x) = \int_{\mathbb{R}} \phi_L(y)\phi_L(y - x)\, dy \tag{1}$$

The scaling function ϕ_L satisfies the following properties:

1. $supp\, \phi_L = [0, 2L + 1]$.
2. $\phi_L \in W^{R/2,\infty}$ for some $R > 0$ (R is proportional to L): $|(d^s/dx^s)\phi_L| \leq C$, for all integers s, with com $0 \leq s \leq R/2$;
3. ϕ_L is orthogonal to all its integer translates: $\int \phi_L(x)\phi_L(x - k)\, dx = \delta_{0k}$
4. All polynomials up to order L can be exactly represented as a linear combination of function ϕ_L and all its integer translates.

As a consequence of the above properties, function ϑ satisfies:

1. $supp\, \vartheta = [-N, N], \quad \text{and} \quad \vartheta \in W^{R,\infty}$;
2. Due to the orthogonality of the translates of ϕ_L, the function ϑ presents the follwing interpolating property:
$$\vartheta(n) = \int_{\mathbb{R}} \phi_L(y)\phi_L(y - n)\, dy = \delta_{n0}. \tag{2}$$
3. All polynomials up to order N can be exactly represented as a linear combination of function ϑ and all its integer translates.

Based on the fundamental function ϑ it is possible to build the complete wavelet system on $\mathbb{R}$. As described in detail in [1], tensor products will lead to wavelet systems on $\mathbb{R}^d$.

Following the ideas and techniques described in [38, 39], it is the possible to build a Deslaurier-Dubuc wavelet system *on the interval* $[0, 1]$. As described in [34] for $j \geq j_0 = [\log_2(N/2)] + 1$ we define

$$\vartheta_{jk} = \vartheta(2^j x - k) + \sum_{n=-N+1}^{-1} a_{nk}\vartheta(2^j x - n), \qquad k = 0, .., L \tag{3}$$

$$\vartheta_{jk} = \vartheta(2^j x - k), \qquad k = L + 1, ..., 2^j - L - 1, \tag{4}$$

$$\vartheta_{jk} = \vartheta(2^j x - k) + \sum_{n=2^j+1}^{2^j+N-1} b_{nk}\vartheta(2^j x - n), \qquad k = 2^j - L, .., 2^j, \tag{5}$$

where the coefficients a_{nk} and b_{nk} are defined by:

$$a_{nk} = l^1_{jk}(n2^{-j}), \qquad b_{nk} = l^2_{jk}(n2^{-j}), \tag{6}$$

and where l^1_{jk} and l^2_{jk} represent Lagrange interpolation polynomials of degree L, defined by:

$$l^1_{jk} = \prod_{\substack{i=0\\ i\neq k}}^{L} \frac{x - i2^{-j}}{k2^{-j} - i2^{-j}}, \qquad l^2_{jk} = \prod_{\substack{i=2^j-L\\ i\neq k}}^{2^j} \frac{x - i2^{-j}}{k2^{-j} - i2^{-j}}. \tag{7}$$

An interpolating multiresolution analysis (MRA) on the interval $[0,1]$ is defined by a set of closed subspaces $V_j = span < \vartheta_{jk}, \quad k = 0,..,2^j > \quad \subset \quad L^2(0,1)$. By using tensor products it is then possible to define a multiresolution on the square $[0,1]^2$. The two dimensional scaling functions $\vartheta_{j,\mathbf{k}}, \ \ \mathbf{k} = (k_1,k_2) \in G_j = \{0,..,2^j\}^2$ are defined by

$$\vartheta_{j,\mathbf{k}} = \vartheta_{jk_1} \otimes \vartheta_{jk_2} \tag{8}$$

The subspace $\mathbb{V}_j$ is the defined by:

$$\mathbb{V}_j = span < \vartheta_{j,\mathbf{k}}, \ \ \mathbf{k} = (k_1,k_2) \in \{0,..,2^j\}^2 > \tag{9}$$

It is easy to define an interpolation operator $L_j : C^0([0,1]^2) \rightarrow \mathbb{V}_j$

$$L_j f = \sum_{\mathbf{k}\in G_j} f(\mathbf{k}/2^j)\theta_{j,\mathbf{k}}. \tag{10}$$

