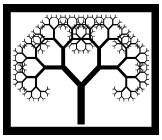
Paper 123



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Numerical Analysis of Density Changing Flows using the Semi-Lagrange Galerkin Method

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Abstract

The purpose of this paper is an analysis of adiabatic flows using the semi-Lagragian Galerkin method. In the study presented in this paper, conservations of mass and momentum of the adiabatic flows are employed as the governing equations. The Birch-Murnaghan equation is assumed for the equation of state. The semi-Lagrange method is used, in which the governing equations are divided into the advection and non-advection calculations. The advection calculation is transformed using the characteristic method. In both advection and non-advection calculations, the Hermite interpolation function which is complete third order approximation for a triangular element is used for both the velocity and density.

Keywords: finite element method, density changing, conservation of mass, conservation of momentum, semi-Lagragian Galerkin method, Hermite interpolation function, characteristic method.

1 Introduction

The adiabatic flows mean compressible flows assuming adiabatic state. In actual natural phenomena, almost flows are compressible flows. Considering the fluid flow around a body, the density changing phenomenon is important. The heat is changed only inside the computational domain, even if the adiabatic state is assumed.

In the governing equation of the flow problems, the advection term and the diffusion term are included. In case of either term is superior, the characteristic of flows are different. Depending on the characteristic of flows, the suitable appropriate technique is required. If the advection term is superior, the computation has an inclination to be unstable. For preventing this problem, the characteristic method is used in this study. The terms of temporal differentiation and advection are shown in the form of mate-

rial differentiation and transformed by the characteristic method. In addition, in the semi-Lagrange method, the advection calculation is forwarded by the non-advection calculation. After having calculated the advection term by the semi-Lagrange method, the non-advection term is calculated by the implicit method. This technique is called the semi-Lagragian Galerkin method. In the advection and non-advection calculation, the Hermite interpolation function is used for velocity and density in this study. The Hermite interpolation function is composed of 10 degrees of freedom of which function values at the three nodes, values of the first derivative, and a function value in the center of gravity. Therefore, the Hermite interpolation function is the complete third order triangular element. The characteristics of this study has three points. Firstly, the characteristic method is used. The advection term need not be discretized by the characteristic method. Secondly, the Hermite interpolation function is used. The Hermite interpolation function which has ten degrees of freedom can be applied to the third order approximation. Finally, the semi-Lagragian Galerkin method is used.

As an example of numerical analysis in this study, a cavity flow is carried out for showing that the semi-Lagragian Galerkin method is effective. In addition, as an example of numerical analysis, the adiabatic flows are analyzed using semi-Lagragian Galerkin method in a circular computational area and a circular computational area which has a body at the center. At the time, the uniform velocity is given in the direction of tangent to the circle. The exact velocity in the X-direction and Y-direction on the boundary is calculated by the B-spline.

2 Governing equations

As the governing equation, conservation of mass for density is expressed as:

$$\frac{\partial \rho}{\partial t} + u_i \rho_{,i} + \rho u_{i,i} = 0, \tag{1}$$

Conservation of momentum is expressed as:

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j u_{i,j} \right) - \tau_{ij,j} = 0, \tag{2}$$

$$\tau_{ij} = -p\delta_{ij} + \lambda \delta_{ij} u_{k,k} + \mu (u_{i,j} + u_{j,i}), \tag{3}$$

In this study, a fluid is assumed as liquid. Therefore, the Birch-Murnaghan equation of state is applied to the governing equation.

$$p = \frac{3}{2}\kappa_0 \left[\left(\frac{\rho}{\rho_0} \right)^{\frac{7}{3}} - \left(\frac{\rho}{\rho_0} \right)^{\frac{5}{3}} \right] \tag{4}$$

The equation (4) is differentiated as follows.

