Paper 42

©Civil-Comp Press, 2012
Proceedings of the Eighth International Conference
on Engineering Computational Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Dynamic Behaviour of Steffensen-Type Methods

F. Chicharro, A. Cordero and J.R. Torregrosa
Institute for Multidisciplinary Mathematics
Universitat Politècnica de València, Spain

Abstract

The dynamic behaviour of two iterative derivative-free schemes, Steffensen and M4 methods, is studied in case of quadratic and cubic polynomials. The parameter plane is analysed for both procedures on quadratic polynomials. Different dynamic planes are shown when the mentioned methods are applied to particular cubic polynomials with real or complex coefficients. The property of immersion of the basins of attraction in all cases is analysed.

Keywords: nonlinear equation, iterative method, derivative-free, complex dynamics, parameters plane, fixed point, critical point, immersed basin of attraction.

1 Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$, gives rise to rational functions whose dynamics are not well-known. There is an extensive literature on the study of iteration of rational mappings of a complex variable (see, for example, [1, 2]). The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative process is Newton's method. The dynamics of this iterative scheme has been widely studied (see, for instance, [2, 3, 4]).

The analysis of the dynamics of Newton's method has been extended to other point-to-point iterative methods, used for solving nonlinear equations with convergence higher than two (see, for example, [5, 6, 7, 8, 9]).

The most of the iterative methods analyzed from the dynamic point of view are schemes with derivatives in their iterative expressions. Unlike Newton's methods, the derivative-free scheme of Steffensen has been little studied. We can find some dynamic comments on this method in [5, 10].

In this paper, we analyze the dynamics of two derivative-free iterative procedures, optimal in the sense of Kung-Traub's conjecture, of order two and four.

As it is well-known, if we replace the derivative of Newton's iterative expression by the progressive finite difference, we obtain Steffensen's method (see [11]), whose iterative expression is

$$z_{n+1} = z_n - \frac{f^2(z_n)}{f(v_n) - f(z_n)}, \quad n = 1, 2, \dots \tag{1}$$

where $v_n = z_n + f(z_n)$. As in Newton's method, Steffensen's one has quadratic convergence and the same efficiency index. From Kung-Traub's conjecture, Steffensen's method is optimal.

The fixed point operator of Steffensen's method on a polynomial $p(z)$ is

$$S_p(z) = z - \frac{p^2(z)}{p(v) - p(z)}. \tag{2}$$

A common guideline used to improve the local order of convergence is the composition of two iterative methods, as shown in [11]. This technique obtains iterative schemes or order $c_1 \cdot c_2$, where c_1 and c_2 are the convergence order of the involved methods. The method M4 (see [12]) is obtained by composing Newton's and Steffensen's methods and using the Pade's approximant of degree one in order to avoid the last evaluation of the derivative. The iterative scheme is

$$\begin{array}{rcl} y_n & = & z_n - \frac{f^2(z_n)}{f(v_n) - f(z_n)}, \\ z_{n+1} & = & y_n - \frac{f(y_n) f[z_n, v_n]}{f[z_n, y_n] f[y_n, v_n]}, \end{array} \quad n = 1, 2, \dots \tag{3}$$

where $f[\cdot, \cdot]$ denotes the divided difference of order one.

This method is fourth-order convergent and it is optimal from Kung-Traub's conjecture.

The fixed point operator of M4 on $p(z)$ is

$$M_p(z) = y - \frac{p(y) p[z, v]}{p[z, y] p[y, v]}. \tag{4}$$

In order to study the dynamic behaviour of an iterative method when is applied to a polynomial $p(z)$, it is necessary to recall some basic dynamic concepts. For a more extensive and comprehensive review of these concepts, see [13, 14].

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function, where $\hat{\mathbb{C}}$ is the Riemann sphere. The orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as

$$\{z_0, R(z_0), \dots, R^n(z_0), \dots\}.$$

The dynamic behaviour of the orbit of a point on the complex plane can be classified depending on its asymptotic behaviour. In this way, a point $z_0 \in \mathbb{C}$ is a fixed point of R if $R(z_0) = z_0$. A fixed point is attracting, repelling or neutral if $|R'(z_0)|$ is less than, greater than or equal to 1, respectively. Moreover, if $|R'(z_0)| = 0$, the fixed point is superattracting.

Let z_f^* be an attracting fixed point of the rational function R. The basin of attraction of an attracting fixed point $\mathcal{A}(z_f^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(z_f^*) = \left\{ z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z_f^*, n \to \infty \right\}.$$

The set of points whose orbits tends to an attracting fixed point z_f^* is defined as the Fatou set, $\mathcal{F}(R)$. The complementary set, the Julia set $\mathcal{J}(R)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

It is possible to find the fixed and critical points from the fixed point operator associated to each method, $O_p(z)$ on a polynomial $p(z)$. The fixed points z_f verify

$$O_p(z) = z, \tag{5}$$

and the critical points z_c validate

$$O_p'(z) = 0. \tag{6}$$

The attracting fixed points z_f^* are the points z_f such that

$$|O_p'(z_f)| < 1. \tag{7}$$

If $|O_p'(z_f)| = 0$, the fixed point is superattracting.

Mayer and Schleicher define in [15] the immediate basin of attraction of a superattracting fixed point z_f^*, $\mathcal{A}^\#$, as the connected component of the basin containing z_f^*. It is well-known if z_f^* is an superattracting fixed point, the immediate basin of attraction $\mathcal{A}^\#$ contains at least a critical point.

In order to study the affine conjugacy classes of the iterative methods, the following relevant result must be mentioned.

