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# Comparison of the Continuity Vorticity Pressure, Auxiliary Potential and Implicit Potential Methodologies for Incompressible Flow in Straight Ducts

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## Abstract

The scope of this paper is to compare the CVP, the auxiliary potential and IPOT methodologies on the solution of laminar incompressible developing flow with buoyancy forces in a straight square duct. The comparison concerns the convergence properties, the robustness, the accuracy and the functionality of each method and highlights the relative advantages and disadvantages of each approach. The results show that the CVP method is the most demanding in terms of the programming effort, but it is also characterized by increased robustness and computational efficiency; the auxiliary potential method has great functionality but it can be slower than the CVP; the IPOT method is the simplest, fastest and most flexible of the three but its lacks robustness and requires the use of a relaxation parameter.

**Keywords:** finite volumes, incompressible flow, internal flow.

# 1 Introduction

The need to create accurate and functional methods for flow simulation has led to the development of various methods and computational schemes. The main difficulty that must be overcome in incompressible flow simulation in particular, is the creation of a robust way to achieve the velocity-pressure coupling. The utilization of mathematical and numerical techniques along with physical approximations has provided a variety of different approaches concerning the solution of incompressible flow problems. Some of the most recent developments in the this direction are the CVP, the Auxiliary Potential and IPOT methods.

The CVP (Continuity Vorticity Pressure) variational equations method was pioneered by Hatzikonstantinou et al. [1] and it has been applied to several internal flow problems, such as flow in curved and helical ducts of various cross-sectional geometries[2],[3]. Its formulation is based on a well known vector identity which facilitates the computation of the velocity corrections based on the continuity residual. The velocity corrections are then used for the determination of the pressure correction and this procedure essentially achieves the velocity-pressure coupling.

The Auxiliary Potential method [4] bares many similarities to the SMAC (Simplified Marker and Cell) scheme, as it involves a potential velocity through which the velocity and pressure corrections are determined. However, it is not based on a time-step technique as SMAC and it can be easily applied to steady state problems. The procedure of the Auxiliary Potential method is based on a potential velocity correction that accounts for the mass residual and it is obtained from the solution of a Poisson equation. The method has been applied up to now only to benchmark flow problems and it has been found to be very robust and accurate.

The IPOT (Implicit Potential) method is a recently created method which is characterized by great simplicity. Its main property is that it does not involve the solution of any partial differential equations apart from the ones for the momentum. The pressure correction that is required in its formulation is directly linked to the continuity residual. Due to the fact that the method is very recent, it has only been applied to benchmark internal and external flows and the results have designated that, due to its simplicity, it is a very promising approach for incompressible flow problems.

The scope of the paper is to compare the CVP, the Auxiliary Potential and IPOT methodologies on the solution of laminar incompressible developing flow with buoyancy forces in a straight square duct. The comparison concerns the convergence properties, the robustness, the accuracy and the functionality of each methodology and highlights the advantages and disadvantages of each approach. The results show that although all three methods can predict accurately the development of the flow for low Reynolds numbers, they have their unique properties: the CVP method is the most demanding in terms of the equations that must be solved, but it is also characterized by increased robustness and computational efficiency; the Auxiliary Potential method has great functionality but it can be slower than CVP; the IPOT method is the simplest, fastest and most flexible of the three but its lacks robustness and requires the use of a relaxation parameter. The results of the comparison lead to useful conclusions concerning the characteristics of the three methods and they are crucial to the determination of the most suitable one for the various technological problems.

## 2 Analysis

In order to outline the steps of the three computational procedures we consider the general case of three-dimensional incompressible flow. For simplicity, we express the governing equations in terms of the non-dimensional variables

$$\mathbf{v} = \frac{\mathbf{v}'}{\mathrm{U}}, \mathrm{p} = \frac{\mathrm{p}'}{\rho \mathrm{U}^2}, \nabla = \mathrm{D_h}\nabla', \mathrm{Re} = \frac{\mathrm{UD_h}}{\nu} \tag{1}$$

where $\mathbf{v}', \mathrm{p}', \rho, \nu, \mathrm{U}, \mathrm{D_h}$ are the dimensional velocity, pressure, density, kinematic viscosity, characteristic velocity and hydraulic diameter respectively. The governing equations are the continuity and momentum equations. We shall describe the three schemes for the steady state case, omitting the time derivative. Using the lagging of the coefficient technique to linearize the momentum equation, the governing equations at the n+1 iteration become

$$\nabla \cdot \mathbf{v}^{n+1} = 0 \tag{2}$$

$$\left(\mathbf{v}^{n} \cdot \nabla\right)\mathbf{v}^{n+1} = -\nabla p^{n+1} + \frac{1}{Re}\nabla^{2}\mathbf{v}^{n+1} \tag{3}$$