The wavelet basis for the complement space $\mathbb{W}_j = (L_{j+1} - L_j)\mathbb{V}_{j+1}$ is composed by the functions

$$\begin{aligned}
\psi^{(1,0)}_{j,\mathbf{k}} &= \vartheta_{j+1,2k_1-1} \otimes \vartheta_{j,2k_2} && (11)\\
\psi^{(0,1)}_{j,\mathbf{k}} &= \vartheta_{j,2k_1} \otimes \vartheta_{j+1,2k_2-1} && (12)\\
\psi^{(1,1)}_{j,\mathbf{k}} &= \vartheta_{j+1,2k_1-1} \otimes \vartheta_{j+1,2k_2-1} && (13)
\end{aligned}$$

and a hierarchical basis for $\mathbb{V}_j$ can be assembled as

$$\{\vartheta_{j_0,\mathbf{k}}, \ \ \mathbf{k} = (k_1,k_2) \in \{0,..,2^{j_0}\}^2 \bigcup_{m=j_0}^{j-1} \{\psi^{(1,0)}_{m,\mathbf{k}}, \psi^{(0,1)}_{m,\mathbf{k}}, \psi^{(1,1)}_{m,\mathbf{k}}, \ \ \mathbf{k} = (k_1,k_2) \in \{0,..,2^m\} \tag{14}$$

The grid points corresponding to the scaling functions and the wavelets are defined by:

$$\zeta_{j,\mathbf{k}} = (k_1 2^{-j}, k_2 2^{-j}). \tag{15}$$

For the sake of simplicity we will use the following compact notation: given $\lambda = (\eta, j, \mathbf{k})$ with $\eta \in \Xi = \{0,1\}^2 \backslash \{0,0\}$, $j \geq j_0$, and $\mathbf{k}$ such that $\xi^\eta_{j,\mathbf{k}} \in [0,1]^2$, we define

$$\psi_\lambda = \psi^\eta_{j,\mathbf{k}}, \qquad \qquad \xi_\lambda = \xi^\eta_{j,\mathbf{k}}. \tag{16}$$

Any continuous function $f \in C^0([0,1]^2)$ can be expanded in the form

$$f = \sum_{\mathbf{k} \in \{0,..,2^{j_0}\}^2} \beta_{j_0 \mathbf{k}} \vartheta_{j_0 \mathbf{k}} + \sum_{\lambda \in \Lambda} \alpha_\lambda \psi_\lambda, \tag{17}$$

where

$$\Lambda = \{(\eta, j, \mathbf{k}),\ \eta \in \Xi, j \geq j_0, \mathbf{k} \text{ such that } \xi^\eta_{j,\mathbf{k}} \in [0,1]^2\} \tag{18}$$

denotes the set of compact indexes.

It can be shown [34] that the scaling functions are responsible for representing f at a given level of resolution and the wavelets define the *detail* that is necessary to add to switch from one level of resolution to the following. Consequently, the value of the wavelet coefficents, α_λ, allow for the identification of the region of the domain where details are important, which correspond to the regions where the discretization should be improved.

3 Collocation technique

This section briefly describes the collocation method based on Deslaurier-Dubuc interpolating wavelets. We consider here an uniform discretization, though the collocation method that we present does not a priori require the uniformity of the grid and can easily be adapted to the case of non uniform grids of dyadic points. For any $j \geq j_0$, let the dyadic grid G_j be defined by

$$G_j := \{\zeta_{j,\mathbf{k}}, \quad \mathbf{k} \in \{0, \cdots, 2^j\}^2\}. \tag{19}$$

In order to take into account the boundary conditions, the grid G_j is subdivided into a set of interior nodes and sets of Neumann and Dirichlet boundary nodes. It is then possible to write:

$$G_j = G_j^{(i)} \cup G_j^{(N)} \cup G_j^{(D)}$$

with

$$G_j^{(i)} = G_j \cap]0,1[^2, \qquad G_j^{(N)} = G_j \cap \Gamma_\sigma, \qquad G_j^{(D)} = G_j \cap \Gamma_u.$$

Problem (P) can be discretized as follows:

Find $\mathbf{u} \in \mathbb{V}_j$ such that

$$\mathcal{A}\mathbf{u}_h(p) = f(p) \qquad \text{for all nodes p } \in G_j^{(i)} \tag{20}$$

$$\mathbf{u}_h(p) = g(\mathbf{x}_\lambda) \qquad \text{for all nodes p } \in G_j^{(D)} \tag{21}$$

$$\mathcal{B}\mathbf{u}_h(p) = \mathbf{t}(p) \qquad \text{for all nodes p } \in G_j^{(N)} \tag{22}$$

4 The Unified Formulation

The unified formulation (UF) proposed by Carrera [3, 4, 5, 6], also known as CUF, is a powerful framework for the analysis of beams, plates and shells. This formulation has been applied in several finite element analyses, either using the Principle of Virtual Displacements, or by using the Reissner's Mixed Variational theorem. The stiffness matrix components, the external force terms or the inertia terms can be obtained directly with this UF, irrespective of the shear deformation theory being considered.