$$\begin{aligned} p_{,i} &= \frac{2}{3}\kappa_0 \left[\left(\frac{7}{3}\right) \left(\frac{1}{\rho_0}\right)^{\frac{7}{3}} \rho^{\frac{4}{3}} - \left(\frac{5}{3}\right) \left(\frac{1}{\rho_0}\right)^{\frac{5}{3}} \rho^{\frac{2}{3}} \right] \rho_{,i} \\ &\approx \frac{2}{3}\kappa_0 \left[\left(\frac{7}{3}\right) \left(\frac{1}{\rho_0}\right)^{\frac{7}{3}} \bar{\rho}^{\frac{4}{3}} - \left(\frac{5}{3}\right) \left(\frac{1}{\rho_0}\right)^{\frac{5}{3}} \bar{\rho}^{\frac{2}{3}} \right] \rho_{,i} \\ &= \bar{\kappa}\rho_{,i}, \end{aligned}$$

$$\bar{\rho} = \frac{1}{2}\left(3\rho^n - \rho^{n-1}\right),$$

In this study, $\Delta\rho$ is variation of density. Therefore, ρ is expressed as:

$$\rho = \rho_0 + \Delta\rho,$$

The equation (1) is expressed as:

$$\frac{\partial \Delta\rho}{\partial t} + u_i \Delta\rho_{,i} + (\rho_0 + \Delta\rho)\, u_{i,i} = 0, \tag{5}$$

The equation (2) is expressed as:

$$(\rho_0 + \Delta\rho)\left(\frac{\partial u_i}{\partial t} + u_j u_{i,j}\right) - \tau_{ij,j} = 0, \tag{6}$$

where u_i, ρ, μ, ρ_0 and p_0 are velocity, density, viscosity coefficient, reference density, and reference pressure respectively. Kronecker delta is denoted by δ_{ij}. Coefficient of bulk viscosity is λ, and is expressed as:

$$\lambda = -\frac{2}{3}\mu,$$

3 Numerical study

3.1 Characteristic method

The position of the virtual fluid particle which was at position x at time t is $X_i(x_i,t;\tau)$ at time τ. The trajectory of particle is shown by the following ordinary differential equations.

$$\frac{dX_i}{d\tau} = u_i\left(X_i\left(x_i,t;\tau\right),\tau\right), \quad X_i\left(x_i,t;\tau\right) = x_i \tag{7}$$

The time derivative and advection terms are expressed by the Lagragian derivative as:

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j u_{i,j} = \frac{d}{d\tau} u_i\left(X_i\left(x_i,t;\tau\right),\tau\right)|_{\tau=t} \tag{8}$$

By the characteristic method, the time derivative and advection terms are as follows in the conservation of mass:

$$\frac{D\Delta\rho}{Dt} = \frac{\partial\Delta\rho}{\partial t} + u_i\Delta\rho_{,i} \equiv \Delta\dot{\rho} \tag{9}$$

Similarly, they are as in the conservation of momentum:

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j u_{i,j} \equiv \dot{u}_i \tag{10}$$

The dot expresses as the time differentiation.

3.2 Semi-Lagrange Galerkin method

The semi-Lagragian Galerkin method divides the governing equations into the advection and non-advection calculations. The advection calculation can be forwarded. The values at the upstream point by advection calculation are assumed to be $\Delta\tilde{\rho}$ and $\tilde{u_i}$ in the governing equation.

$$\Delta\rho^{n+1} = \Delta\tilde{\rho} + \frac{1}{2}\Delta t\Delta\dot{\rho}^{n+1} \tag{11}$$

$$u_i^{n+1} = \tilde{u_i} + \frac{1}{2}\Delta t\dot{u}_i^{n+1} \tag{12}$$

Therefore, the equation (5) is transformed into equation (13).

$$\frac{2}{\Delta t}\Delta\rho^{n+1} + (\rho_0 + \Delta\rho)\, u_{i,i}^{n+1} - \frac{2}{\Delta t}\Delta\tilde{\rho} = 0 \tag{13}$$

Similarly, the equation (6) is transformed into equation (14).

