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Dynamic Equations for a Spherical Shell

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Abstract

Using a series expansion technique together with recursion relations the dynamic equations for an elastic spherical shell are derived. The starting point is an expansion of the displacement components into power series in the thickness direction relative the mid-surface of the shell. The three-dimensional elastodynamic equations yield recursion relations among these that can be used to eliminate all but the six of lowest order. The boundary conditions on the surfaces of the shell then give the shell equations as a power series in the thickness that can in principle be truncated to any order. The method is believed to asymptotically exact to any order. Comparisons are made with correct three-dimensional theory and other shell theories.

Keywords: spherical shell, shell equations, dynamic, eigenfrequency, power series.

1 Introduction

Shells are important structures in many branches of engineering and have therefore been investigated for a number of different types of shells. Spherical shells appear in some applications, and some dynamic shell theories have thus been developed for this case. All these theories seem to depend on more or less ad hoc kinematical assumptions and/or other approximations. Here the dynamic equations for a spherical shell are derived by using a method developed during the last decade for bars, plates, and beams. It has, in particular, been developed for a number of different plate structures, like anisotropic, layered, and piezoelectric plates, see e.g. [1, 2, 3]. The main advantage with the method is that it is very systematic and can be developed to any order. It also seems that the resulting structural equations are asymptotically correct to any order [1]. The method has also been applied to a cylindrical shell [4].

The literature on shells is significant. For the present purposes the most relevant references seem to be those of Shah et al. [5, 6] and Niordson [7]. Shah et al. [5] seem to be the first to give the exact three-dimensional solution for the eigenfrequencies of a spherical shell (of arbitrary thickness), drawing on earlier work by Morse and Feshbach [8]. They also give a higher-order bending theory including a shear correction factor and relevant references to the older literature. Niordson [7] uses an asymptotic method that has similarities (but also differences) to the present approach. This method also has the benefits of not using any ad hoc assumptions and the possibility to go to, in principle, any order in the thickness.

The plan of the present paper is as follows. In the next section the problem is stated and the three-dimensional equations of elasticity are given. Next, the expansion of the displacement components in series in the thickness coordinate is performed, leading to the recursion relations for the expansion functions, this being the key ingredient in the present approach. Applying the boundary conditions at the inner and outer surfaces of the spherical shell and using the recursion relation to eliminate all but the six lowest-order expansion functions give the six shell equations. These can in principle be given to any order and are believed to be asymptotically correct to any order. A few numerical results for eigenfrequencies comparing with the exact three-dimensional solution concludes the paper.

2 Problem formulation

Consider a spherical shell with mean radius R and thickness $2h$. The material is assumed to be isotropic and linearly elastic with Lamé constants λ and μ and density ρ. Introduce spherical coordinates r, θ, and ϕ, where r is the radial coordinate, θ the polar coordinate, and ϕ the azimuthal coordinate. The main goal is to derive a set of dynamic shell equations for this case, i.e. a set of differential equations that depend on the two angular spherical coordinates and time, but where the radial dependence has disappeared.

The starting point is the three-dimensional dynamic equations of elasticity for the displacement field $\mathbf{u}$

$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\nabla \times (\nabla \times \mathbf{u}) = \rho\frac{\partial^2 \mathbf{u}}{\partial^2 t}. \tag{1}$$

This equation is written in a way that clearly shows the decoupling into compression and shear waves. In spherical coordinates this equation is written for the radial component u, the polar component v, and the azimuthal component w. The equations then become more lengthy

$$\mu\left[\partial_\theta^2 u + \cot\theta\,\partial_\theta u + \frac{\partial_\varphi^2 u}{\sin^2\theta}\right] + (\lambda+\mu)\left[r\partial_r\partial_\theta v + r\cot\theta\,\partial_r v + \frac{r\partial_r\partial_\varphi w}{\sin\theta}\right] + (\lambda+2\mu)\left[r^2\partial_r^2 u + 2r\partial_r u - 2u\right] - (\lambda+3\mu)\left[\partial_\theta v + \cot\theta\, v + \frac{\partial_\varphi w}{\sin\theta}\right] = \rho r^2 \partial_t^2 u, \tag{2}$$