In this section the Carrera's unified formulation [3, 4, 5, 6] is briefly reviewed. It is shown how to obtain the fundamental nuclei, which allows the derivation of the equations of motion and boundary conditions, in weak form for the finite element analysis; and in strong form for the present RBF collocation.

4.1 Governing equations and boundary conditions in the framework of Unified Formulation

Although one can use the UF for a one-layer, isotropic plate, a multi-layered plate with N_l layers is considered. The Principle of Virtual Displacements (PVD) for the pure-mechanical case reads:

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \left\{ \delta \epsilon_{pG}^{k}{}^T \sigma_{pC}^k + \delta \epsilon_{nG}^{k}{}^T \sigma_{nC}^k \right\} d\Omega_k dz = \sum_{k=1}^{N_l} \delta L_e^k \tag{23}$$

where Ω_k and A_k are the integration domains in plane (x,y) and z direction, respectively. Here, k indicates the layer and T the transpose of a vector, and δL_e^k is the external virtual work for the kth layer. G means geometrical relations and C constitutive equations.

The steps to obtain the governing equations are:

- Substitution of the geometrical relations (subscript G)
- Substitution of the appropriate constitutive equations (subscript C)
- Introduction of the unified formulation

Stresses and strains are separated into in-plane and through-the-thickness components, denoted respectively by the subscripts p and n. The mechanical strains in the kth layer can be related to the displacement field $\mathbf{u}^k = \{u_x^k, u_y^k, u_z^k\}$ via the geometrical relations:

$$\begin{aligned} \epsilon_{pG}^k &= [\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy}]^{kT} = \mathbf{D}_p^k \mathbf{u}^k \ , \\ \epsilon_{nG}^k &= [\gamma_{xz}, \gamma_{yz}, \epsilon_{zz}]^{kT} = \left(\mathbf{D}_{np}^k + \mathbf{D}_{nz}^k\right) \mathbf{u}^k \ , \end{aligned} \tag{24}$$

wherein the differential operator arrays are defined as follows:

$$\mathbf{D}_p^k = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix}, \quad \mathbf{D}_{np}^k = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{nz}^k = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}, \tag{25}$$

The 3D constitutive equations are given as:

$$\begin{aligned} \sigma_{pC}^k &= \mathbf{C}_{pp}^k \ \epsilon_{pG}^k + \mathbf{C}_{pn}^k \ \epsilon_{nG}^k \\ \sigma_{nC}^k &= \mathbf{C}_{np}^k \ \epsilon_{pG}^k + \mathbf{C}_{nn}^k \ \epsilon_{nG}^k \end{aligned} \tag{26}$$

with

$$\begin{aligned} \mathbf{C}_{pp}^k &= \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \qquad \mathbf{C}_{pn}^k = \begin{bmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ 0 & 0 & C_{36} \end{bmatrix} \\ \mathbf{C}_{np}^k &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{13} & C_{23} & C_{36} \end{bmatrix} \qquad \mathbf{C}_{nn}^k = \begin{bmatrix} C_{55} & C_{45} & 0 \\ C_{45} & C_{44} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \end{aligned} \tag{27}$$

According to the unified formulation by Carrera, the three displacement components u_x, u_y and u_z and their relative variations can be modelled as:

$$(u_x, u_y, u_z) = F_\tau \ (u_{x\tau}, u_{y\tau}, u_{z\tau}) \qquad (\delta u_x, \delta u_y, \delta u_z) = F_s \ (\delta u_{xs}, \delta u_{ys}, \delta u_{zs}) \tag{28}$$

with Taylor expansions from first up to 4^{th} order: $F_0 = z^0 = 1$, $F_1 = z^1 = z$, ..., $F_N = z^N$, ..., $F_4 = z^4$ if an Equivalent Single Layer (ESL) approach is used.

