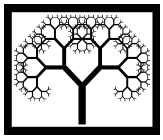
Paper 89



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A Multigrid Method using Explicit Approximate Inverses for the Numerical Solution of Two-Dimensional Time-Dependent Problems

C.K. Filelis-Papadopoulos and G.A. Gravvanis
Department of Electrical and Computer Engineering
School of Engineering
Democritus University of Thrace, Xanthi, Greece

Abstract

During the recent decades, multigrid methods have been extensively used in order to solve large scale linear systems derived from the discretization of partial differential equations using the finite difference method. The purpose of this work is the derivation of hybrid heterogeneous schemes based on time implicit approximating schemes, namely backward differences and Crank–Nicolson method, with the multigrid method in conjunction with explicit approximate inverses for the efficient iterative solution of unsymmetric linear systems. Thus, a new class of smoothers, based on approximate inverses, has been derived. Moreover, a dynamic relaxation scheme is proposed based on the dynamic over - under relaxation (DOUR) algorithm. In addition, a new class of Krylov subspace iterative methods in conjunction with multigrid preconditioning based on explicit approximate inverses is used for solving initial value problems. Finally, the applicability of the new proposed hybrid heterogeneous implicit time - explicit approximate inverses combined with multigrid methods for parabolic PDEs is discussed by solving characteristic time-dependent problems in two dimensions and the numerical results are given.

Key words: multigrid method, finite difference method, V – cycle, linear systems, approximate inverse smoothing, dynamic over-under relaxation method, preconditioned conjugate gradient method, initial value problems.

1 Introduction

Let us consider a class of problems defined by the following parabolic partial differential equation (PDE):

$$\frac{\partial u}{\partial t}+\sum_{i,j=1}^{N=2}\frac{\partial}{\partial x_i}\left[a_{i,j}(x)\frac{\partial u}{\partial x_j}\right]+\sum_{j=1}^{N=2}\left[b_j(x)\frac{\partial u}{\partial x_j}\right]+c(x)u=f(x,t),\quad (x,t)\in\Omega\times[0\le t\le T] \quad (1)$$

subject to the following general boundary conditions

$$\alpha(x,t)u(x,t)+\beta(x,t)\frac{\partial u(x,t)}{\partial \eta}=\gamma(x,t),\quad (x,t)\in \partial\Omega\times[0\le t\le T] \tag{1.a}$$

and initial conditions

$$u(x,0)=u_0(x),\quad x\in\Omega \tag{1.b}$$

where Ω is a closed bounded domain, $\partial\Omega$ denotes the boundary of Ω, $\partial\eta$ is the direction of the outward normal and $a_{i,j}(x)>0, b_j(x)>0, c(x)\ge 0$ are sufficiently smooth functions on Ω.

The numerical solution of the parabolic PDE using "time-implicit" methods, i.e. backward differences or the Crank – Nicolson scheme requires the solution of a sparse linear system of algebraic equations, i.e.

$$Au=s. \tag{2}$$

Explicit preconditioning methods have been extensively used for solving sparse linear systems on multiprocessor systems, and the preconditioned form of the sparse linear system (2) is

$$MAu=Ms, \tag{3}$$

where M is a suitable preconditioner. The preconditioner M has to satisfy the following conditions: (i) MA should have a clustered spectrum, (ii) M can be efficiently computed in parallel, (iii) "$M\times$ vector" should be fast to compute in parallel. The effectiveness of explicit approximate inverse preconditioning relies on the use of suitable preconditioners that are close approximants to the inverse of the coefficient matrix and are fast to compute in parallel, cf. [6, 7, 11, 13, 17]. In this article we present a parameterized "smoother" based on the explicit approximate inverse preconditioner and the explicit preconditioned Richardson iterative method.

During the last decade, multigrid methods, have gained substantial interest for solving large sparse linear systems amongst the scientific community because of their efficiency and convergence behavior. Multigrid methods are based on the observation that the high frequency components of the error are damped effectively by a stationary iterative method (namely Jacobi or Gauss - Seidel), however the low – frequency components are not damped effectively, cf. [1, 2, 19]. In order for low frequency components of the error to be handled, a series of coarser grids with higher mesh size are used as shown in Figure 1. In this series of coarser grids the low–frequency modes of the error are more oscillatory and can be damped efficiently by a stationary iterative method, cf. [1, 2, 3, 19]. Multigrid methods are composed by four discrete elements: stationary iterative method, restriction operator, prolongation operator and cycle strategy. The stationary iterative methods are first order iterative methods such as Richardson, Jacobi and Gauss – Seidel method. Restriction and prolongation are transfer operators from finer to coarser grids and from coarser to finer grids respectively. The cycle strategy refers to the sequence in which the grids are visited until a solution to a prescribed tolerance is computed.

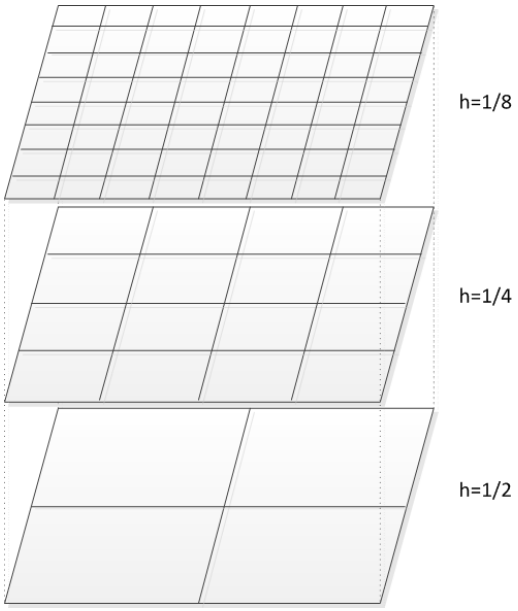

Figure 1: Square domain with different discretization step h

In order to accelerate convergence and increase robustness, multigrid can be used as a preconditioner to the Krylov subspace iterative methods, cf. [19]. Krylov subspace methods such as Bi–CGSTAB with multigrid preconditioning in conjunction with Explicit Approximate Inverse Smoothing are presented along with comparative results.