Theorem 1 *(Scaling Theorem for Newton's method, [2]) Let $g(z)$ be an analytic function, and let $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, be an affine map. Let $h(z) = \lambda(g \circ A)(z)$, with $\lambda \neq 0$. Let $O_p(z)$ be the fixed point operator of Newton's method. Then, $A \circ O_h \circ A^{-1}(z) = O_g(z)$, i.e., O_g and O_h affine conjugated by A.*

This result allows up the knowledge of a family of polynomials with just the analysis of a few cases, from a suitable scaling.

For the derivative-free iterative methods whose complex dynamics are going to be studied, it can be proved that there is no scaling theorem for them, so the dynamics of the methods will not be generalized in this way. In order to get this aim, the parameter space will be analyzed in case of quadratic polynomials and, when cubic polynomials are considered, particular cases are studied.

So, let us consider an affine map $A(z) = \alpha z + \beta$, with $\alpha \neq 0$. Let also be $h(z) = \lambda(g \circ A)(z)$, with $\lambda \neq 0$, and $S_h(z)$ the fixed-point operator of Steffensen's method on $h(z)$. Since $A(x+y) = A(x) + A(y) - \beta$, $A(x-y) = A(x) - A(y) + \beta$ and $h(A^{-1}(z)) = \lambda g(z)$,

$$\begin{aligned}
A \circ S_h \circ A^{-1}(z) &= A\left(A^{-1}(z) - \frac{\lambda^2 \left[g(z)\right]^2}{h\left(A^{-1}(z) + \lambda g(z)\right) - g(z)}\right) = \\
&= A(A^{-1}(z)) - A\left(\frac{\lambda^2 \left[g(z)\right]^2}{h\left(A^{-1}(z) + \lambda g(z)\right) - \lambda g(z)}\right) + \beta = \\
&= z - \alpha \frac{\lambda^2 \left[g(z)\right]^2}{h\left(A^{-1}(z) + \lambda g(z)\right) - \lambda g(z)} = \\
&= z - \frac{\alpha\lambda^2 \left[g(z)\right]^2}{g\left(A\left(A^{-1}(z) + \lambda g(z)\right)\right) - \lambda g(z)} = \\
&= z - \frac{\alpha\lambda^2 \left[g(z)\right]^2}{g\left(z + \alpha\lambda g(z)\right) - \lambda g(z)}.
\end{aligned}$$

Then, the operators $S_g(z)$ and $S_h(z)$ are affine conjugated if and only if $\alpha = \lambda = 1$. So that, the scaling is not possible. The reason is the derivative-free nature of Steffensen's method. Therefore, a relation between a few instances of polynomials and the complete family behaviour cannot be established. Indeed, a similar result can be proved for M4.

2 Quadratic polynomials

In this section the Steffensen's (2) and M4 (4) fixed point methods are studied, when they are applied to a quadratic polynomial $p_c(z) = z^2 + c$. In Steffensen case, the fixed point operator and the fixed and critical points are introduced as dependent on c, since the numerator of $S_{p_c}(z)$ is a polynomial of degree 3. Nevertheless, for M4 the fixed and critical points are obtained in specific cases of c, due to the high degree of the involved polynomials.

The fixed point operator of Steffensen (2) when is applied to $p_c(z) = z^2 + c$ is

$$S_{p_c}(z) = \frac{z^3 + z^2 + cz - c}{z^2 + 2z + c}. \tag{8}$$

It is easy to proof that $S_{p_c}(z)$ satisfies the symmetry property

$$S_{p_{\bar{c}}}(\bar{z}) = \overline{S_{p_c}(z)}, \qquad \forall c, z \in \mathbb{C}. \tag{9}$$

This property implies that the dynamic plane of the iterative method presents one of the following appearances:

- if $\mathrm{Im}\{c\} = 0$, there exists a symmetry about the abscissas axis, and

- if $\mathrm{Im}\{c\} \neq 0$, there exists a symmetry between the methods applied to polynomials $p_c(z) = z^2 + c$ and $p_{\bar{c}}(z) = z^2 + \bar{c}$.

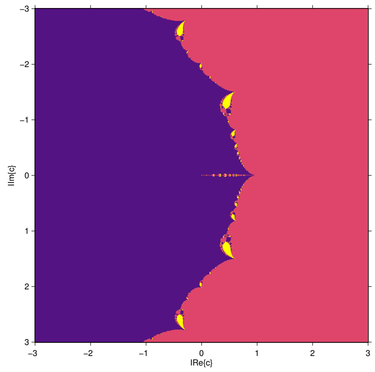
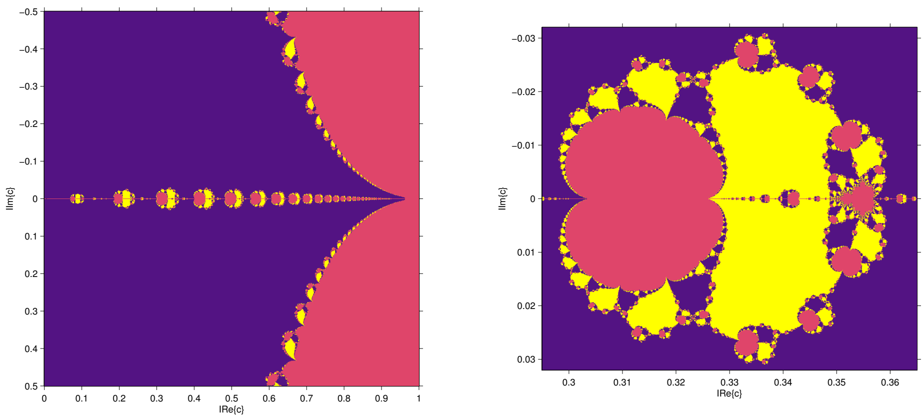