$$\begin{aligned}&(\lambda+2\mu)\left[\partial_\theta^2 v+\cot\theta\,\partial_\theta v-\frac{v}{\sin^2\theta}+2\partial_\theta u\right]+\mu\left[r^2\partial_r^2 v+2r\partial_r v+\frac{\partial_\varphi^2 v}{\sin^2\theta}\right]\\&+(\lambda+\mu)\left[r\partial_r\partial_\theta u+\frac{\partial_\theta\partial_\varphi w}{\sin\theta}\right]-(\lambda+3\mu)\frac{\cos\theta\,\partial_\varphi w}{\sin^2\theta}=\rho r^2\partial_t^2 v,\end{aligned}\tag{3}$$

$$\begin{aligned}&(\lambda+2\mu)\left[\frac{\partial_\varphi^2 w}{\sin^2\theta}+\frac{2\partial_\varphi u}{\sin\theta}\right]+\mu\left[r^2\partial_r^2 w+2r\partial_r w+\partial_\theta^2 w+\cot\theta\,\partial_\theta w-\frac{w}{\sin^2\theta}\right]\\&+(\lambda+\mu)\left[\frac{r\partial_r\partial_\varphi u}{\sin\theta}+\frac{\partial_\theta\partial_\varphi v}{\sin\theta}\right]+(\lambda+3\mu)\frac{\cos\theta\,\partial_\varphi v}{\sin^2\theta}=\rho r^2\partial_t^2 w,\end{aligned}\tag{4}$$

where ∂_r, ∂_θ, and ∂_φ denote partial derivatives with respect to r, θ, and φ, respectively. These equations are to be supplemented with boundary conditions on the inner and outer surfaces of the shell. These are for simplicity taken as vanishing traction on both surfaces, although, e.g., an applied pressure would also be possible.

3 The shell equations

To derive shell equations the first step is to substitute $r = R + \xi$, where the variable ξ is located in the shell, i.e. $-h < \xi < h$. It is assumed that h is small both relative to the radius R and to relevant wavelengths. The displacement components are then expanded as

$$u(r,\theta,\varphi,t)=\sum_{k=0} u_k(\theta,\varphi,t)\xi^k,\tag{5}$$

$$v(r,\theta,\varphi,t)=\sum_{k=0} v_k(\theta,\varphi,t)\xi^k,\tag{6}$$

$$w(r,\theta,\varphi,t)=\sum_{k=0} w_k(\theta,\varphi,t)\xi^k.\tag{7}$$

Formally the sums are infinite but in practice only a few terms are used. But as compared to most other methods no truncation is performed at this stage, the truncation scheme is discussed later on.

The displacement expansions are inserted into the governing equations and equal powers of ξ are identified. Solving for the highest orders this gives

$$\begin{aligned}u_{k+2}=&\frac{1}{(k+1)(k+2)(\lambda+2\mu)R^2}\left[\rho\left(R^2\partial_t^2 u_k+2R\partial_t^2 u_{k-1}+\partial_t^2 u_{k-2}\right)\right.\\&-(\lambda+2\mu)\left(2R(k+1)^2 u_{k+1}+(k+2)(k-1)u_k\right)-\mu\left(\partial_\theta^2 u_k+\cot\theta\,\partial_\theta u_k\right)\\&-\frac{\mu\partial_\varphi^2 u_k}{\sin^2\theta}-R(k+1)(\lambda+\mu)\left(\partial_\theta v_{k+1}+\cot\theta\,v_{k+1}+\frac{\partial_\varphi w_{k+1}}{\sin\theta}\right)\\&\left.-((k-1)\lambda+(k-3)\mu)\left(\partial_\theta v_k+\cot\theta\,v_k+\frac{\partial_\varphi w_k}{\sin\theta}\right)\right],\end{aligned}\tag{8}$$