$$\frac{2\,(\rho_0 + \Delta\rho)}{\Delta t}u_i^{n+1} + \frac{p_0\gamma}{\rho_0}\rho_{v,j}^{n+1}\delta_{ij} - \mu u_{i,jj}^{n+1} - \mu u_{j,ij}^{n+1} + \frac{2}{3}\mu u_{k,kj}^{n+1}\delta_{ij} - \frac{2\,(\rho_0 + \Delta\rho)}{\Delta t}\tilde{u}_i = 0 \tag{14}$$

The calculation of upstream point is used non-linear iterative calculation method in order to give higher precision. The calculation of the upstream point is expressed as:

$$\mathcal{X}_i^n = x - u_i^n\Delta t \tag{15}$$

$$X_{i(l)}^n = x - \frac{1}{2}\{u_i^n\,(\mathcal{X}_i^n) + u_{i(l-1)}^{n+1}\}\Delta t \tag{16}$$

$$u_{i(0)}^{n+1} = u_i^n \;\;,\;\; \dot{u}_{i(0)}^{n+1} = \dot{u}_i^n \tag{17}$$

Iterative calculation number is denoted by $l(1M_{max})$. The figure 1 is shown outline of the advection velocity. The positions of upstream points of node and center of gravity are X_l^n and X_e^n. The elements which the upstream point belongs are expressed by

$K(X_l^n)$ and $K(X_e^n)$. The function values are updated by the advection calculation at node and at center of gravity. The element that upstream point belongs is interpolated by using the Hermite function.

$$\tilde{u}_{i(l)} = u_i^n(X_{i(l)}^n) + \left(1 - \frac{1}{2}\right) \Delta t \, \dot{u}_i^n(X_{i(l)}^n) \tag{18}$$

$$\frac{\partial \tilde{u}_{i(l)}}{\partial x_j} = \left(\delta_{jk} - \Delta t \frac{\partial u_{k(l-1)}^{n+1}}{\partial x_j}\right) \left\{ \frac{\partial u_i^n(X_{i(l)}^n)}{\partial X_k} + \left(1 - \frac{1}{2}\right) \Delta t \frac{\partial \dot{u}_i^n(X_{i(l)}^n)}{\partial X_k} \right\} \tag{19}$$

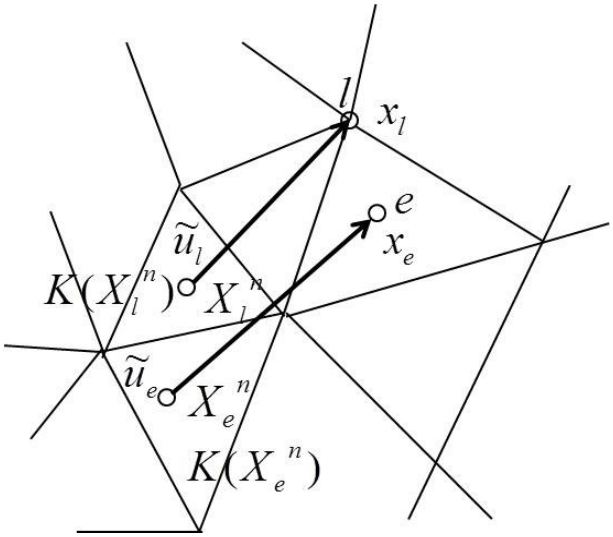

Figure 1: outline of advection velocity

4 Finite element interpolation

As for the interpolation, the Hermite interpolation function is applied. Figure 2 shows the Hermite interpolation function. The function value at the nodes of element, value of the first derivative, and function value at center of gravity are assumed to be degree of freedom. The interpolation is the complete third order element. The Hermite interpolation function is shown as follows:

$$\begin{aligned}
H_{0i} &= L_i^2(3 - 2L_i) - 7L_1L_2L_3, \\
H_{xi} &= L_i^2(x_{ji}L_j - x_{ik}L_k) - (x_{ji} - x_{ik})L_1L_2L_3, \\
H_{yi} &= L_i^2(y_{ji}L_j - y_{ik}L_k) - (y_{ji} - y_{ik})L_1L_2L_3, \\
H_{0e} &= 27L_1L_2L_3,
\end{aligned}$$

$$x_{ij} = x_i - x_j,\ y_{ij} = y_i - y_j$$

where (x_i, y_i) means the nodal coordinate and H_{0i}, H_{xi}, H_{yi}, H_{0e} are interpolation functions, in which L_1, L_2, L_3, are area coordinates. The permutation is used for (i, j, k).

The finite element approximation in Ω_e of each element is expressed as follows:

$$u_i = H_\alpha U_{\alpha i} \tag{20}$$

$$H_\alpha = \begin{bmatrix} H_{0i} & H_{xi} & H_{yi} & H_{0e} \end{bmatrix}^T$$

$$U_{\alpha i} = \begin{bmatrix} u_i & \left.\frac{\partial u}{\partial x}\right|_i & \left.\frac{\partial u}{\partial y}\right|_i & u_e \end{bmatrix}^T$$

$$u_{\alpha i} = \sum_{i=1}^{3} \left(H_{0i} u_i + H_{xi} \left.\frac{\partial u}{\partial x}\right|_i + H_{yi} \left.\frac{\partial u}{\partial y}\right|_i \right) + H_{0e} u_e \tag{21}$$

The values of the first derivative are:

$$\frac{\partial u}{\partial x} = \frac{\partial H_e}{\partial x} U_e,\ \frac{\partial u}{\partial y} = \frac{\partial H_e}{\partial y} U_e \tag{22}$$

$$\frac{\partial H_e}{\partial x} = \begin{bmatrix} \frac{\partial H_{0i}}{\partial x} & \frac{\partial H_{xi}}{\partial x} & \frac{\partial H_{yi}}{\partial x} & \frac{\partial H_{0e}}{\partial x} \end{bmatrix}$$

$$\frac{\partial H_e}{\partial y} = \begin{bmatrix} \frac{\partial H_{0i}}{\partial y} & \frac{\partial H_{xi}}{\partial y} & \frac{\partial H_{yi}}{\partial y} & \frac{\partial H_{0e}}{\partial y} \end{bmatrix}$$

Each component of the first derivative can be obtained in the following form.

$$x \begin{cases} \frac{\partial H_{0i}}{\partial x} = 2b_i L_i(3 - 2L_i) - 7(b_1 L_2 L_3 + b_2 L_1 L_3 + b_3 L_1 L_2) - 2b_i L_i^2 \\ \frac{\partial H_{xi}}{\partial x} = 2b_i L_i(c_k L_j - c_j L_k) - b_1(c_k - c_j) L_2 L_3 - b_2(c_k - c_j) L_1 L_3 - b_3(c_k - c_j) L_1 L_2 + (b_j c_k - b_k c_j) L_i^2 \\ \frac{\partial H_{yi}}{\partial x} = 2b_i L_i(b_j L_k - b_k L_j) - b_1(b_j - b_k) L_2 L_3 - b_2(b_j - b_k) L_1 L_3 - b_3(b_j - b_k) L_1 L_2 \\ \frac{\partial H_{0e}}{\partial x} = 27(b_1 L_2 L_3 + b_2 L_1 L_3 + b_3 L_1 L_2) \end{cases}$$

$$\begin{cases} \frac{\partial H_{0i}}{\partial y} = 2c_i L_i(3 - 2L_i) - 7(c_1 L_2 L_3 + c_2 L_1 L_3 + c_3 L_1 L_2) - 2c_i L_i^2 \\ \frac{\partial H_{xi}}{\partial y} = 2c_i L_i(c_k L_j - c_j L_k) - c_1(c_k - c_j) L_2 L_3 - c_2(c_k - c_j) L_1 L_3 - c_3(c_k - c_j) L_1 L_2 \\ \frac{\partial H_{yi}}{\partial y} = 2c_i L_i(b_j L_k - b_k L_j) - c_1(b_j - b_k) L_2 L_3 - c_2(b_j - b_k) L_1 L_3 - c_3(b_j - b_k) L_1 L_2 + (b_j c_k - b_k c_j) L_i^2 \\ \frac{\partial H_{0e}}{\partial y} = 27(c_1 L_2 L_3 + c_2 L_1 L_3 + c_3 L_1 L_2) \end{cases}$$

Similarly, the Hermite interpolation function is used for density.