Figure 1: Parameters plane of $S_{p_c}(z)$, $\mathrm{Re}\{c\} \in (-3, 3)$, $\mathrm{Im}\{c\} \in (-3, 3)$

By definition, the fixed points of $S_{p_c}(z)$ are $z_{f_{1,2}} = \pm i\sqrt{c}$. So, the only finite fixed points are the roots of the polynomial, which favors the method convergence. These points are classified from the absolute value of the derivative operator in the fixed points. For both fixed points, $|S_{p_c}(z_{f_{1,2}})| = 0$, so $z_{f_{1,2}}$ are superattracting fixed points, denoted as $z^*_{f_{1,2}}$. In this case, it is easy to prove that the infinity is also a superattracting fixed point. The critical points null the derivative operator, as deduced from (6). In this way,

$$S'_{p_c}(z) = 0 \leftrightarrow \begin{cases} z_{c1,2} &= \pm i\sqrt{c}, \\ z_{c3,4} &= -2 \pm \sqrt{2-c}. \end{cases}$$

Note for $c = 2$, both critical points meet at $z = -2$.



(a) $\mathrm{Re}\{c\} \in (0, 1)$, $\mathrm{Im}\{c\} \in (-0.5, 0.5)$

(b) $\mathrm{Re}\{c\} \in (0.295, 0.365)$, $\mathrm{Im}\{c\} \in (-0.032, 0.032)$

Figure 2: Parameters plane of $S_{p_c}(z)$

Although there is no scaling theorem, it is possible to generalize some behaviours of the family $S_{p_c}(z)$ with the associated parameters plane. In Figure 1 the parameter

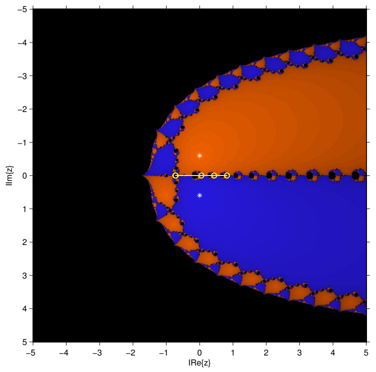


Figure 3: A periodic orbit for $S_{p_c}(z)$, $\mathrm{Re}\{c\} \in (-5, 5)$, $\mathrm{Im}\{c\} \in (-5, 5)$

plane of $S_{p_c}(z)$ is shown, where $\mathrm{Re}\{c\} \in (-3, 3)$ and $\mathrm{Im}\{c\} \in (-3, 3)$. Two details of Figure 1 are represented in Figure 2, focussing on the central antenna (Figure 2a) and the cell (Figure 2b).

The critical point $z_{c_4} = -2 - \sqrt{2-c}$ is always at the basin of attraction of the infinity, $\mathcal{A}(\infty)$. The parameter plane in Figure 1 reports the basin of attraction that contains the critical point $z_{c_3} = -2 + \sqrt{2-c}$. When c is in the yellow region of Figure 1, $z_{c_3} \in \mathcal{A}(z^*_{f_1})$. If c is in the purple region, $z_{c_3} \in \mathcal{A}(z^*_{f_2})$. The pink region places both critical points in $\mathcal{A}(\infty)$.

For every value of c, there is convergence to one of the superattracting fixed points, including the infinity, although periodic orbits can be found (see Figure 3).

The fixed point operator of M4 (see 3) when is applied to $p_c(z) = z^2 + c$ is

$$M_{p_c}(z) = \frac{N_{p_c}(z)}{D_{p_c}(z)}, \tag{10}$$

where $N_{p_c}(z) = z^{10} + 7z^9 + (16+3c)z^8 + (15+12c)z^7 + (5+2c^2)z^6 - (31c+6c^2)z^5 - (23c+48c^2+2c^3)z^4 - (43c^2+20c^3)z^3 + (3c^2-32c^3-3c^4)z^2 + (3c^3-9c^4)z - c^3 - c^5$ and $D_{p_c}(z) = (z^4+4z^3+(3+2c)z^2+4cz+c^2-c)(2z^3+3z^2+2cz-c)(z^2+2z+c)$.

As in Steffensen's method, M4 satisfies the symmetry property

$$M_{p_{\bar{c}}}(\bar{z}) = \overline{M_{p_c}(z)}, \qquad \forall c, z \in \mathbb{C}. \tag{11}$$

Respect to the critical points, the analysis of the respective parameter plane has showed that $z_{c_1} = -1 + \sqrt{-c}$ remains in the basin of attraction of $z^*_{f_2}$, and also $z_{c_2} = -1 - \sqrt{-c} \in \mathcal{A}(z^*_{f_1})$. As no other explicit critical points have been found, due to the complexity of the expression (10), different cases of $M_{p_c}(z)$ are studied.

The fixed point operator of M4, when is applied to $p_1(z) = z^2 + 1$, is

$$M_{p_1}(z) = \frac{z^{10} + 7z^9 + 19z^8 + 27z^7 + 7z^6 - 37z^5 - 73z^4 - 63z^3 - 32z^2 - 6z - 2}{z(z+1)^2(z^3+4z^2+5z+4)(2z^3+3z^2+2z-1)}. \tag{12}$$

By applying (5), the fixed points of $M_{p_1}(z)$ are

$$\begin{aligned} z_{f_{1,2}} &= \pm i, & z_{f_{7,8}} &= -0.939241 \pm i0.884754, \\ z_{f_{3,4}} &= -0.0247075 \pm i0.347611, & z_{f_{9,10}} &= -2.51928 \pm i1.27208. \\ z_{f_{5,6}} &= -0.516776 \pm i0.987314, \end{aligned}$$

By evaluating the fixed points over the modulus of the derivative of $M_{p_1}(z)$, the fixed points are classified in (super)attracting, repelling or neutral points. In this case, the fixed points $z_{f_{1,2}} = z^*_{f_{1,2}}$ are superattracting, and the rest of fixed points $z_{f_{3-10}}$ are repelling. Then, the dynamic plane has two basins of attraction, $\mathcal{A}(z^*_{f_1})$ and $\mathcal{A}(z^*_{f_2})$. In this way, when an initial value in any of both basins is taken, the orbit of this value converges to its corresponding superattracting value. The only superattracting fixed points are the polynomial roots, favoring the method convergence.