Substituting the geometrical relations, the constitutive equations and the unified formulation into the variational statement PVD, for the kth layer, one has:

$$\begin{aligned} \int_{\Omega_k}\int_{A_k} \Big[&(\mathbf{D}_p^k \ F_s \delta \mathbf{u}_s^k)^T (\mathbf{C}_{pp}^k \mathbf{D}_p^k \ F_\tau \mathbf{u}_\tau^k + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) F_\tau \mathbf{u}_\tau^k) \\ &+ ((\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) F_s \delta \mathbf{u}_s^k)^T (\mathbf{C}_{np}^k \mathbf{D}_p^k \ F_\tau \mathbf{u}_\tau^k + \mathbf{C}_{nn}^k (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k) F_\tau \mathbf{u}_\tau^k) \Big] d\Omega_k dz = \delta L_e^k \end{aligned} \tag{29}$$

At this point, the formula of integration by parts is applied:

$$\int_{\Omega_k} \left((\mathbf{D}_\Omega) \delta \mathbf{a}^k \right)^T \mathbf{a}^k d\Omega_k = - \int_{\Omega_k} \delta \mathbf{a}^{k^T} \left((\mathbf{D}_\Omega^T) \mathbf{a}^k \right) d\Omega_k + \int_{\Gamma_k} \delta \mathbf{a}^{k^T} \left((\mathbf{I}_\Omega) \mathbf{a}^k \right) d\Gamma_k \tag{30}$$

where the $\mathbf{I}_\Omega$ matrix is obtained applying the *Divergence theorem*:

$$\int_\Omega \frac{\partial \psi}{\partial x_i} dv = \oint_\Gamma n_i \psi ds \tag{31}$$

In (31) n_i are the components of the normal $\widehat{n}$ to the boundary along the direction i. After integration by parts, the governing equations and boundary conditions for the plate in the mechanical case are obtained:

$$
\begin{aligned}
&\int_{\Omega_k}\int_{A_k} (\,\delta\mathbf{u}_s^k)^T\Big[\Big(\big(-\mathbf{D}_p^k\big)^T\big(\mathbf{C}_{pp}^k(\mathbf{D}_p^k)\ +\mathbf{C}_{pn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k\big)\\
&+\big(-\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k\big)^T\big(\mathbf{C}_{np}^k(\mathbf{D}_p^k)+\mathbf{C}_{nn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k)\big)\Big)\mathbf{F}_\tau\mathbf{F}_s\mathbf{u}_\tau^k\Big]dxdydz\\
&+\int_{\Omega_k}\int_{A_k} (\,\delta\mathbf{u}_s^k)^T\Big[\Big(\mathbf{I}_p^{kT}\big(\mathbf{C}_{pp}^k(\mathbf{D}_p^k)\ +\mathbf{C}_{pn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k)\big)\\
&+\mathbf{I}_{np}^{kT}\big(\mathbf{C}_{np}^k(\mathbf{D}_p^k)\ +\mathbf{C}_{nn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k)\big)\Big)\mathbf{F}_\tau\mathbf{F}_s\mathbf{u}_\tau^k\Big]dxdydz=\int_{\Omega_k}\delta\mathbf{u}_s^{kT}F_s\mathbf{p}_u^k d\Omega_k\ .
\end{aligned}
\tag{32}
$$

where $\mathbf{I}_p^k$ and $\mathbf{I}_{np}^k$ depend on the boundary geometry:

$$
\mathbf{I}_p^k=\begin{bmatrix} n_x & 0 & 0\\ 0 & n_y & 0\\ n_y & n_x & 0\end{bmatrix}\ ,\quad \mathbf{I}_{np}^k=\begin{bmatrix} 0 & 0 & n_x\\ 0 & 0 & n_y\\ 0 & 0 & 0\end{bmatrix}\ . \tag{33}
$$

The normal to the boundary of domain Ω is:

$$
\widehat{\mathbf{n}}=\begin{bmatrix} n_x\\ n_y\end{bmatrix}=\begin{bmatrix} cos(\varphi_x)\\ cos(\varphi_y)\end{bmatrix} \tag{34}
$$

where φ_x and φ_y are the angles between the normal $\widehat{n}$ and the direction x and y respectively.