$$
\begin{aligned}
v_{k+2} =& \frac{1}{(k+1)(k+2)\mu R^2} \left[\rho \left(R^2 \partial_t^2 v_k + 2R\partial_t^2 v_{k-1} + \partial_t^2 v_{k-2}\right)\right. \\
&- (\lambda + 2\mu)\left(\partial_\theta^2 v_k + \cot\theta\, \partial_\theta v_k - \frac{v_k}{\sin^2\theta} + 2\partial_\theta u_k\right) \\
&- \mu\left(2R(k+1)^2 v_{k+1} + k(k+1)v_k + \frac{\partial_\varphi^2 v_k}{\sin^2\theta}\right) - (\lambda+3\mu)\frac{\cos\theta}{\sin^2\theta}\partial_\varphi w_k \\
&\left. -(\lambda+\mu)\left(R(k+1)\partial_\theta u_{k+1} + k\partial_\theta u_k + \frac{\partial_\varphi \partial_\theta w_k}{\sin\theta}\right)\right],
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
w_{k+2} =& \frac{1}{(k+1)(k+2)\mu R^2} \left[\rho \left(R^2 \partial_t^2 w_k + 2R\partial_t^2 w_{k-1} + \partial_t^2 w_{k-2}\right)\right. \\
&- \mu\left(2R(k+1)^2 w_{k+1} + k(k+1)w_k + \partial_\theta^2 w_k + \cot\theta\,\partial_\theta w_k\right) \\
&+ \frac{1}{\sin^2\theta}\left(\mu w_k - (\lambda+2\mu)\partial_\varphi^2 w_k\right) - \frac{2(\lambda+2\mu)}{\sin\theta}\partial_\varphi u_k \\
&\left. - \frac{\lambda+\mu}{\sin\theta}\left(R(k+1)\partial_\varphi u_{k+1} + k\partial_\varphi u_k + \partial_\varphi\partial_\theta v_k\right) - (\lambda+3\mu)\frac{\cos\theta}{\sin^2\theta}\partial_\varphi v_k\right].
\end{aligned}
\tag{10}
$$

These equations can be used recursively to express all higher order expansion functions in the six lowest-order ones u_0, u_1, v_0, v_1, w_0, and w_1. It is noticed that the procedure so far does not depend on any assumption about the thickness of the shell. The recursion relations can be used also to get a good representation of the displacements in the shell for other purposes.

To obtain the shell equations the boundary conditions on the shell are now applied. The relevant stress components in spherical coordinates are

$$
\sigma_{rr} = (\lambda+2\mu)\partial_r u + \lambda\left(\frac{2u}{r} + \frac{\partial_\theta v}{r} + \frac{\cot\theta v}{r} + \frac{\partial_\varphi w}{r\sin\theta}\right), \tag{11}
$$

$$
\sigma_{r\theta} = \mu\left(\partial_r v - \frac{v}{r} + \frac{\partial_\theta u}{r}\right), \tag{12}
$$

$$
\sigma_{r\varphi} = \mu\left(\partial_r w - \frac{w}{r} + \frac{\partial_\varphi u}{r\sin\theta}\right). \tag{13}
$$

Insertion of the field expansion gives

$$
\begin{aligned}
\sigma_{rr} =& \frac{1}{R+\xi}\sum_{k=0}\left[(\lambda+2\mu)\left((k+1)Ru_{k+1} + ku_k\right)\right. \\
&\left. +\lambda\left(2u_k + \partial_\theta v_k + \cot\theta v_k + \frac{\partial_\varphi w_k}{\sin\theta}\right)\right]\xi^k,
\end{aligned}
\tag{14}
$$

$$
\sigma_{r\theta} = \frac{\mu}{R+\xi}\sum_{k=0}\left[(k+1)Rv_{k+1} + (k-1)v_k + \partial_\theta u_k\right]\xi^k, \tag{15}
$$

$$\sigma_{r\varphi} = \frac{\mu}{R+\xi} \sum_{k=0} \left[(k+1)Rw_{k+1} + (k-1)w_k + \frac{\partial_\varphi u_k}{\sin\theta} \right] \xi^k. \tag{16}$$

The boundary conditions at $\xi = \pm h$ are then applied, usually this means that all three stress components are zero. By using the recursion relations all but the six lowest order expansion functions can be eliminated. It is also convenient to take the sum and difference between the equations at the two boundaries, and this results in the six shell equations. These are given as an expansion in h, which can, in principle, be given to any order. With increasing orders the equations become extremely complex, so here only the lowest, h-independent, terms are given. Assuming homogeneous boundary conditions the six equations then become

$$\begin{aligned} &R^2\rho\partial_t^2 u_0 + (\lambda+2\mu)u_0 - \mu\left(\partial_\theta^2 + \cot\theta\,\partial_\theta + \frac{\partial_\varphi^2}{\sin^2\theta}\right)u_0 + \frac{2\mu}{\sin\theta}\,\partial_\varphi w_0 \\ &+ (\lambda+3\mu)\left(\cot\theta + \partial_\theta\right)v_0 + R(\lambda-2\mu)u_1 - \frac{R\mu}{\sin^2\theta}\,\partial_\varphi w_1 = 0, \end{aligned} \tag{17}$$