The critical points are obtained by forcing the derivative of the fixed point operator to be null.

$$\begin{aligned} z_{c_{1-6}} &= \pm i, & z_{c_{10,11}} &= -0.899354 \pm i0.750718, \\ z_{c_7} &= 0.143866, & z_{c_{12,13}} &= -0.776073 \pm i0.944697, \\ z_{c_8} &= -0.241736, & z_{c_{14,15}} &= -1 \pm i, \\ z_{c_9} &= -2.02613, & z_{c_{16,17}} &= -3.26257 \pm i1.78652. \end{aligned}$$

The dynamic plane of M4 when it is applied to polynomial $p_1(z) = z^2 + 1$ is shown in Figure 4b. The orange basin of attraction belongs to the fixed point $z^*_{f_1} = i$, while the blue one belongs to the root $z^*_{f_2} = -i$. Let us note that the roots of the polynomial are always plotted with white stars. Unlike Steffensen's method, the infinity has no basin of attraction in M4. So, every point of the dynamic plane belongs to one of the basins of attraction of the roots, except for the Julia set.

When M4 is applied to $p_{-1}(z) = z^2 - 1$, the numerator of the fixed point operator is a polynomial of degree 10 and 10 fixed points are obtained. The only superattracting points are the roots of the polynomial, i.e., $z^*_{f_{1,2}} = \pm 1$; the other fixed points are repelling. In Figure 5b the dynamic plane of $M_{p_{-1}}(z)$ is on view.

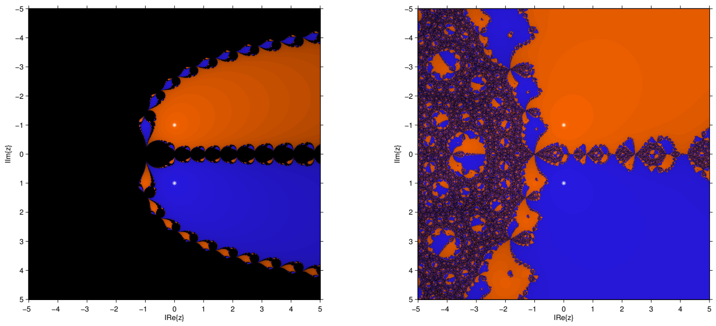
The fixed point operator of M4 on $p_0(z) = z^2$ is

$$M_{p_0}(z) = \frac{z^4 + 6z^3 + 10z^2 + 5z}{(z+2)(2z^2 + 9z + 9)}. \tag{13}$$

The degree of the numerator decreases compared with the operators associated to $c = \pm 1$. One of the four fixed points is the root of $p_0(z)$, that is the only superattracting point. We can see its dynamic plane in Figure 6b.

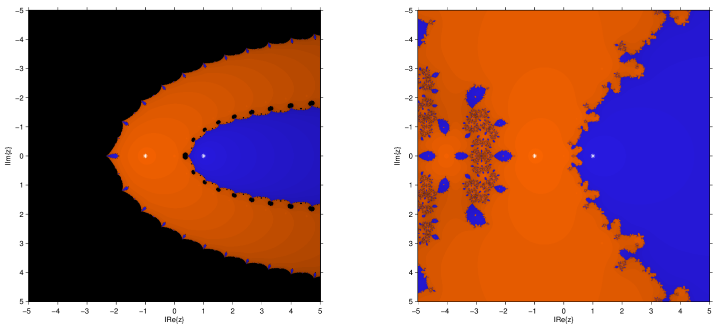
When $p_{\pm i}(z) = z^2 \pm i$, there are 10 fixed points. The fixed points of $p_i(z)$ and $p_{-i}(z)$ are conjugated. As in previous cases, the roots of $p_{\pm i}(z)$ are the only superattracting fixed points, and the rest are repelling. In this way, $M_{p_i}(z)$ has its basins of attraction at $z^*_{f_{1,2}} = \pm\sqrt{-i}$ and $M_{p_{-i}}(z)$ at $z^*_{f_{1,2}} = \pm\sqrt{i}$. The dynamic plane of both methods is displayed in Figures 7b and 8b, respectively.

In order to describe rigorously the different behaviours observed, let us define an immediate basin of attraction $\mathcal{A}^{\#}_1$ as immersed in other basin of attraction $\mathcal{A}^{\#}_2$ when



(a) STF (b) M4

Figure 4: $p_1(z) = z^2 + 1$



(a) STF (b) M4

Figure 5: $p_{-1}(z) = z^2 - 1$

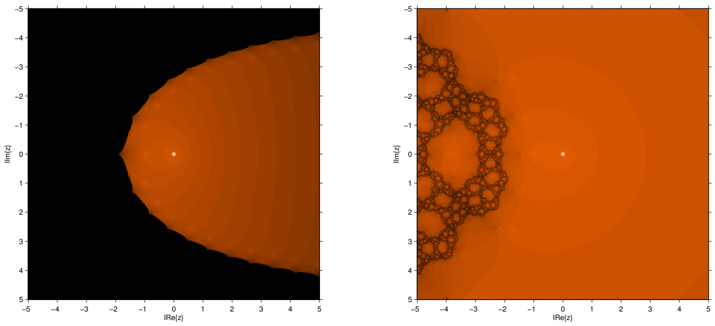
it is possible to find two points of $\mathcal{A}_2^{\#}$ such that the line that connect both points goes across $\mathcal{A}_1^{\#}$.