The governing equations for a multi-layered plate subjected to mechanical loadings are:

$$
\delta\mathbf{u}_s^{k^T}\ :\qquad \mathbf{K}_{uu}^{k\tau s}\,\mathbf{u}_\tau^k\ =\ \mathbf{P}_{u\tau}^k \tag{35}
$$

where the fundamental nucleus $\mathbf{K}_{uu}^{k\tau s}$ is obtained as:

$$
\begin{aligned}
\mathbf{K}_{uu}^{k\tau s}=&\Big[\big(-\mathbf{D}_p^k\big)^T\big(\mathbf{C}_{pp}^k(\mathbf{D}_p^k)\ +\mathbf{C}_{pn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k\big)\\
&+\big(-\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k\big)^T\big(\mathbf{C}_{np}^k(\mathbf{D}_p^k)+\mathbf{C}_{nn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k)\big)\Big]\mathbf{F}_\tau\mathbf{F}_s
\end{aligned}
\tag{36}
$$

and the corresponding Neumann-type boundary conditions on Γ_k are:

$$
\mathbf{\Pi}_d^{k\tau s}\,\mathbf{u}_\tau^k\ =\ \mathbf{\Pi}_d^{k\tau s}\,\bar{\mathbf{u}}_\tau^k\ , \tag{37}
$$

where:

$$
\begin{aligned}
\mathbf{\Pi}_d^{k\tau s}=&\Big[\mathbf{I}_p^{kT}\big(\mathbf{C}_{pp}^k(\mathbf{D}_p^k)\ +\mathbf{C}_{pn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k)\big)+\\
&\mathbf{I}_{np}^{kT}\big(\mathbf{C}_{np}^k(\mathbf{D}_p^k)\ +\mathbf{C}_{nn}^k(\mathbf{D}_{n\Omega}^k+\mathbf{D}_{nz}^k)\big)\Big]\mathbf{F}_\tau\mathbf{F}_s
\end{aligned}
\tag{38}
$$

and $\mathbf{P}_{u\tau}^k$ are variationally consistent loads with applied pressure.

4.2 Fundamental nuclei

The fundamental nuclei in explicit form are then obtained as:

$$
\begin{aligned}
K^{k\tau s}_{uu_{11}} =&(-\partial^\tau_x\partial^s_x C_{11} - \partial^\tau_x\partial^s_y C_{16} + \partial^\tau_z\partial^s_z C_{55} - \partial^\tau_y\partial^s_x C_{16} - \partial^\tau_y\partial^s_y C_{66}) F_\tau F_s \\
K^{k\tau s}_{uu_{12}} =&(-\partial^\tau_x\partial^s_y C_{12} - \partial^\tau_x\partial^s_x C_{16} + \partial^\tau_z\partial^s_z C_{45} - \partial^\tau_y\partial^s_y C_{26} - \partial^\tau_y\partial^s_x C_{66}) F_\tau F_s \\
K^{k\tau s}_{uu_{13}} =&(-\partial^\tau_x\partial^s_z C_{13} - \partial^\tau_y\partial^s_z C_{36} + \partial^\tau_z\partial^s_y C_{45} + \partial^\tau_z\partial^s_x C_{55}) F_\tau F_s \\
K^{k\tau s}_{uu_{21}} =&(-\partial^\tau_y\partial^s_x C_{12} - \partial^\tau_y\partial^s_y C_{26} + \partial^\tau_z\partial^s_z C_{45} - \partial^\tau_x\partial^s_x C_{16} - \partial^\tau_x\partial^s_y C_{66}) F_\tau F_s \\
K^{k\tau s}_{uu_{22}} =&(-\partial^\tau_y\partial^s_y C_{22} - \partial^\tau_y\partial^s_x C_{26} + \partial^\tau_z\partial^s_z C_{44} - \partial^\tau_x\partial^s_y C_{26} - \partial^\tau_x\partial^s_x C_{66}) F_\tau F_s \\
K^{k\tau s}_{uu_{23}} =&(-\partial^\tau_y\partial^s_z C_{23} - \partial^\tau_x\partial^s_z C_{36} + \partial^\tau_z\partial^s_y C_{44} + \partial^\tau_z\partial^s_x C_{45}) F_\tau F_s \\
K^{k\tau s}_{uu_{31}} =&(\partial^\tau_z\partial^s_x C_{13} + \partial^\tau_z\partial^s_y C_{36} - \partial^\tau_y\partial^s_z C_{45} - \partial^\tau_x\partial^s_z C_{55}) F_\tau F_s \\
K^{k\tau s}_{uu_{32}} =&(\partial^\tau_z\partial^s_y C_{23} + \partial^\tau_z\partial^s_x C_{36} - \partial^\tau_y\partial^s_z C_{44} - \partial^\tau_x\partial^s_z C_{45}) F_\tau F_s \\
K^{k\tau s}_{uu_{33}} =&(\partial^\tau_z\partial^s_z C_{33} - \partial^\tau_y\partial^s_y C_{44} - \partial^\tau_y\partial^s_x C_{45} - \partial^\tau_x\partial^s_y C_{45} - \partial^\tau_x\partial^s_x C_{55}) F_\tau F_s
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
\Pi^{k\tau s}_{11} =&(n_x\partial^s_x C_{11} + n_x\partial^s_y C_{16} + n_y\partial^s_x C_{16} + n_y\partial^s_y C_{66}) F_\tau F_s \\
\Pi^{k\tau s}_{12} =&(n_x\partial^s_y C_{12} + n_x\partial^s_x C_{16} + n_y\partial^s_y C_{26} + n_y\partial^s_x C_{66}) F_\tau F_s \\
\Pi^{k\tau s}_{13} =&(n_x\partial^s_z C_{13} + n_y\partial^s_z C_{36}) F_\tau F_s \\
\Pi^{k\tau s}_{21} =&(n_y\partial^s_x C_{12} + n_y\partial^s_y C_{26} + n_x\partial^s_x C_{16} + n_x\partial^s_y C_{66}) F_\tau F_s \\
\Pi^{k\tau s}_{22} =&(n_y\partial^s_y C_{22} + n_y\partial^s_x C_{26} + n_x\partial^s_y C_{26} + n_x\partial^s_x C_{66}) F_\tau F_s \\
\Pi^{k\tau s}_{23} =&(n_y\partial^s_z C_{23} + n_x\partial^s_z C_{36}) F_\tau F_s \\
\Pi^{k\tau s}_{31} =&(n_y\partial^s_z C_{45} + n_x\partial^s_z C_{55}) F_\tau F_s \\
\Pi^{k\tau s}_{32} =&(n_y\partial^s_z C_{44} + n_x\partial^s_z C_{45}) F_\tau F_s \\
\Pi^{k\tau s}_{33} =&(n_y\partial^s_y C_{44} + n_y\partial^s_x C_{45} + n_x\partial^s_y C_{45} + n_x\partial^s_x C_{55}) F_\tau F_s
\end{aligned}
\tag{40}
$$

4.3 Dynamic governing equations

The PVD for the dynamic case is expressed as:

$$
\sum_{k=1}^{N_l} \int_{\Omega_k}\int_{A_k} \left\{ \delta\epsilon^{k}_{pG}{}^T \sigma^k_{pC} + \delta\epsilon^{k}_{nG}{}^T \sigma^k_{nC} \right\} d\Omega_k dz = \sum_{k=1}^{N_l} \int_{\Omega_k}\int_{A_k} \rho^k \delta\mathbf{u}^{kT} \ddot{\mathbf{u}}^k d\Omega_k dz + \sum_{k=1}^{N_l} \delta L^k_e
\tag{41}
$$

where ρ^k is the mass density of the k-th layer and double dots denote acceleration.

By substituting the geometrical relations, the constitutive equations and the unified formulation, we obtain the following governing equations:

$$
\delta\mathbf{u}^{k^T}_s : \qquad \mathbf{K}^{k\tau s}_{uu}\,\mathbf{u}^k_\tau = -\mathbf{M}^{k\tau s}\ddot{\mathbf{u}}^k_\tau + \mathbf{P}^k_{u\tau}
\tag{42}
$$

In the case of free vibrations one has:

$$
\delta\mathbf{u}^{k^T}_s : \qquad \mathbf{K}^{k\tau s}_{uu}\,\mathbf{u}^k_\tau = -\mathbf{M}^{k\tau s}\ddot{\mathbf{u}}^k_\tau
\tag{43}
$$

where $\mathbf{M}^{k\tau s}$ is the fundamental nucleus for the inertial term. The explicit form of that is:

$$M_{11}^{k\tau s} = \rho^k F_\tau F_s; \qquad M_{12}^{k\tau s} = 0; \quad M_{13}^{k\tau s} = 0 \tag{44}$$
$$M_{21}^{k\tau s} = 0; \qquad M_{22}^{k\tau s} = \rho^k F_\tau F_s; \quad M_{23}^{k\tau s} = 0 \tag{45}$$
$$M_{31}^{k\tau s} = 0; \qquad M_{32}^{k\tau s} = 0; \quad M_{33}^{k\tau s} = \rho^k F_\tau F_s \tag{46}$$

The geometrical and mechanical boundary conditions are the same of the static case.