2 A class of hybrid implicit – explicit schemes

In order to derive a numerical scheme, the region Ω is covered with a square mesh with spacing h and Δt is the time step across the T direction. Using the five point stencil across the space direction and the backward or Crank – Nicolson scheme, results in solving the following compact linear system, viz.

$$Bu^{(k+1)} = Cu^{(k)} + d, \quad i \geq 0 \tag{4}$$

where the matrices B and C are of the following form:

$$B = \begin{array}{c} \overset{\leftarrow \qquad m \qquad \rightarrow}{} \\ \begin{bmatrix} 1+\rho\theta b_1 & -\rho\theta c_1 & & -\rho\theta w_1 & & \\ -\rho\theta a_2 & \ddots & \ddots & & \ddots & \\ & \ddots & \ddots & \ddots & & -\rho\theta w_{n-m+1} \\ -\rho\theta v_m & & \ddots & \ddots & \ddots & \\ & \ddots & & \ddots & \ddots & -\rho\theta c_{n-1} \\ & & -\rho\theta v_n & & -\rho\theta a_n & 1+\rho\theta b_n \end{bmatrix} \end{array} \tag{5}$$

$$
\begin{array}{c}
\leftarrow \quad m \quad \rightarrow \\
C = \begin{bmatrix}
1-\rho(1-\theta)b_1 & \rho(1-\theta)c_1 & & \rho(1-\theta)w_1 & & \\
\rho(1-\theta)a_2 & \ddots & \ddots & & \ddots & \\
 & \ddots & \ddots & \ddots & & \rho(1-\theta)w_{n-m+1} \\
\rho(1-\theta)v_m & & \ddots & \ddots & \ddots & \\
 & \ddots & & \ddots & \ddots & \rho(1-\theta)c_{n-1} \\
 & & \rho(1-\theta)v_n & & \rho(1-\theta)a_n & 1-\rho(1-\theta)b_n
\end{bmatrix}
\end{array} \tag{6}
$$

where $\rho = \Delta t/h^2$ is the mesh ratio, Δt is the time step, h is the space mesh size and various values of the parameter θ denote the nature of the finite difference method. In case of $\theta=1$ the above parametric form results to the "time – implicit" (Backward Differences) scheme, while in case of $\theta=1/2$ the parametric form results to the "time - implicit" Crank – Nicolson scheme. The "time - implicit" schemes are unconditionally stable.

The linear system (4) can be expressed in the following compact matrix form

$$Au = s \tag{7}$$

where A (n×n) is a large sparse coefficient matrix of the same profile as (5) while u and s are n-column vectors, which consist respectively of the (FD) solution and the source terms plus the known initial/boundary values.

The resulting linear system (7) is then solved by the proposed multigrid schemes in conjunction with explicit approximate inverses resulting in hybrid heterogeneous implicit – explicit schemes, cf. [8, 12, 18].

3 Elements of the Multigrid

Let us consider the linear system derived from the discretization of a PDE on a unit square domain with mesh size h:

$$A_h u_h = f_h \tag{8}$$

In order to solve the linear system (8) it is important to introduce the basic elements of the multigrid algorithm as discussed in [2, 3, 15 ,19]. Let us assume the following spaces:

$$X_\ell = R^{n_\ell}, \ell = 0,1,2... \; n_0 < n_1 < ... \tag{9}$$

where ℓ is the index of each level of the multigrid algorithm. The grid transfer operators (prolongation and restriction) can also be defined as follows:

$$P : X_{\ell-1} \to X_\ell, \quad R : X_\ell \to X_{\ell-1}, \quad \ell = 1,2,... \tag{10}$$

In each space X_ℓ a non-singular coefficient matrix can be defined, that arises from the discretization of equation (1) with different mesh sizes $h = 1/n, n = 2^k, k = 0,1,2,....$.

$$A_\ell : X_\ell \to X_\ell \tag{11}$$

The linear system (8) can be solved iteratively with a multigrid method. A multigrid method can be formulated by the recursive call of the two – grid method. The two grid coarse grid correction method with mesh size h consists of ν_1 pre–smoothing steps and ν_2 post–smoothing steps on the level with mesh size h as well as the coarse grid correction on the level with mesh size 2h. The error propagation at the m+1 iteration can be described as follows:

$$e_h^{(m+1)} = \left[S_h^{\nu_2}\left(I - PA_{2h}^{-1}RA_h\right)S_h^{\nu_1}\right]e_h^{(m)} \tag{12}$$

where $S_h = \left(I_h - M_h A_h\right)$ denotes the iteration matrix of the stationary iterative method (smoother), cf. [3, 16].

3.1 Smoothers

Multigrid smoothers are stationary iterative solvers that can be described by the following recurrence relation known as general iterative scheme:

$$x_\ell^{(k+1)} = x_\ell^{(k)} + M_\ell r_\ell, \quad r_\ell = f_\ell - A_\ell x_\ell, \tag{13}$$

There are various choices of smoothers, such as Jacobi and Gauss – Seidel method, discussed in [2, 5, 16, 17, 19]. In the case, where $M_\ell = \left(M_\ell\right)_r^{\delta l}$ then we have the Explicit Approximate Inverse Smoothing scheme, cf. [5].

3.2 Approximate Inverse Smoothing

In order to accelerate convergence of multigrid algorithms, a new class of stationary iterative methods is proposed. Finite difference explicit approximate inverses are used in conjunction with Richardson iterative method. Hence,

$$x_\ell^{(k+1)} = x_\ell^{(k)} + \omega\left(M_\ell\right)_r^{\delta l}\left(f_\ell - A_\ell x_\ell^{(k)}\right) \tag{14}$$

where $\left(M_\ell\right)_r^{\delta l}$ is a class of finite difference explicit approximate inverses and ω is the damping parameter with $0 < \omega \le 1$.

During the past decades explicit approximate inverse preconditioners have been extensively used due to their inherent parallelism and convergence behavior, cf. [6, 10, 11, 13, 17].

Let us assume the sparse approximate factorization, such that

$$A \approx L_r U_r, \quad r \in [1, m-1] \tag{15}$$

where r is the so called "fill–in" parameter and L_r, U_r are upper and lower matrices respectively, of the same profile as the coefficient matrix A, cf. [4]. The elements of the decomposition factors L_r, U_r can be computed by the ALUBOT algorithm, cf. [4].

Let $M_r^{\delta l} = \left(\mu_{i,j}\right), i \in [1, n], j \in [i - \delta l + 1, i + \delta l - 1]$ be the approximate inverse of the coefficient matrix A. The elements of a class of banded forms of the approximate inverse, by retaining δl and $\delta l - 1$ elements in the lower and upper parts can be computed by solving recursively the following systems:

$$M_r^{\delta l} L_r = \left(U_r\right)^{-1} \text{ and } U_r M_r^{\delta l} = \left(L_r\right)^{-1}, r = [1, m-1), \delta l = [1, \rho m) \tag{16}$$

where $\rho = 1,2,...m-1$, cf. [8, 17].