The dynamic planes associated to Steffensen's and M4 method when they are applied to quadratic polynomial $p_c(z) = z^2 + c$ are shown in Figures 4-8, taking different values of c, and represented in the region $\mathrm{Re}\{\mathrm{z}\} \in (-5, 5)$ and $\mathrm{Im}\{z\} \in (-5, 5)$. Moreover, the basin of attraction of $z_{f_1} = i\sqrt{c}$ is pictured in blue, and the corresponding one of $z_{f_2} = -i\sqrt{c}$ is in orange. The basin of attraction of the infinity is in black, when it exists.

When the dynamic planes of Steffensen's method are studied according to the parameters plane, the critical point $z_{c_1} = -2 + \sqrt{2 - c}$ is in the basin of attraction $\mathcal{A}(z_{f_1}^*)$, $\mathcal{A}(z_{f_2}^*)$ or $\mathcal{A}(\infty)$ when the c value is in the yellow, purple or pink region of Figure 1, respectively. Wherever the c value is, z_{c_2} is always in $\mathcal{A}(\infty)$.

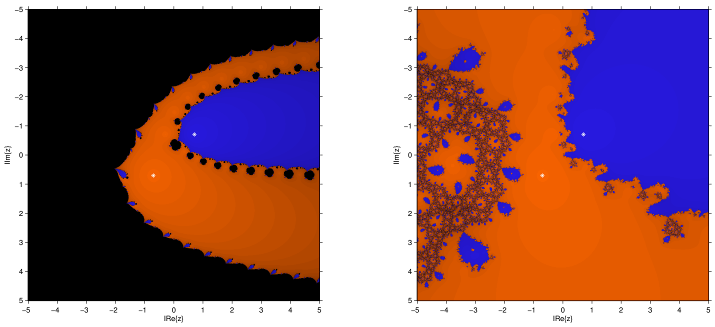
Moreover, $\mathcal{A}(z_{f_2}^*)$ is immersed in $\mathcal{A}(z_{f_1}^*)$ when the c value belongs to the yellow or purple region from parameters plane. This is the case of c equals to -1 and $\pm i$ (Figures 5a, 7a and 8a). If a c value of the pink region is taken, $\mathcal{A}(z_{f_1}^*)$ and $\mathcal{A}(z_{f_2}^*)$ are not immersed, as in Figure 4a, where $c = 1$. The same immersion behaviour happens for each case of M4 method.

It can be concluded that the stability of M4 method is greater than Steffensen's.



(a) STF (b) M4

Figure 6: $p_0(z) = z^2$



(a) STF (b) M4

Figure 7: $p_i(z) = z^2 + i$

When any point on the Fatou set is chosen in M4, the orbit converges to one of the superattracting fixed points, because there is no basin of attraction in the infinity.

About the symmetries of Steffensen (9) and M4 (11) expressions, it is easy to check the abscissas symmetry for methods with $c \in \mathbb{R}$ (Figures 4-6). In Figures 7 and 8 the symmetry is between the methods, because c values are conjugated.

3 Cubic polynomials

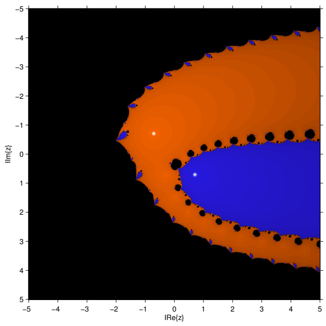
Now, we are going to analyze the dynamic behaviour of the derivative-free schemes Steffensen and M4 on cubic polynomials, showing their similarities and differences.

The fixed point operator of Steffensen's method when it is applied to the polynomial $q_c(z) = z^3 + c$ is

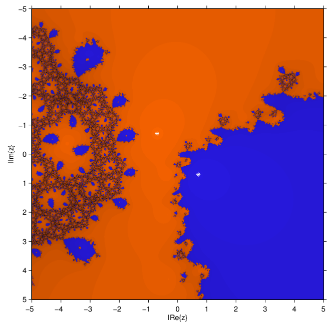
$$S_{q_c}(z) = \frac{z^7 + 3z^5 + 2cz^4 + 2z^3 + 3cz^2 + c^2 z - c}{z^6 + 3z^4 + 2cz^3 + 3z^2 + 3cz + c^2}. \tag{14}$$

It can be proved that the operator $S_{q_c}(z)$ satisfies the symmetry property

$$S_{q_{\bar{c}}}(\bar{z}) = \overline{S_{q_c}(z)}, \qquad \forall c, z \in \mathbb{C}, \tag{15}$$



(a) STF



(b) M4

Figure 8: $p_{-i}(z) = z^2 - i$

that will be observed in the different dynamic planes. As in case of quadratic polynomials, five particular cases of $q_c(z) = z^3 + c$ are studied, taking into account that the infinity is a superattracting point, also for cubic polynomials.

When $q_1(z) = z^3 + 1$ the fixed points of the operator are

$$z_{f_1} = -1, \quad z_{f_2} = i^{2/3}, \quad z_{f_3} = -i^{4/3}. \tag{16}$$

These fixed points $z_{f_{1,2,3}} = z^*_{f_{1,2,3}}$ are superattracting, and coincide with the roots of the polynomial $q_1(z)$. The dynamic plane has four basins of attraction, $\mathcal{A}(z^*_{f_1})$, $\mathcal{A}(z^*_{f_2})$, $\mathcal{A}(z^*_{f_3})$ and $\mathcal{A}(\infty)$.