## 2.1 CVP method

The implementation of the CVP method begins with the distribution of the pressure and the velocity field at the $n^{th}$ iteration, which we denote by $p^n$ and $\mathbf{v}^n$ respectively. The momentum equation can therefore be solved to give an estimation of the velocity field which is denoted by $\mathbf{v}^*$.

$$\left(\mathbf{v}^{n} \cdot \nabla\right)\mathbf{v}^{*} = -\nabla p^{n} + \frac{1}{Re}\nabla^{2}\mathbf{v}^{*} \tag{4}$$

The estimated velocity field $\mathbf{v}^*$ does not in general satisfy the continuity equation. Hence, we introduce the velocity correction $\boldsymbol{\delta}\mathbf{v}$ and the pressure correction $\delta p$ which are defined by the relations

$$\mathbf{v}^{n+1} = \mathbf{v}^{*} + \boldsymbol{\delta}\mathbf{v} \;,\;\; p^{n+1} = p^{n} + \delta p \tag{5}$$

In the context of the CVP method, the next step is to compute the velocity correction field from the general velocity correction equation

$$\nabla^{2}\boldsymbol{\delta}\mathbf{v} = -\nabla\left(\nabla \cdot \mathbf{v}^{*}\right) \tag{6}$$

After the determination of the velocity correction field, the pressure correction needs to be computed. According to the CVP methodology, in order to construct the pressure variational equation we must first express the momentum equation in terms of the velocity correction by subtracting Eq.(4) from Eq.(3) and using the relations of Eq.(5). Subsequently we solve for the pressure gradient correction which is denoted by $\mathbf{f}$. This yields the following relation

$$\mathbf{f} \equiv \nabla \delta p = -\left(\mathbf{v}^{n} \cdot \nabla\right)\boldsymbol{\delta}\mathbf{v} + \frac{1}{Re}\nabla^{2}\boldsymbol{\delta}\mathbf{v} \tag{7}$$

The value of the term $\mathbf{f}$ may be easily computed from the known values of the correction velocity $\boldsymbol{\delta}\mathbf{v}$. The pressure is then determined from the pressure variational equation

$$\nabla^2 \delta p = \nabla \cdot \mathbf{f} + \nabla \cdot \mathbf{v}^* + \nabla \cdot \boldsymbol{\delta v} \tag{8}$$

The solution of the pressure variational equation provides the pressure correction field. The velocity and pressure corrections are then substituted in the relations of Eq.(5) to give the updated velocity and pressure fields and this completes an iteration of the CVP scheme.

## 2.2 Auxiliary Potential method

Similarly to the CVP implementation, the procedure for the Auxiliary Potential method begins with the distribution of the pressure and the velocity field at the $n^{th}$ iteration. Equation (4) is solved to give an estimation $\mathbf{v}^*$ of the of the velocity field and in order to satisfy the continuity restriction, we introduce the velocity correction $\boldsymbol{\delta v}$ and the pressure correction $\delta p$ which are defined by the relations of Eq.(5).

The next step is to introduce the potential correction $\delta\varphi$ which is defined by the relation

$$\nabla \delta\varphi = \boldsymbol{\delta v} \tag{9}$$

It is noted that the velocity correction field is by definition irrotational, since it is introduced to impose the mass conservation condition to the solution and not to correct the velocity in general, as has been shown in [4]. The potential correction $\delta\varphi$ is then found from the solution of the auxiliary potential correction equation

$$\nabla^2 \delta\varphi = -\nabla \cdot \mathbf{v}^* \tag{10}$$

The velocity correction $\boldsymbol{\delta v}$ can then be determined from Eq.(9) and the pressure correction $\delta p$ can be computed from the algebraic relation

$$\delta p = -\frac{1}{2}\mathbf{v}^n \cdot \nabla \delta\varphi + \frac{1}{\mathrm{Re}} \nabla^2 \delta\varphi \tag{11}$$

The velocity and pressure corrections are then substituted in the relations of Eq.(5) to give the updated velocity and pressure fields and this completes an iteration of the Auxiliary Potential scheme.