Then, the elements of the optimized form, of the approximate inverse based on a shifted window from top to bottom of the generalized approximate inverse are computed by the Optimized Banded Generalized Approximate Inverse (OBGAIM) algorithm, cf. [7, 11]. The computational work of the OBGAIM algorithm is $O(n \times \delta l \times r)$.

The OBGAIM algorithm has proved to be particularly effective for solving "banded" sparse FD systems of large order, i.e. $(2\delta l - 1) < n/2$, or "narrow – banded " sparse FD system of very large order, i.e. $(2\delta l - 1) << n/2$. It should be noted that the largest in magnitude elements of the approximate inverse matrix are clustered around the diagonals at distances ρm, where ρ=1,2,…, from the main diagonal in a "recurring wave"–like pattern. Therefore, it is reasonable to assume and has been proven that the value of the "retention" parameter δl can be chosen as multiples of the semi–bandwidth m, cf. [7, 10, 17].

The approximate inverse matrix $M_r^{\delta l}$ represents a class of approximate inverses that includes various families of approximate inverses according to the requirements of accuracy, storage and computational work, as can be seen by the following diagrammatic relation:

$$\begin{array}{ccccccc} & & \underline{\text{class I}} & & \underline{\text{class II}} & & \underline{\text{class III}} \\ A^{-1} \equiv M & \leftarrow & M_{r=m-1}^{\delta l} & \leftarrow & M_r^{\delta l} & \leftarrow & M_i \end{array} \qquad (17)$$

where M is the exact inverse resulting in a direct method, i.e. r=m-1 and δl=n with the disadvantage of high order memory requirements and computational work for large order systems. The entries of the class I inverse have been retained after the computation of the exact inverse (r=m-1, δl=n) by retaining only δl and δl-1 elements in the lower and upper part of the exact inverse. The entries of the class II inverse have been computed and retained during the computational procedure of the approximate inverse (r≤m-1, δl<n), while the entries of the class III of the generalized approximate inverse retains only the diagonal elements i, i∈[1,n], i.e. δl=1, requiring only the inversion of the diagonal entries of the sparse lower matrix L_r, cf. [6, 7], resulting in a fast inverse algorithm.

In multigrid convergence theory, cf. [2, 3, 15, 19], two properties must be satisfied in order for the two – grid cycle to converge:

1. The smoothing property, cf. [2, 3, 15, 19]

$$\left\| A_\ell S_\ell^\nu \right\|_2 \le \eta(\nu) \left\| A_\ell \right\|_2, \; 0 \le \nu < \infty, \; \ell \ge 1 \qquad (18)$$

$\eta(\nu)$ is any function with $\lim_{\nu \to \infty} \eta(\nu) = 0$.

2. The approximation property, cf. [2, 3, 15, 16, 19]

$$\left\| A_\ell - P A_{\ell-1} R \right\|_2 \le \frac{C_A}{\left\| A_\ell \right\|_2}, \ell \ge 1. \qquad (19)$$

If the smoothing and approximation properties are satisfied, convergence independent of the levels ℓ for the W – cycle is also implied. In [16] is stated that for symmetric positive definite coefficient matrices, convergence for the V – cycle independent of the levels ℓ, is also implied. The approximation property is

independent of the smoother and depends only on the discretization, the prolongation and restriction operators. Hackbush in [16] has shown that the approximation property is satisfied for various elliptic boundary value problems. Moreover, the smoothing property for S.P.D problems is satisfied for the classical smoothers such as the Jacobi or Gauss–Seidel iterative scheme. The smoothing property of the Optimized Banded Generalized Approximate Finite Difference Inverse matrix has been proven in [5]. Moreover, Optimized Banded Generalized Approximate Finite Difference Inverse matrix is used in conjunction with the Dynamic Over / Under Relaxation, cf. [5, 14] in order to make the smoothing scheme independent of the choice of the damping parameter, cf. [5].

3.3 Prolongation and Restriction

The prolongation operator P (10) is an interpolation procedure that transfers vectors from a coarse to a finer discretized grid. An effective choice for the prolongation operator is the linear interpolation because of the low cost and effectiveness in computation. Higher order interpolation schemes such as bicubic or biquadratic interpolation can also be used, but the computational cost increases rapidly.

The bilinear interpolation for a square discretized grid can be described as

$$\begin{cases} u^{h}_{2i,2j} = u^{2h}_{i,j} \\ u^{h}_{2i+1,2j} = \frac{1}{2}\left(u^{2h}_{i,j} + u^{2h}_{i+1,j}\right) \\ u^{h}_{2i,2j+1} = \frac{1}{2}\left(u^{2h}_{i,j} + u^{2h}_{i,j+1}\right) \\ u^{h}_{2i+1,2j+1} = \frac{1}{4}\left(u^{2h}_{i,j} + u^{2h}_{i+1,j} + u^{2h}_{i,j+1} + u^{2h}_{i+1,j+1}\right) \end{cases} ,0 \le i, j \le \frac{n}{2} - 1 \qquad (20)$$

where u is the solution vector and n is the number of unknowns of the finer grid in each direction incremented by 1. Discussions for different prolongation strategies can be found in [2, 15, 19]. Restriction operator R (10) is used to transfer vectors from finer discretized grids to coarser discretized grids. An effective choice for the restriction operator is the full–weighting, because of its robustness for the multigrid iteration. The full–weighting restriction operator can be defined as follows:

$$u^{2h}_{i,j} = \frac{1}{16}\begin{bmatrix} u^{h}_{2i-1,2j-1} + u^{h}_{2i-1,2j+1} + u^{h}_{2i+1,2j-1} + u^{h}_{2i+1,2j+1} \\ +2\left(u^{h}_{2i,2j-1} + u^{h}_{2i,2j+1} + u^{h}_{2i-1,2j} + u^{h}_{2i+1,2j}\right) + 4u^{h}_{2i,2j} \end{bmatrix}, 0 \le i, j \le \frac{n}{2} - 1 \qquad (21)$$

where n is the number of unknowns of the finer grid in each direction incremented by 1. Both transfer operators can be presented also in sparse matrix form. Furthermore, the full–weighting restriction operator in conjunction with bilinear interpolation satisfies the Galerkin condition $[P] = c([R])^{T}$, $c \in R$. Transfer operators related through the Galerkin condition are benefited, due to simplification of their mapping on the data, cf. [2]. Further information on various restriction operators can be found in [2, 15, 19].

3.4 Cycle Strategy

The cycle strategy is an essential component of the multigrid algorithm and refers to the sequence in which the various grids are visited and the respective coarse grid corrections are obtained. The common cycle strategy is the V – Cycle algorithm and can be represented in Figure 2.