By applying (6) to the operator $S_{q_1}(z)$, the critical points are

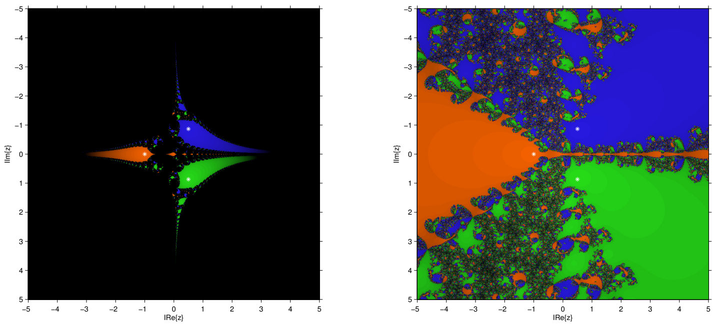
$$\begin{array}{llll} z_{c_1} &= -1, & z_{c_{5,6}} &= -0.295567 \pm i0.649277, \\ z_{c_2} &= i^{2/3}, & z_{c_{7,8}} &= 0.283565 \pm i1.34654 \\ z_{c_3} &= -i^{4/3}, & z_{c_{9,10}} &= -0.573381 \pm i1.44982, \\ z_{c_4} &= -0.471717, & z_{c_{11,12}} &= 0.821242 \pm i1.71625. \end{array} \tag{17}$$

The dynamic plane of $S_{q_1}(z)$ is shown in Figure 9a. The three basins of attraction, $\mathcal{A}(z^*_{f_1})$, $\mathcal{A}(z^*_{f_2})$ and $\mathcal{A}(z^*_{f_3})$, are pictured with orange, green and blue colors, respectively. The basin of the infinity is in black.

If $c = -1$ in (14), the numerator of the fixed point operator is a polynomial of degree 7 (as in $c = 1$). Three finite superattracting fixed points are obtained. All of them are the roots of $q_{-1}(z) = z^3 - 1$, i.e., $z^*_{f_1} = 1$, $z^*_{f_2} = -i^{1/3}$ and $z^*_{f_3} = i^{2/3}$. The four basins of attraction and the rest of the dynamic plane of $S_{q_{-1}}(z)$ are represented in Figure 10a.

Applying (2) to $q_0(z) = z^3$, the degree of the numerator is 5. In this case, there is an only superattracting finite fixed point in $z_{f_1} = z^*_{f_1} = 0$. In Figure 11a the dynamic plane of $S_{q_0}(z)$ is represented.

The last two cases correspond to $q_{\pm i}(z) = z^3 \pm i$. The polynomial of the numerator of $S_{q_{\pm i}}(z)$ is of degree 7. The only finite superattracting points are the three roots. These roots are conjugated between $S_{q_i}(z)$ and $S_{q_{-i}}(z)$. According to the property



(a) STF (b) M4

Figure 9: $q_1(z) = z^3 + 1$

(15), Figures 12a and 13a (the dynamic planes of $S_{q_i}(z)$ and $S_{q_{-i}}(z)$, respectively) are symmetrical.

When M4 is applied to polynomial $q_c(z) = z^3 + c$, the $M_{q_c}(z)$ general expression is

$$M_{q_c}(z) = \frac{N_{q_c}(z)}{D_{q_c}(z)}, \tag{18}$$

where $N_{q_c}(z)$ is a polynomial of degree 39 and $D_{q_c}(z)$ a polynomial of degree 38. This operator satisfies the symmetry property

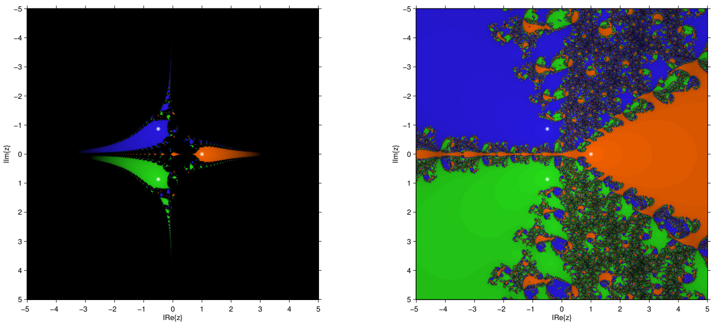
$$M_{q_{\bar{c}}}(\bar{z}) = \overline{M_{q_c}(z)}, \qquad \forall c, z \in \mathbb{C}. \tag{19}$$

When the fixed points are evaluated in the modulus of the derivative of $M_{q_1}(z)$, the superattracting behaviour of $z_{f_1} = -1$ and $z_{f_{2,3}} = 0.5000 \pm i0.8660$ is obtained (so, $z_{f_{1,2,3}} = z^*_{f_{1,2,3}}$). Their basins of attraction can be observed in Figure 9b. The rest of the fixed points are repelling. By applying (6) to the operator $M_{q_1}(z)$, 76 critical points are obtained, three of which agree with the roots of $q_1(z)$.

If (18) is applied to polynomial $q_{-1}(z) = z^3 - 1$, the numerator of the fixed point operator is a polynomial of degree 39. The superattracting fixed points are the roots of $q_{-1}(z)$, i.e., $z^*_{f_1} = 1$ and $z^*_{f_{2,3}} = -0.5 \pm 0.866$. The other 36 fixed points are repelling. The dynamic plane of $M_{q_{-1}}(z)$ is shown in Figure 10b.