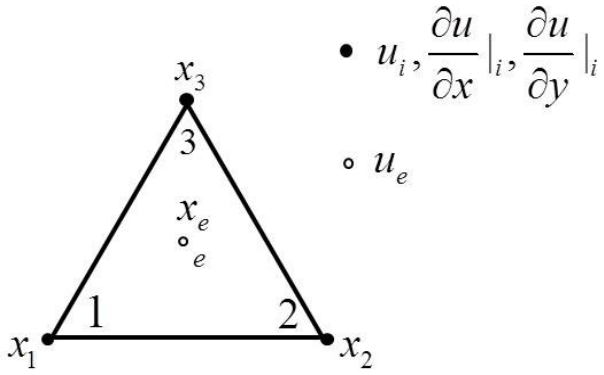

Figure 2: Hermite interpolation function

5 Finite element equation

The Galerkin method is applied to the conservation of mass, equation (13). The finite element equation of the conservation of mass is expressed as:

$$M_{\alpha\beta}\Delta\rho_\beta^{n+1} + (\rho_0 + \Delta\rho)\, G'_{\alpha\beta i} U_{\beta i}^{n+1} - M_{\alpha\beta}\tilde{\rho}_\beta = 0 \tag{23}$$

Similarly, the Galerkin method is applied to the conservation of momentum, equation (14). The finite element equation of the conservation of momentum is expressed as:

$$(\rho_0 + \Delta\rho)\, M_{\alpha\beta} U_{\beta i}^{n+1} - \frac{1}{2}\bar{\kappa}\Delta t G_{\alpha i\beta}\Delta\rho_\beta^{n+1}\delta_{ij} + \frac{1}{2}\mu\Delta t D_{\alpha j\beta j} U_{\beta i}^{n+1} + \frac{1}{2}\mu\Delta t D_{\alpha j\beta i} U_{\beta j}^{n+1} - \frac{1}{3}\mu\Delta t D_{\alpha j\beta k} U_{\beta k}^{n+1}\delta_{ij} - (\rho_0 + \Delta\rho)\, M_{\alpha\beta}\tilde{U}_{\beta i} = 0 \tag{24}$$

Each matrix is expressed as:

$$M_{\alpha\beta} = \int_{\Omega_e} H_\alpha H_\beta d\Omega, \;\; G_{\alpha i\beta} = \int_{\Omega_e} H_{\alpha,i} H_\beta d\Omega, \;\; D_{\alpha j\beta j} = \int_{\Omega_e} H_{\alpha,j} H_{\beta,j} d\Omega$$

$$D_{\alpha j\beta i} = \int_{\Omega_e} H_{\alpha,j} H_{\beta,i} d\Omega, \;\; D_{\alpha j\beta k} = \int_{\Omega_e} H_{\alpha,j} H_{\beta,k} d\Omega, \;\; G'_{\alpha\beta i} = \int_{\Omega_e} H_\alpha H_{\beta,i} d\Omega$$

6 Numerical examples

6.1 Case 1 : Analysis of a cavity flow

A numerical study is an analysis of a cavity flow of adiabatic flows using the semi-Lagragian Galerkin method. Numerical parameters are $\Delta t = 0.001$ and $\gamma = 1.4$.

Reynolds number is 1000 and Mach number is 0.2. A computational domain and boundary conditions are expressed in Figure 3. The velocity is zero at the corners of top of the computational domain. The finite element mesh is represented in Figure 4. Total numbers of nodes and elements are 14,641 and 28,800, respectively.