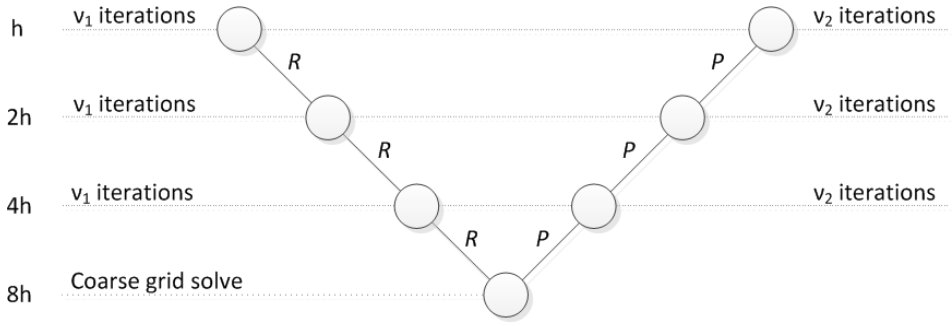

Figure 2: The V – Cycle with finest grid Ω^{h}, coarsest grid Ω^{8h}, and ν_1, ν_2 pre–smoothing and post–smoothing steps, respectively

The solution of a model problem can be achieved by successive applications of the V – Cycle according to arbitrary termination criterion. The proposed multigrid scheme descends to the coarsest possible level, where the exact solution is obtained. The coarsest level has only one unknown at the center of the grid. The iterative algorithm for the V – Cycle multigrid method is presented in [2]. Various cycle strategies and their corresponding algorithms are presented in [2, 15, 19].

3.5 Approximate Inverse Smoothing with Dynamic Over-Under Relaxation

The proposed explicit approximate inverse smoothing scheme (14) requires a relaxation parameter ω in order to be more effective and efficient. The choice of the relaxation parameter is non-trivial for a wide variety of problems and various choices of the "retention" parameter as discussed in [5]. A method to compute the relaxation parameter dynamically is a "predictor–corrector" like scheme. This scheme is based on the dynamic over-under relaxation (DOUR) method, cf. [14].

Let us consider the proposed relaxation scheme

$$x_{\ell}^{(k+1)} = x_{\ell}^{(k)} + \omega_e \left(S\left(x_{\ell}^{(k)}\right) - x_{\ell}^{(k)} \right), \omega_e = \omega(1+\kappa) \tag{22}$$

where $\kappa = \dfrac{\left\langle \Delta x_{\ell}^{(k)}, f_{\ell} - A_{\ell} \tilde{x}_{\ell}^{(k)} \right\rangle}{\left\langle \Delta x_{\ell}^{(k)}, A_{\ell} \Delta x_{\ell}^{(k)} \right\rangle}$, and ω_e is the effective relaxation parameter.

3.6 Multigrid preconditioned Bi-CGSTAB

In order to accelerate the convergence, multigrid methods are used in conjunction with Krylov subspace iterative solvers. For complex applications the use of multigrid as a preconditioner to a Krylov subspace method results in an efficient iterative solution method, cf. [19]. The proposed V-Cycle multigrid scheme, with explicit approximate inverses as smoothers, is used in conjunction with Bi–Conjugate Gradient Stabilized method resulting in MGV(ν_1, ν_2, ν_3) Bi–CGSTAB, where ν_1 denotes the pre–smoothing steps, ν_2 denotes the post–smoothing steps and ν_3 denotes the number of cycles performed during the preconditioning. The algorithm of the MGV(ν_1, ν_2, ν_3) Bi–CGSTAB is presented in [5].

4 Numerical Results

In this section numerical results are presented for the proposed multigrid schemes by considering two model problems. The results were obtained using the MATLAB environment. The termination criterion for the inner multigrid solver was $\|r_i\|_2 / \|r_0\|_2 < 10^{-4}$, while for convergence to the steady state solution (s.s.s.) the criterion was $\|u_{i+1} - u_i\|_\infty < 10^{-4}$. The pre–smoothing and post–smoothing iterations for the model problems are set to $\nu_1 = 2$ and $\nu_2 = 1$ respectively. Furthermore, the number of cycles for preconditioning in MGV(ν_1, ν_2, ν_3) Bi–CGSTAB method is set to $\nu_3 = 1$.

4.1 Model Problem I

The first problem to be solved with the proposed multigrid schemes is a 2D linear diffusion equation, i.e.

$$\frac{\partial u}{\partial t} - c(\Delta u) = 0, \ (x, y, t) \in \Omega \times [0 \le t \le T] \tag{23}$$

$$u(t, x, y) = 0, \ (x, y, t) \in \partial\Omega \times [0 \le t \le T] \tag{23.a}$$

with initial condition

$$u(0, x, y) = \cos(\pi x)\cos(\pi y), \quad (x, y) \in \Omega \tag{23.b}$$

where Δ is the Laplace operator, Ω is the unit square and $\partial\Omega$ denotes the boundary of Ω. The value of the diffusion parameter is chosen arbitrarily to $c = 0.3$.

In Table 1, the number of inner and outer iterations for the backward differences scheme as well as the Crank-Nicolson scheme in conjunction with multigrid V(2,1) cycle, with explicit approximate inverse smoothing, for various values of the

"retention" parameter δl and time step Δt with space mesh size h=1/64 is presented. In Table 2, the number of inner and outer iterations for both the backward differences and Crank-Nicolson scheme in conjunction with MGV(2,1,1) Bi-CGSTAB method, with explicit approximate inverse smoothing, for various values of the "retention" parameter δl and various values of the time step Δt with space mesh size h=1/64 is presented.

In Figure 3, the computed solution at t=1 with multigrid V(2,1) method and explicit approximate inverse smoothing, time step Δt=0.01, space mesh size h=1/64 and "retention" parameter δl=1, with the backward differences scheme, is presented. In Figure 4, the exact solution for time step Δt=0.01, space mesh size h=1/64 at t=1 is presented. In Figure 5, the absolute error (between the computed and exact solution) of the backward differences scheme is presented.

In Figure 6, the computed solution at t=1 with multigrid V(2,1) method with explicit approximate inverse smoothing, time step Δt=0.01, space mesh size h=1/64 and "retention" parameter δl=1, with the Crank - Nicolson scheme, is presented. In Figure 7, the absolute error of the Crank - Nicolson scheme is presented.