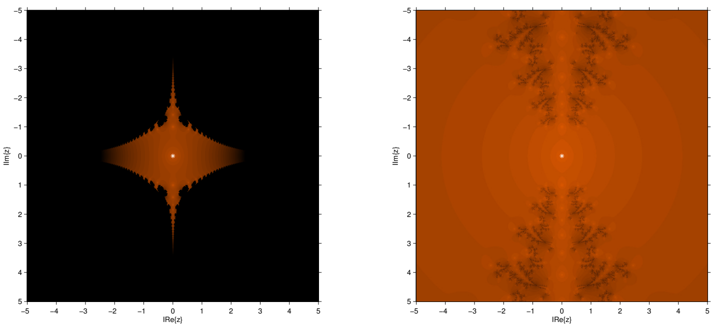
When $c = 0$, the numerator of the fixed point operator is a polynomial of degree 21. The only superattracting fixed point is the root of the polynomial $q_0(z)$, $z^*_{f_1} = 0$. The dynamic behaviour of $M_{q_0}(z)$ can be observed in Figure 11b. The repelling fixed points belong to the Julia set.

Replacing the c parameter by pure imaginary values, such that $c = \pm i$, the fixed point function associated to M4 method when is applied to $q_{\pm i}(z) = z^3 \pm i$ has a polynomial of degree 39 in the numerator. The only superattracting fixed points are the roots of $q_{\pm i}(z)$, i.e., $z^*_{f_1} = \pm i$ and $z^*_{f_{2,3}} = 0.8660 \mp i0.5$. In Figures 12b and 13b is represented the dynamic plane of $M_{q_i}(z)$ and $M_{q_{-i}}(z)$, respectively. The repelling points belong to the Julia set.



(a) STF (b) M4

Figure 10: $q_{-1}(z) = z^3 - 1$



(a) STF (b) M4

Figure 11: $q_0(z) = z^3$

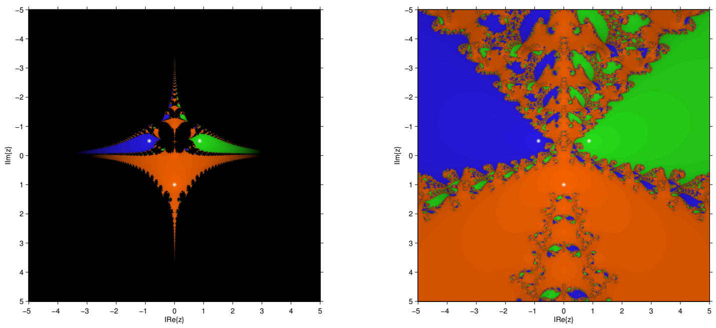
Summarizing, the dynamic planes of Steffensen's and M4 methods when they are applied to cubic polynomial $q_c(z) = z^3 + c$ are represented in Figures 9-13, with different values of c. As in quadratic polynomials, the superattracting fixed points are plotted with white stars. Its basins of attraction are drawn in blue, orange and green. The basin of attraction of the infinity is black, if it exists.

When Steffensen and M4 methods are compared in terms of stability, the convergence region of the fourth-order method is greater than the second-order one. The basin of attraction of the infinity –superattracting in Steffensen– disappears in M4. Applying M4, every point on the complex plane converge to either of the roots of $q_c(z)$, but the Julia set.

The symmetry property on the abscissas axis is an evidence in Figures 9-11, where the c value is real. Furthermore, $S_{q_i}(z)$ and $S_{q_{-i}}(z)$ (Figures 12a and 13a) on the one hand, and $M_{q_i}(z)$ y $M_{q_{-i}}(z)$ (Figures 12b and 13b), on the other hand, are symmetrical between them.

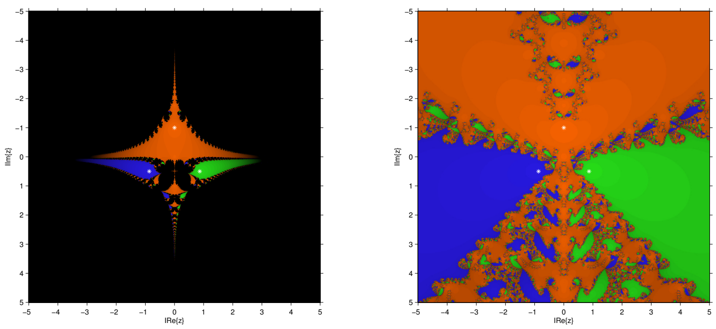
It is interesting to note that, as the absolute value of c increases, M4 method tends to the dynamic behaviour of Newton, as it is showed in Figure 14.

The immediate basins of attraction of every Steffensen case are not immersed. In M4 method, the basins of attraction are immersed in $c = \pm i$ cases. In the rest of