$$4\lambda u_0 + (\lambda+2\mu)Ru_1 + 2\lambda\cot\theta\, v_0 + \frac{2\lambda}{\sin\theta}\,\partial_\varphi w_0 + 2\lambda\partial_\theta v_0 = 0, \tag{18}$$

$$\begin{aligned} &R^2\rho\partial_t^2 v_0 + \frac{\lambda+2\mu}{\sin^2\theta}\,v_0 - \frac{\mu}{\sin^2\theta}\,\partial_\varphi^2 v_0 - (\lambda+2\mu)\left(\cot\theta+\partial_\theta\right)\partial_\theta v_0 - 2\mu R v_1 \\ &- 2(\lambda+2\mu)\partial_\theta u_0 - \lambda R\partial_\theta u_1 + (\lambda+3\mu)\frac{\cot\theta}{\sin\theta}\,\partial_\varphi w_0 - \frac{\lambda+\mu}{\sin\theta}\,\partial_\theta\partial_\varphi w_0 = 0, \end{aligned} \tag{19}$$

$$\partial_\theta u_0 - v_0 + Rv_1 = 0, \tag{20}$$

$$\begin{aligned} &R^2\rho\partial_t^2 w_0 + \frac{\mu}{\sin^2\theta}\,w_0 - \frac{\lambda+2\mu}{\sin^2\theta}\,\partial_\varphi^2 w_0 - \mu\left(\cot\theta+\partial_\theta\right)\partial_\theta w_0 - 2\mu R w_1 \\ &- \frac{2(\lambda+2\mu)}{\sin\theta}\,\partial_\varphi u_0 - \frac{\lambda+2\mu}{\sin\theta}R\partial_\theta u_1 + (\lambda+3\mu)\frac{\cot\theta}{\sin\theta}\,\partial_\varphi v_0 - \frac{\lambda+\mu}{\sin\theta}\,\partial_\theta\partial_\varphi v_0 = 0, \end{aligned} \tag{21}$$

$$\frac{1}{\sin\theta}\partial_\varphi u_0 - w_0 + Rw_1 = 0. \tag{22}$$

From the second, fourth, and sixth equation it is of course straightforward to eliminate u_1, v_1, and w_1 and this leads to the classical membrane equations. Including also the quadratic terms, a type of bending theory is obtained, and it is noted this theory do not include any shear correction factor and that it is believed to be asymptotically correct. For reasons of length these equations are not given in full. Following the procedure of Shah et al. [5] or Niordson [7] it should be possible to introduce some sort of potentials which should greatly reduce the length of the equations.

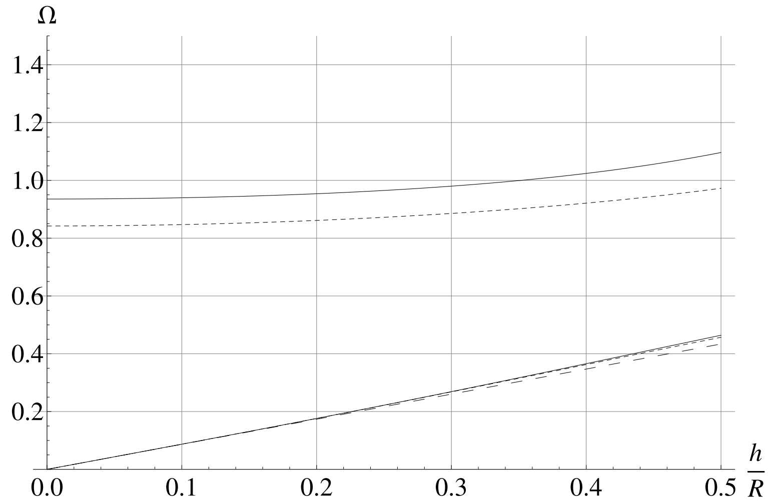

Figure 1: The first two spherically symmetric eigenfrequencies as a function of shell thickness: exact 3D theory (full-drawn), present theory (short-dashed), membrane theory (long-dashed).