Method	**δl** / **Δt**	**1**	**2**	**2m**	**4m**
		Total number of inner (outer) iterations.			
Implicit Scheme	0.0100	359(111)	370(111)	259(111)	259(111)
	0.0050	511(196)	539(196)	355(196)	348(196)
	0.0010	1018(692)	1159(692)	865(692)	864(692)
	0.0005	1314(1147)	1600(1147)	1147(1147)	1147(1147)
Crank - Nicolson Scheme	0.0100	355(109)	308(109)	185(109)	184(109)
	0.0050	485(193)	441(194)	300(193)	287(193)
	0.0010	892(697)	910(690)	690(690)	690(690)
	0.0005	1147(1147)	1145(1145)	1145(1145)	1145(1145)

Table 1: Convergence behavior of the V(2,1) multigrid method with explicit approximate inverse smoothing for various values of the "retention" parameter δl and time step Δt with space mesh size h=1/64.

Method	**δl** / **Δt**	**1**	**2**	**2m**	**4m**
		Total number of inner (outer) iterations.			
Implicit Scheme	0.0100	333(111)	333(111)	222(111)	222(111)
	0.0050	392(196)	392(196)	196(196)	196(196)
	0.0010	692(692)	692(692)	692(692)	692(692)
	0.0005	1147(1147)	1147(1147)	1147(1147)	1147(1147)
Crank - Nicolson Scheme	0.0100	218(109)	218(109)	109(109)	109(109)
	0.0050	386(193)	386(193)	193(193)	193(193)
	0.0010	690(690)	690(690)	690(690)	690(690)
	0.0005	1145(1145)	1145(1145)	1145(1145)	1145(1145)

Table 2: Convergence behavior of the MGV(2,1,1) Bi-CGSTAB method with explicit approximate inverse smoothing for various values of the "retention" parameter δl and time step Δt with space mesh size h=1/64.

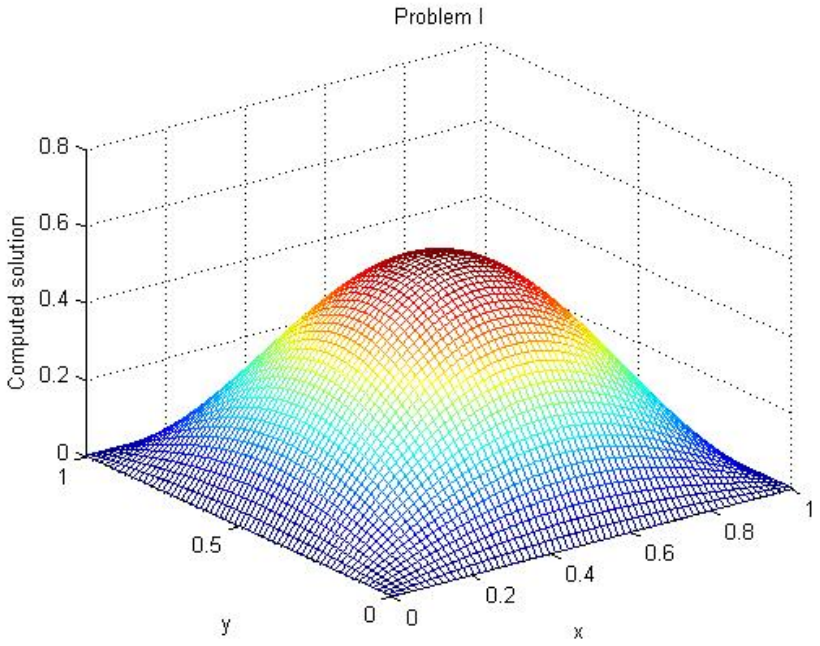

Figure 3: The computed solution for problem I obtained with V(2,1) multigrid method with explicit approximate inverse smoothing for δl=1, time step Δt=0.01 and space mesh size h=1/64 at t=1, using the backward differences scheme.

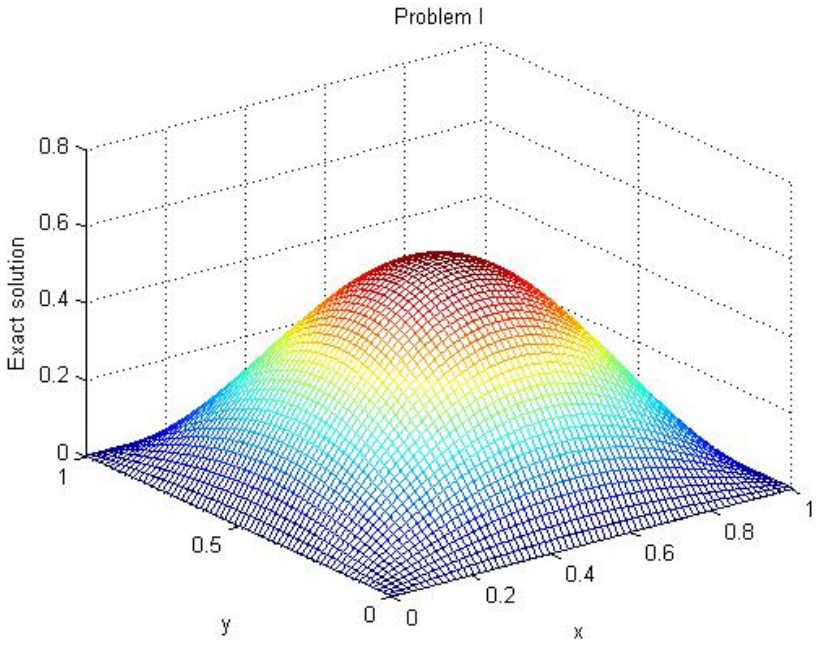

Figure 4: The exact solution for problem I with time step Δt=0.01 and space mesh size h=1/64 at t=1.

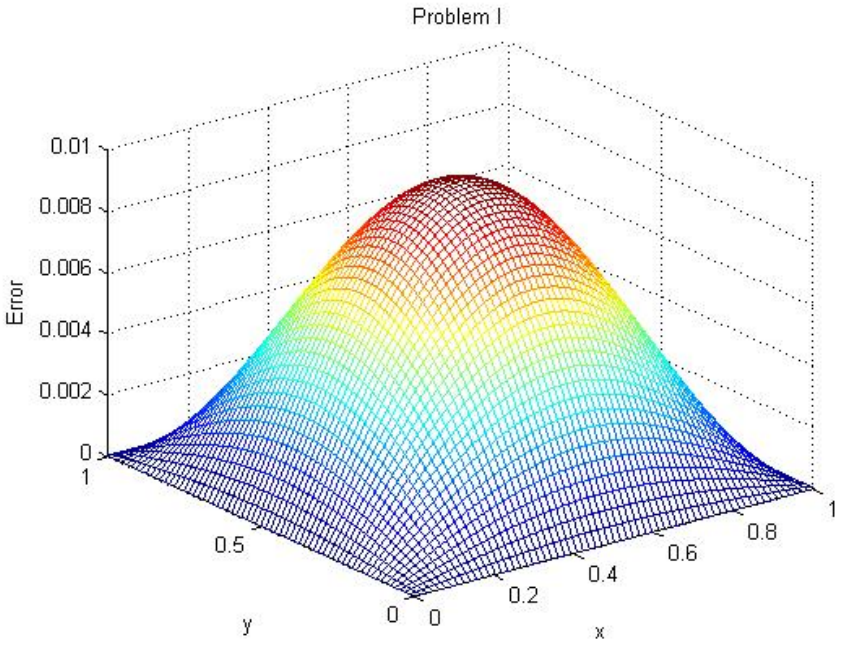

Figure 5: The absolute error between the exact and computed solution for problem I with time step $\Delta t=0.01$ and space mesh size $h=1/64$ at $t=1$, with the backward differences scheme.