Taking into account a sinusoidal higher-order shear deformation theory, we choose vectors $F_t = [1 \quad z \quad sin(\pi z/h)]$ for displacements u, v, w.

5 Numerical examples

5.1 Natural frequencies of composite plates

We now consider square laminated plates, where all layers of the laminate are assumed to be of the same thickness, density and made of the same linearly elastic composite material. The following material parameters of a layer are used:

$$\frac{E_1}{E_2} = 10, 20, 30 \text{ or } 40; G_{12} = G_{13} = 0.6E_2; G_3 = 0.5E_2; \nu_{12} = 0.25$$

The subscripts 1 and 2 denote the directions normal and transverse to the fiber direction in a lamina, which may be oriented at an angle to the plate axes. The ply angle of each layer is measured from the global x-axis to the fiber direction.

The example considered is a simply supported square plate of the cross-ply lamination $[0°/90°/90°/0°]$. The thickness and length of the plate are denoted by h and a, respectively. The thickness-to-span ratio $h/a = 0.2$ is employed in the computation. Table 1 lists the fundamental frequency of the simply supported laminate made of various modulus ratios of E_1/E_2. It is found that the results are in very close agreement with the values of [88] and the meshfree results of Liew [90] based on the FSDT. The relative errors between the analytical and present solutions are below 1%.

In Table 2, we consider a three-layer laminate $[0°/90°/0°]$, with clamped bords with $E_1/E_2 = 40$. The normalized frequencies are obtained as $\bar{w} = (wb^2/\pi^2)\sqrt{\rho h/D_0}$, where $D_0 = E_2h^3/12(1 - \nu_{12}\nu_{21})$. Square $(a/b = 1.0)$ and rectangular $(a/b = 2.0)$ plates are considered. Results are compared with solutions by Liew [21] and Zhen and Wanji [32], as well as radial basis functions and pseudospectrals by Ferreira and Fasshauer [17], and show excellent agreement with these solutions.

Method	Grid	E_1/E_2			
		10	20	30	40
Liew [90]		8.2924	9.5613	10.320	10.849
Exact [88][33]		8.2982	9.5671	10.326	10.854
Radial basis functions	9×9	8.2540	9.4986	10.2320	10.7341
	13×13	8.2525	9.4974	10.2308	10.7329
	17×17	8.2526	9.4974	10.2308	10.7329
Wavelets (present)	9×9	12.2487	13.1468	13.6379	13.9596
	17×17	8.2794	9.5375	10.2889	10.8117
	33×33	8.2793	9.5375	10.2889	10.8117

Table 1: The normalized fundamental frequency of the simply-supported cross-ply laminated square plate $[0°/90°/90°/0°]$ ($\bar{w} = (wa^2/h)\sqrt{\rho/E_2}, h/a = 0.2$)

6 Conclusions

A study of the free vibration of shear flexible isotropic and laminated composite plates with a unified formulation was presented. The analysis is based on a collocation method by wavelets.

The results show excellent accuracy of the present method in the free vibration analysis of composite and sandwich plates.

The present method shows excellent agreement with finite element solutions.