4 Results

To validate the shell equations some comparisons with other results are now given. The simplest case is that of spherically symmetric vibrations for which there is no angular dependence at all and only the radial component u is nonzero. For this case the shell equations, including also terms in h^2, simplify to

$$\begin{aligned}&2(\lambda+2\mu)u_0+(\lambda-2\mu)Ru_1+\rho\partial_t^2u_0+\frac{h^2}{6R^2(\lambda+2\mu)}[32\mu(\lambda+2\mu)(u_0-Ru_1)\\&+2\rho R^2(\lambda+6\mu)\partial_t^2u_0+\rho R^3(\lambda-2\mu)\partial_t^2u_1+\rho^2R^2\partial_t^4u_0]=0,\end{aligned} \tag{23}$$

$$\begin{aligned}&2\lambda u_0+(\lambda+2\mu)Ru_1+\frac{h^2}{2R^2(\lambda+2\mu)}[8\mu(\lambda+2\mu)(Ru_1-u_0)\\&+\rho R^2\lambda\partial_t^2u_0+\rho R^3(\lambda+2\mu)\partial_t^2u_1]=0.\end{aligned} \tag{24}$$

With a harmonic time dependence these equations give the lowest eigenfrequency expanded to second order in the shell thickness

$$(k_sR)^2=\frac{4(1+\nu)}{1-\nu}+\frac{4h^2(1+\nu)(-1+9\nu)}{3R^2(1-\nu)^2}, \tag{25}$$

where the wave number $k_s=\omega\sqrt{\rho/\mu}$. This result agrees exactly with that of Niordson [7].

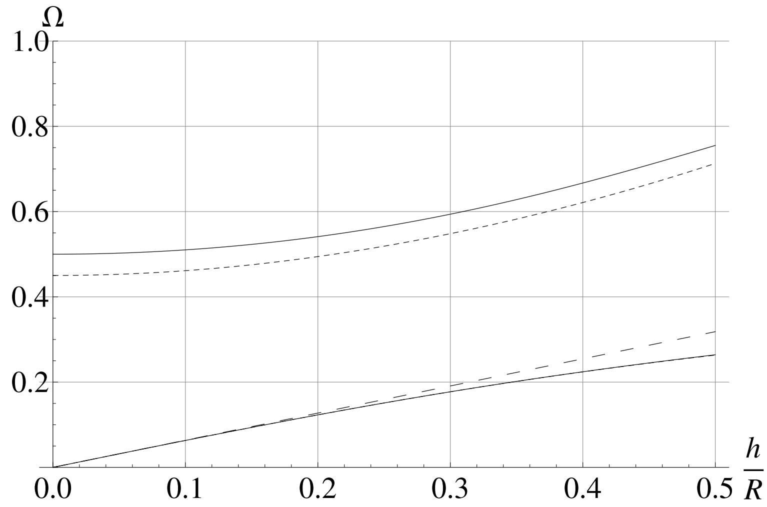

Figure 2: The first two torsional eigenfrequencies as a function of shell thickness: exact 3D theory (full-drawn), present theory (short-dashed), membrane theory (long-dashed).

To further illustrate the spherically symmetric case, Figure 1 shows the two first eigenfrequencies as a function of shell thickness h/R. The eigenfrequency is normalized so that $\Omega = \omega h/(\pi c_s)$, similarly to Shah et al. [5]. Three curves are shown for the first eigenfrequency, namely membrane theory (long-dashed), the present second-order theory (short-dashed), and exact three-dimensional theory (full-drawn) according to Shah et al [5]. For this first eigenfrequency all three curves agree for surprisingly thick shells. For the second eigenfrequency, where membrane theory does not apply, there is some discrepancy, much larger than for the six-mode theory of Shah et al. [5]. But it must be remembered that their theory contains a shear correction factor that is tuned to get good correspondence. It should also be remembered that for the second eigenfrequency the shell is no longer thin (in terms of wavelengths) and that the displacement fields will not be properly described by any shell theory, cf. Boström et al. [1] for a discussion of this point for the case of a plate.

The second case that is investigated is the torsional modes for which only w is nonzero and independent of the azimuthal coordinate (called modes of the first class by Shah et al. [5]). Similarly to Figure 1, Figure 2 shows the first two eigenfrequencies for this case. Exactly the same comments as for Figure 1 apply.

5 Concluding remarks

In this paper the shell equations for a spherical shell are derived using a power series expansion for the displacement components which leads to recursion relations and the shell equations from the boundary conditions. A few results are given showing good agreement with earlier results and with three-dimensional solutions. The present result should be extended in a number of ways. Firstly, it should be useful to introduce potentials according to Shah et al. [5] or Niordson [7] to reduce the equations to a more manageable format. For a shell that is not a complete sphere, the boundary conditions along the edge of the shell should be formulated. It is also possible to extend the present results to more complex situations like an anisotropic, piezoelectric or layered spherical shell, or even to completely general shells (using differential geometrical results).

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