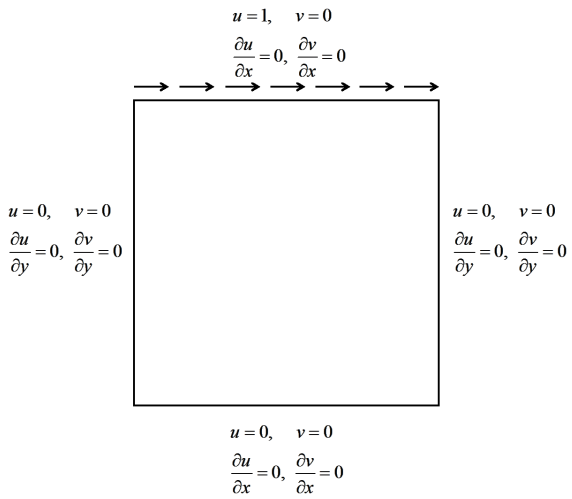

Figure 3: Computational domain

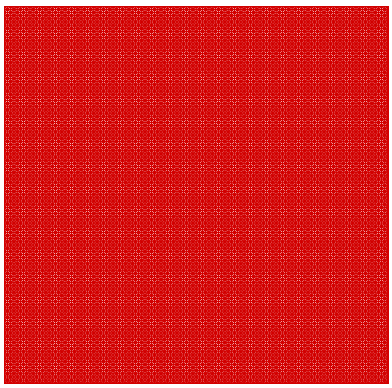

Figure 4: Finite element mesh

As shown in Figure 5, vectors swirl towards the center. The density distribution is expressed in Figure 6. In addition, the velocity profile is expressed in Figure 7. The red line is the result of this study. As shown in Figure 7, the values of this study are almost equal to Ghia's values and Erturk's values. Therefore, the semi-Lagragian Galerkin method is shown to be effective technique.

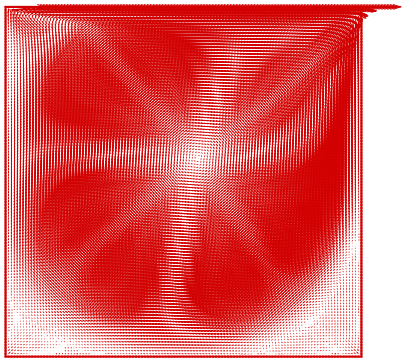

Figure 5: Velocity distribution

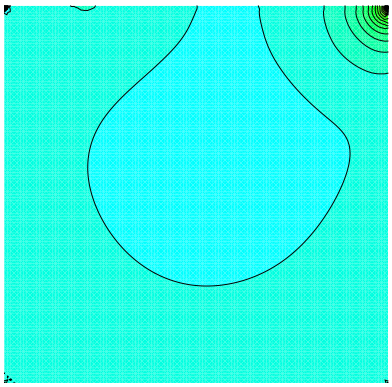

Figure 6: Density distribution

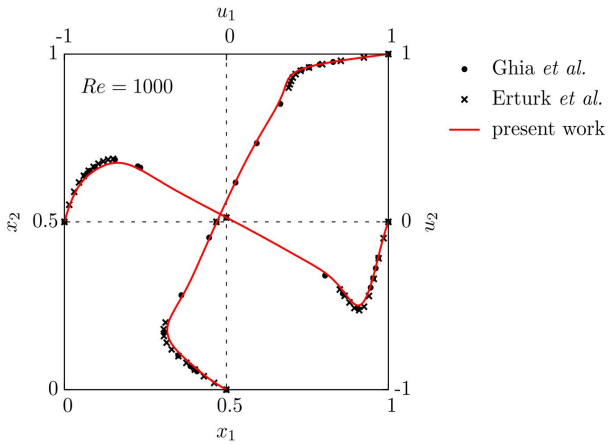

Figure 7: Velocity profile

6.2 Case 2 : Analysis with a circular mesh

A numerical study is an analysis of adiabatic flows with a circular mesh using the semi-Lagragian Galerkin method. Reynolds number is 250 and Mach number is 0.2. A computational domain and boundary conditions are expressed in Figure 8. The uniform velocity is given in the direction of the tangent to the circular computational domain. The velocity is zero in the normal direction of the tangent to the circular computational domain. The finite element mesh is represented in Figure 9. Total numbers of nodes and elements are 9,721 and 19,120, respectively.