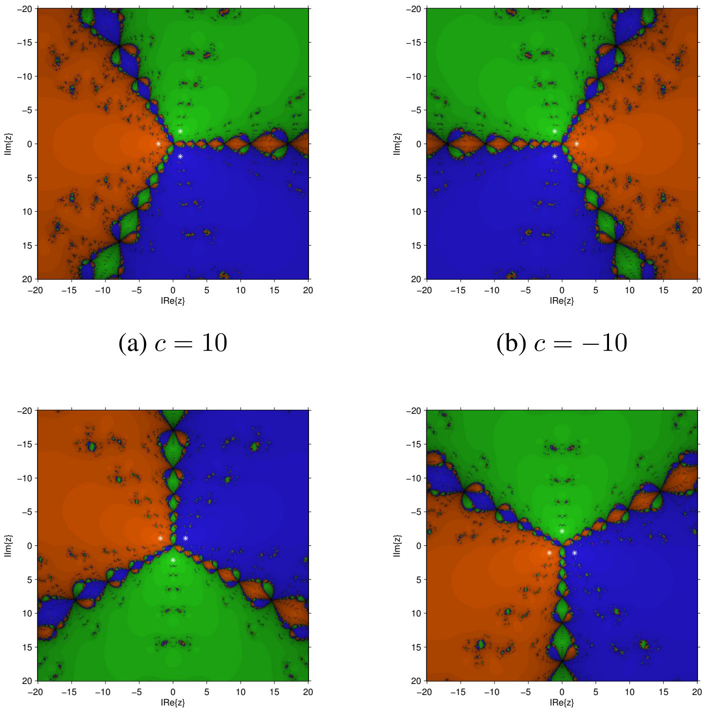
(a) STF (b) M4

Figure 12: $q_i(z) = z^3 + i$



(a) STF (b) M4

Figure 13: $q_{-i}(z) = z^3 - i$



(a) $c = 10$ (b) $c = -10$

(c) $c = i10$ (d) $c = -i10$

Figure 14: $M_{q_c}(z)$ when $q_c(z) = z^3 + c$, $c = \{\pm 10, \pm i10\}$.

particular cases there is no immersion. As the modulus of c increases for pure imaginary values, the basins of attraction are not immersed (see Figures 14c and 14d, where $c = \pm i10$, respectively).

4 Conclusions

The only superattracting fixed points of M4 method are the roots of the polynomial whose fixed point operator is applied to. There is one more superattracting fixed point in Steffensen's method: the infinity. The existence of the basin of attraction of the infinity in Steffensen's method – for either quadratic or cubic polynomials – is a problem to ensure the stability. Nevertheless, M4 method eliminates the basin of attraction of the infinity and has full convergence in the complex plane, but the Julia set.

When the c values of $p_c(z) = z^2 + c$ and $q_c(z) = z^3 + c$ are real, the symmetry property allows up the study of $\mathrm{Im}(z) \geq 0$ semiplane to obtain the complete plane dynamic behaviour. Also, if $S_{p_c}(z)$, $M_{p_c}(z)$, $S_{q_c}(z)$ or $M_{q_c}(z)$, for $c \in \mathbb{C}$ has been studied, the dynamic plane of $S_{p_{\bar{c}}}(z)$, $M_{p_{\bar{c}}}(z)$, $S_{q_{\bar{c}}}(z)$ or $M_{q_{\bar{c}}}(z)$ is immediate.

A useful tool for the dynamic analysis of the methods is the parameters plane. It has been obtained for both methods in quadratic polynomials. In Steffensen's method, the parameters plane sets in which basin of attraction is one of the critical points. Moreover, it states if the basins of attraction are immersed or not. The immersion behaviour in quadratic polynomials for Steffensen's and M4 methods is analogous. However, when the fixed point operators are applied to cubic polynomials, for small modulus complex values of c the immersion behaviour of Steffensen and M4 are not similar.

Acknowledgement

This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-02 and by Vicerrectorado de Investigación, Universitat Politècnica de València PAID-06-2010-2285.

References

[1] A. Douady, J.H. Hubbard, "On the dynamics of polynomials-like mappings", Ann. Sci. Ec. Norm. Sup., 18, 287-343, 1985.

[2] J. Curry, L. Garnet, D. Sullivan, "On the iteration of a rational function: Computer experiments with Newton's method", Comm. Math. Phys., 91, 267-277, 1983.

[3] P. Blanchard, "The dynamics of Newton's method", Proc. of Symposia in Applied Math., 49, 139-154, 1994.

[4] N. Fallega, "Invariants in dinàmica complexa", Butlletí de la Soc. Cat. de Matemàtiques", 23(1), 29-51, 2008.

[5] J. Varona, "Graphic and numerical comparison between iterative methods", The Mathematical Intelligencer, 24(1), 37-46, 2002.

[6] S. Amat, S. Busquier, S. Plaza, "Chaotic dynamics of a third-order Newton-type method", J. Math. Anal. Appl., 366, 24-32, 2010.

[7] J.M. Gutiérrez, M.A. Hernández, N. Romero, "Dynamics of a new family of iterative processes for quadratic polynomials", Journal of Computational and Applied mathematics, 233, 2688-2695, 2010.

[8] S. Plaza, N. Romero, "Attracting cycles for the relaxed newton's method", Journal of Computational and Applied Mathematics, 235, 3238-3244, 2011.

[9] C. Chun, M. Y. Lee, B. Neta, J. Dzunic, "On optimal fourth-order iterative methods free from second derivative and their dynamics", Applied Mathematics and Computation, 218, 6427-6438, 2012.

[10] S. Amat, S. Busquier, S. Plaza, "Review of some iterative root-finding methods from a dynamical point of view", Mathematical Sciences, 10, 3-35, 2004.

[11] J. M. Ortega, W. G. Rheinboldt, "Iterative solutions of nonlinear equations in several variables", New York, Academic Press, 1970.

[12] A. Cordero, J. L. Hueso, E. Martínez, J. R. Torregrosa. "A new technique to obtain derivative-free optimal iterative methods for solving non-

linear equations", Journal of Computational and Applied mathematics, doi:10.1016/j.cam.2012.03.030.
[13] P. Blanchard, "Complex Analytic Dynamics on the Riemann Sphere", Bull. of the AMS, 11(1), 85-141, 1984.
[14] R. L. Devaney. "An Introduction to Chaotic Dynamical Systems", Addison-Wesley Publishing Company, 1989.
[15] S. Mayer, D. Schleicher, "Immediate and Virtual Basins of Newtons Method for Entire Functions", Annales de l'institut Fourier, 56(2), 325-336, 2006.