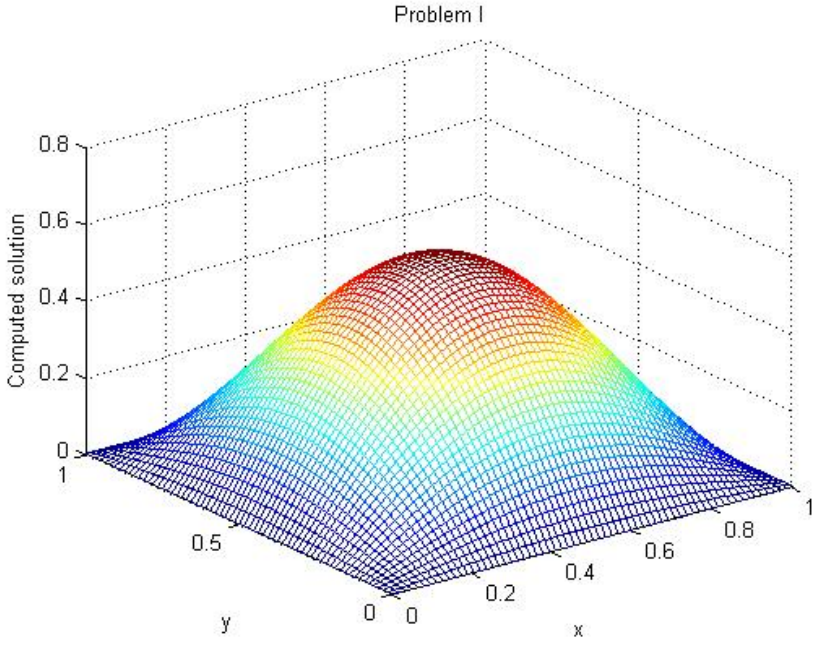

Figure 6: The computed solution for problem I obtained with V(2,1) multigrid method with explicit approximate inverse smoothing for $\delta l=1$, time step $\Delta t=0.01$ and space mesh size $h=1/64$ at $t=1$, using the Crank - Nicolson scheme.

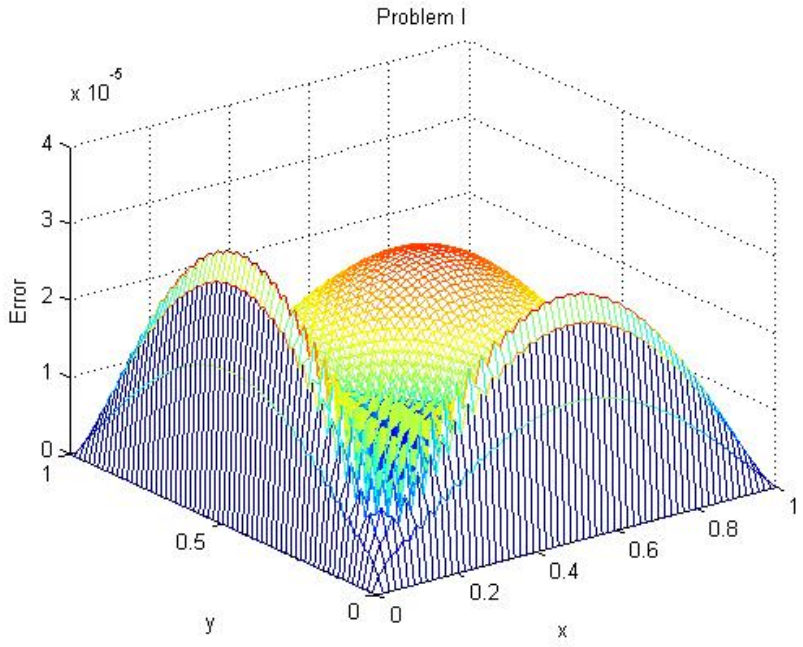

Figure 7: The absolute error between the exact and computed solution for problem I with time step Δt=0.01 and space mesh size h=1/64 at t=1, with the Crank – Nicolson scheme.

4.2 Model Problem II

The second problem to be solved is a parabolic 2D convection–diffusion P.D.E., i.e.

$$\frac{\partial u}{\partial t} - \varepsilon \Delta u + a\frac{\partial u}{\partial x} = Ae^{1-t}\sin(\ell \pi y)\left(\left(1-\varepsilon \ell^2 \pi^2\right)x^2 + \left(\varepsilon \ell^2 \pi^2 - 2a - 1\right)x + \left(a + 2\varepsilon\right)\right), \quad (x,y,t) \in \Omega \times [0 \le t \le T] \tag{24}$$

$$u(t,x,y) = 0, \quad (x,y,t) \in \partial\Omega \times [0 \le t \le T] \tag{24.a}$$

with initial condition

$$u(0,x,y) = Ae\sin(\ell \pi y)\left(\left(1-\varepsilon \ell^2 \pi^2\right)x^2 + \left(\varepsilon \ell^2 \pi^2 - 2a - 1\right)x + \left(a + 2\varepsilon\right)\right), \quad (x,y) \in \Omega \tag{24.b}$$

where Δ is the Laplace operator, Ω is the unit square and $\partial\Omega$ denotes the boundary of Ω. The parameters ε, a, A, ℓ are chosen arbitrarily to ε=0.1, a=2.5, A=1, ℓ=3. The first derivative across the x direction is discretized with the backward stable scheme. In Table 3, the number of inner and outer iterations for the backward differences as well as the Crank-Nicolson scheme in conjunction with multigrid V(2,1) method, with explicit approximate inverse smoothing, for various values of the "retention" parameter δl and time step Δt with space mesh size h=1/64 is presented. In Table 4,

the number of inner and outer iterations for the backward differences and Crank-Nicolson scheme in conjunction with MGV(2,1,1) Bi-CGSTAB method, with explicit approximate inverse smoothing, for various values of the "retention" parameter δl and the time step Δt with space mesh size h=1/64 is presented.