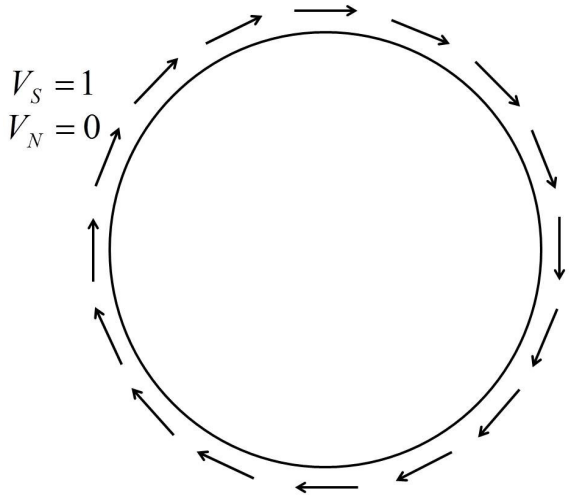

Figure 8: Computational domain

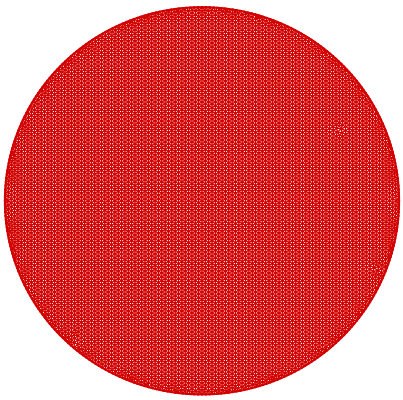

Figure 9: Finite element mesh

The velocity distribution in the quarter of whole domain is expressed in Figure 10. As shown in Figure 10, the vectors swirl. The density distribution is expressed in Figure 11. As shown in Figure 11, the density at the center is lower than at the wall.

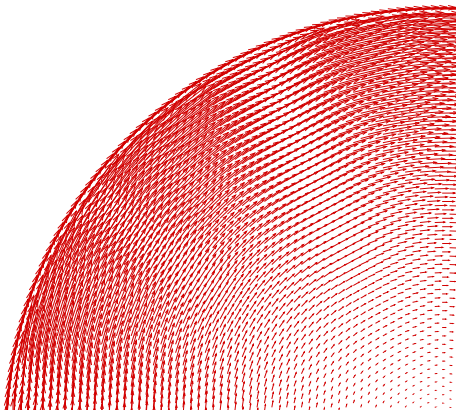

Figure 10: Velocity distribution in the quarter of whole

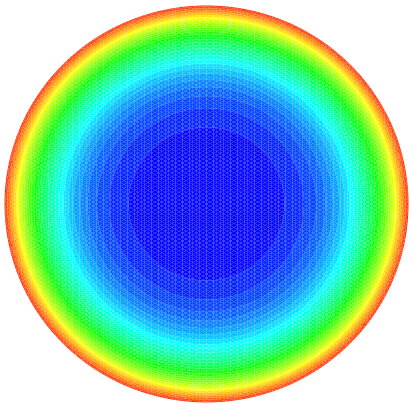

Figure 11: Density distribution

6.3 Case 3 : Analysis with a circular mesh which has a body

A numerical study is an analysis of adiabatic flows with a circular mesh which has a body using the semi-Lagragian Galerkin method. Reynolds number is 250 and Mach number is 0.2. A computational domain and boundary conditions are expressed in Figure 12. In the same way in the case 2, The uniform velocity is given in the direction

of the tangent to the circular computational domain. The velocity is zero in the normal direction of the tangent to the circular computational domain. Moreover, because the body is set at the center, the velocity on the body is zero. The finite element mesh is represented in Figure 13. Total numbers of nodes and elements are 9,223 and 18,046, respectively.