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a/b	b/h	Mode	1	2	3	4	5	6	7	8
1	5	Liew [21]	4.447	6.642	7.700	9.185	9.738	11.399	11.644	12.466
		Zhen and Wanji [32]	4.450	6.524	8.178	9.473	9.492	11.769	12.395	12.904
		RBF-PS 17×17	4.5141	6.5080	8.0361	9.3468	9.3929	11.5749	12.0611	12.6564
		Present 17×17	4.4466	6.6422	7.6996	9.1851	9.7393	11.3988	11.6448	12.4655
		Present 33×33	4.4466	6.6419	7.6996	9.1852	9.7379	11.3992	11.6440	12.4659
	10	Liew [90]	7.411	10.393	13.913	15.429	15.806	19.572	21.489	21.620
		Zhen and Wanji [32]	7.484	10.207	14.340	14.863	16.070	19.508	20.716	22.489
		RBF-PS 17×17	7.4727	10.2544	14.2440	14.9363	15.9807	19.4129	20.6868	22.1851
		Present 17×17	7.4106	10.3944	13.9128	15.4403	15.8061	19.5797	21.4892	21.6855
		Present 33×33	7.4108	10.3928	13.9129	15.4292	15.8056	19.5724	21.4892	21.6227
	20	Liew [90]	10.953	14.028	20.388	23.196	24.978	29.237	29.369	36.266
		Zhen and Wanji [32]	11.003	14.064	20.321	23.498	25.350	29.118	29.679	36.624
		RBF-PS 17×17	10.9680	13.9636	20.0983	23.3572	25.0859	28.6749	29.1620	35.8138
		Present 17×17	10.9528	14.0360	20.4533	23.1974	24.9827	29.2795	29.6910	36.5184
		Present 33×33	10.9529	14.0279	20.3904	23.1960	24.9783	29.2388	29.3789	36.2738
	100	Liew [90]	14.666	17.614	24.511	35.532	39.157	40.768	44.786	50.297
		Zhen and Wanji [32]	14.601	17.812	25.236	37.168	38.528	40.668	45.724	53.271
		RBF-PS 17×17	14.4305	17.3776	24.2662	35.5596	37.7629	39.3756	43.4874	51.7685
		Present 17×17	14.4455	17.5426	25.1868	37.8851	39.5489	39.6519	44.0026	54.1828
		Present 33×33	14.4342	17.3942	24.3148	35.4087	37.7795	39.3921	43.4481	50.4300
2	5	Liew [21]	3.045	4.248	5.792	5.905	6.535	7.688	7.729	9.176
		Zhen and Wanji [32]	2.953	4.288	5.595	6.096	6.446	7.796	8.053	9.005
		RBF-PS 21×21	2.9679	4.2575	5.5406	6.0225	6.3620	7.6737	7.9414	8.7482
		Present 17×17	3.0453	4.2483	5.7921	5.9046	6.5354	7.6901	7.7292	9.1778
		Present 33×33	3.0453	4.2482	5.7916	5.9045	6.5350	7.6882	7.7289	9.1760
	10	Liew [90]	2.9680	4.2576	5.5408	6.0225	6.3620	7.6730	7.9411	8.7462
		Zhen and Wanji [32]	4.119	6.705	8.240	9.916	10.212	12.671	14.066	14.082
		RBF-PS 21×21	4.0924	6.6205	8.0953	9.7047	10.0482	12.3575	13.5224	13.8453
		Present 17×17	4.1410	6.6167	8.3563	9.8970	9.9677	12.4443	13.6695	14.1332
		Present 33×33	4.1408	6.6162	8.3536	9.8945	9.9662	12.4415	13.6586	14.1203
	20	Liew [90]	4.779	8.840	9.847	12.511	14.703	17.300	17.673	19.429
		Zhen and Wanji [32]	4.813	8.954	9.968	12.768	14.960	17.764	18.041	19.993
		RBF-PS 21×21	4.7593	8.8318	9.7221	12.4153	14.7156	17.2484	17.3088	19.1064
		Present 17×17	4.7790	8.8425	9.8600	12.5212	14.7160	17.3161	17.7502	19.4969
		Present 33×33	4.7782	8.8394	9.8463	12.5089	14.7023	17.2979	17.6746	19.4283
	100	Liew [90]	5.105	10.527	10.583	14.324	19.567	19.701	22.148	22.237
		Zhen and Wanji [32]	5.144	10.407	10.929	14.706	18.954	20.799	22.205	23.703
		RBF-PS 21×21	5.0844	10.4349	10.5527	14.2538	19.2727	19.8125	21.9359	22.3671
		Present 17×17	5.1102	10.5341	10.8046	14.4741	19.7425	20.7458	22.3679	23.1373
		Present 33×33	5.0900	10.4388	10.5569	14.2342	19.2840	19.6635	21.8618	22.1442

Table 2: The normalized fundamental frequency of the 3-layer $[0°/90°/0°]$ laminated square and rectangular clamped plate, $\bar{w} = (wb^2/\pi^2)\sqrt{\rho h/D_0}$, where $D_0 = E_2h^3/12(1-\nu_{12}\nu_{21})$

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