In Figure 8, the computed solution at t=1 with multigrid V(2,1) method with explicit approximate inverse smoothing, time step Δt=0.01, space mesh size h=1/64 and "retention" parameter δl=1, with the backward differences scheme, is presented. In Figure 9, the exact solution for time step Δt=0.01, space mesh size h=1/64 at t=1 is presented. In Figure 10, the absolute error of the backward differences scheme is presented.

Method	**δl** / **Δt**	**1**	**2**	**2m**	**4m**
		Total number of inner (outer) iterations.			
Implicit Scheme	0.0100	2301(424)	1985(424)	1053(424)	1017(424)
	0.0050	2571(704)	2193(708)	1414(707)	1414(707)
	0.0010	3628(1922)	3529(1920)	1920(1920)	1920(1920)
	0.0005	2451(2451)	2451(2451)	2451(2451)	2451(2451)
Crank - Nicolson Scheme	0.0100	1630(424)	1235(424)	812(424)	792(424)
	0.0050	2022(707)	1569(707)	1041(707)	993(707)
	0.0010	1919(1919)	1919(1919)	1919(1919)	1919(1919)
	0.0005	2450(2450)	2450(2450)	2450(2450)	2450(2450)

Table 3: Convergence behavior of the V(2,1) multigrid method with explicit approximate inverse smoothing for various values of the "retention" parameter δl and time step Δt with space mesh size h=1/64.

Method	**δl** / **Δt**	**1**	**2**	**2m**	**4m**
		Total number of inner (outer) iterations.			
Implicit Scheme	0.0100	1272(424)	1272(424)	848(424)	848(424)
	0.0050	1414(707)	1414(707)	707(707)	707(707)
	0.0010	1920(1920)	1920(1920)	1920(1920)	1920(1920)
	0.0005	2451(2451)	2451(2451)	2451(2451)	2451(2451)
Crank - Nicolson Scheme	0.0100	848(424)	848(424)	424(424)	424(424)
	0.0050	1414(707)	1414(707)	707(707)	707(707)
	0.0010	1919(1919)	1919(1919)	1919(1919)	1919(1919)
	0.0005	2450(2450)	2450(2450)	2450(2450)	2450(2450)

Table 4: Convergence behavior of the MGV(2,1,1) Bi-CGSTAB method with explicit approximate inverse smoothing for various values of the "retention" parameter δl and time step Δt with space mesh size h=1/64.

In Figure 11, the computed solution at t=1 with multigrid V(2,1) method with explicit approximate inverse smoothing, time step Δt=0.01, space mesh size h=1/64 and "retention" parameter δl=1, with the Crank - Nicolson scheme, is presented. In Figure 12, the absolute error of the Crank - Nicolson scheme is presented.

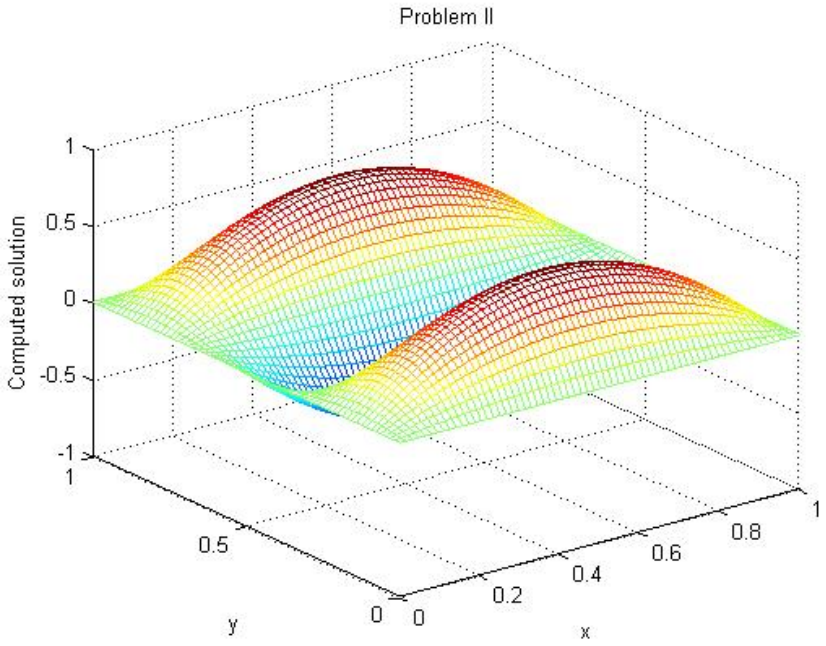

Figure 8: The computed solution for problem II obtained with V(2,1) multigrid method with explicit approximate inverse smoothing for δl=1, time step Δt=0.01 and space mesh size h=1/64 at t=1, using the backward differences scheme.

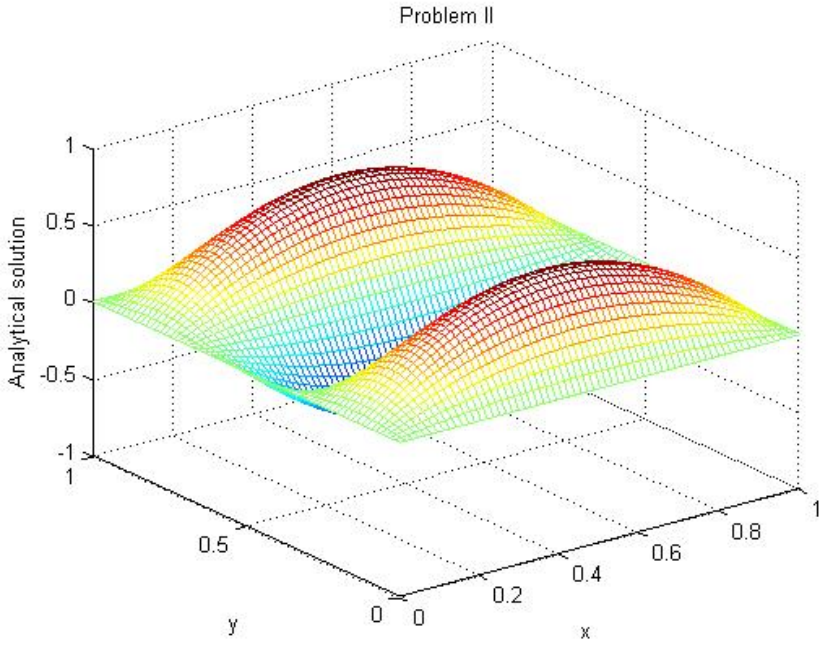

Figure 9: The exact solution for problem II with time step Δt=0.01 and space mesh size h=1/64 at t=1.