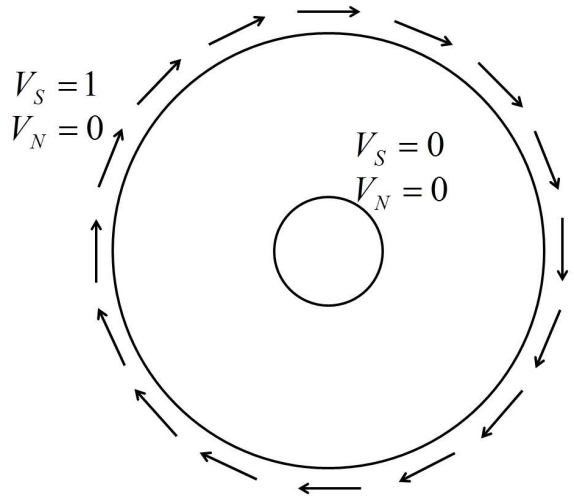

Figure 12: Computational domain

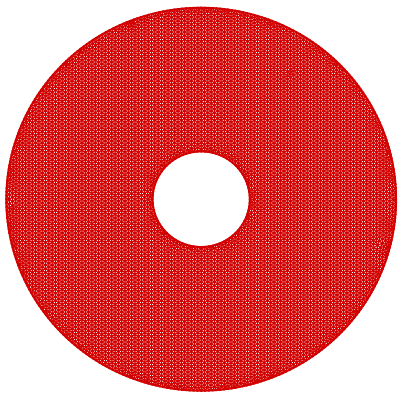

Figure 13: Finite element mesh

The velocity distribution in the quarter of whole domain is expressed in Figure 14. In the same way as the case 2, as shown in Figure 14, the vectors swirl. The density distribution is expressed in Figure 15. As shown in Figure 15, the density near the body is lower than those close to the wall.

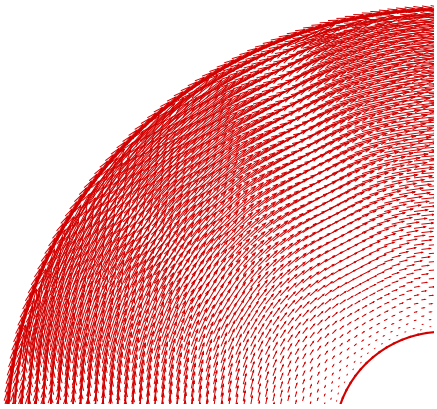

Figure 14: Velocity distribution in the quarter of whole

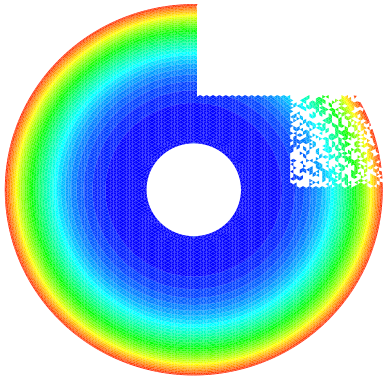

Figure 15: Density distribution

7 Conclusion

In this paper, a numerical study of the compressible viscous fluid flow assuming an adiabatic state using the semi-Lagragian Galerkin method is performed. The Hermite function is used for the interpolation. In the state equations, the advection term is transformed by the characteristic method. The solution is updated by using the value of the upstream point. The flow problem is analyzed using the semi-Lagragian Galerkin method with the Hermite interpolation function. As future work, we will analyse compressible flows with two different densities using the semi-Lagragian Galerkin method.

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