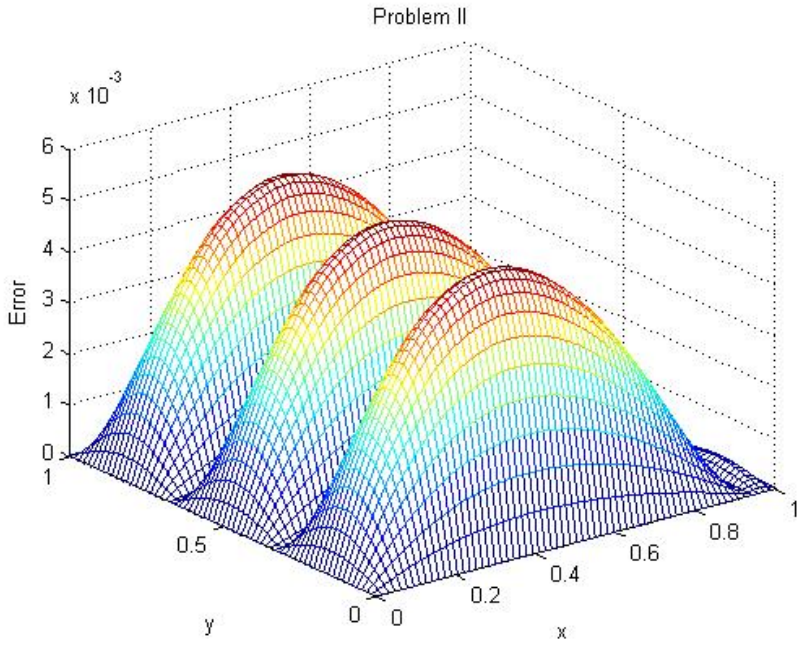

Figure 10: The absolute error between the exact and computed solution for problem II with time step $\Delta t=0.01$ and space mesh size $h=1/64$ at $t=1$, with the backward differences scheme.

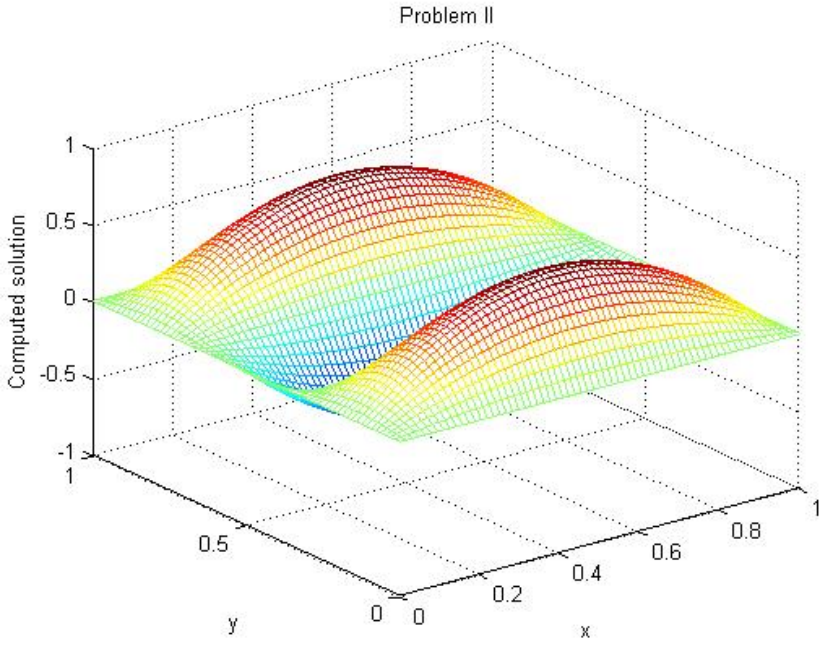

Figure 11: The computed solution for problem II obtained with V(2,1) multigrid method with explicit approximate inverse smoothing for $\delta l=1$, time step $\Delta t=0.01$ and space mesh size $h=1/64$ at $t=1$, using the Crank - Nicolson scheme.

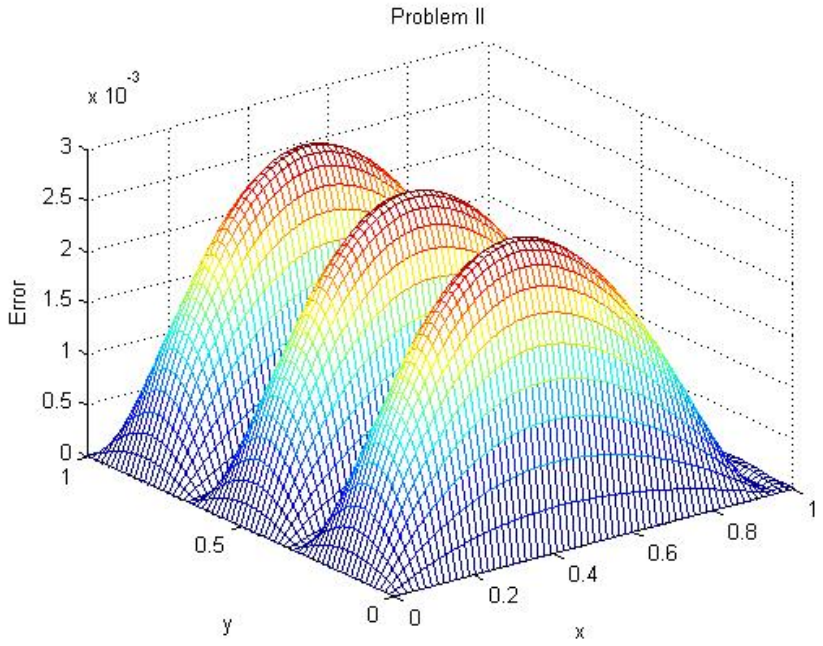

Figure 12: The absolute error between the exact and computed solution for problem II with time step $\Delta t=0.01$ and space mesh size $h=1/64$ at $t=1$, with the Crank – Nicolson scheme.

5 Conclusion

The proposed multigrid schemes as well as the multigrid preconditioned Conjugate Gradient type method have been proven efficient especially for large values of the "retention" parameter δl. Moreover, the explicit approximate inverse can be efficiently computed in parallel [6, 11, 13], and the proposed explicit approximate inverse smoothing schemes are easy to parallelize, since they require matrix–vector multiplications and vector addition computations in contrast to other methods that require complex ordering and coloring schemes. Finally, we state that the new hybrid schemes based on the multigrid approach can be efficiently used for solving non–linear initial or boundary value problems.

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