## 2.3 Implicit Potential method

The IPOT method is the simplest of the three schemes compared in this paper. Its procedure involves the solution of Eq.(3), which gives the velocity field at the n+1 iteration, based on the velocity and pressure fields at the $n^{th}$ iteration, $\mathbf{v}^n$ and $p^n$ respectively. Then based on the new velocity $\mathbf{v}^{n+1}$ we compute the supplementary pressure $p_s$ from the following relation

$$p_s = -\nabla \cdot \mathbf{v}^{n+1} \tag{12}$$

The supplementary pressure acts as an implicit potential that will correct the velocity field through the momentum equation, so it is incorporated in the existing pressure field giving its new updated value

$$p^{n+1} = p^n + p_s \tag{13}$$

Having computed the n+1 values of the velocity and pressure fields, the method can be advanced to the next iteration.

# 3 Problem formulation

The comparison of the three methodologies is conducted on the problem of incompressible, laminar developing flow in a straight duct of square cross-section. The fluid that enters the duct through the inlet has uniform velocity U and temperature $T_{in}$, while the duct's walls have constant temperature $T_w$. With the aid of the dimensionless temperature $\theta$ and the Prandtl and Grashof numbers which are defined as

$$\theta = \frac{T - T_{in}}{T_w - T_{in}}, \quad Gr = \frac{g\beta\left(T_w - T_{in}\right)D_h^{\ 3}}{\nu^2}, \quad Pr = \frac{\nu}{\alpha} \tag{14}$$

where, g is the acceleration of gravity, $\beta$ is the volumetric thermal expansion coefficient and $\alpha$ is the thermal diffusivity, the set of governing equations becomes

$$\nabla \cdot \mathbf{v} = 0 \tag{15}$$

$$\left(\mathbf{v} \cdot \nabla\right)\mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{v} + \frac{Gr}{Re^2}\theta\hat{\mathbf{g}} \tag{16}$$

$$\left(\mathbf{v} \cdot \nabla\right)\theta = \frac{1}{Pr\,Re}\nabla^2\theta \tag{17}$$

The boundary conditions are u=0, v=0, w=U and $\theta = 0$ at the inlet (u, v are the transversal components of the velocity vector, while w is the axial component), $\mathbf{u} = \mathbf{0}, \theta = 1$ at the walls. We solve the parabolic form of the governing equations by neglecting the axial diffusion term, so there is no need for outlet boundary conditions.

# 4 Numerical implementation

The equations involved in the three methods where discretized using second order central differences except from the axial derivatives where we used forward differences. For the solution of the equations, we used the successive line over-relaxation (SLOR) iterative algorithm. The convergence criterion for the momentum equation was

$$\frac{1}{N}\sum_{i=1}^{i_{\max}}\sum_{j=1}^{j_{\max}}\left|\frac{\Phi_{i,j}^{n+1}-\Phi_{i,j}^{n}}{\Phi_{i,j}^{n+1}}\right|\leq 10^{-5} \tag{18}$$

where $\Phi = u, v, w$, the total number of grid points is N and the subscripts i, j represent the nodes of the cross-sectional grid. It must be noted that since we use the parabolic form of the equations with respect to the axial direction, the solution progresses axially through a marching technique. Details about this technique can be found in [3].

In order to assess the dependence of the solutions obtained with the three methods from the grid size and the axial step, we conducted numerical experiments for flow with Re=200, Pr=0.7 and Gr=100. The results obtained with the Auxiliary Potential method are shown in Fig.1 and Fig.2. Concerning the cross-sectional mesh density, the grid sizes that where tested were 20x20, 25x25 and 30x30. Figure 1 depicts the development of the dimensionless axial pressure gradient $\mathrm{Pz} \equiv |dp/dz|$ (z denotes the axial direction) along the duct's entrance. It is observed that after an initial stage where the pressure gradient decreases, its value is stabilized at approximately 0.125. The differences between the three grid sizes are negligible. Similar comparisons for the same three grid sizes were conducted with the CVP and the IPOT methods. The results showed negligible differences for all the case, so it is concluded that predictions are independent from the cross-sectional grid size.

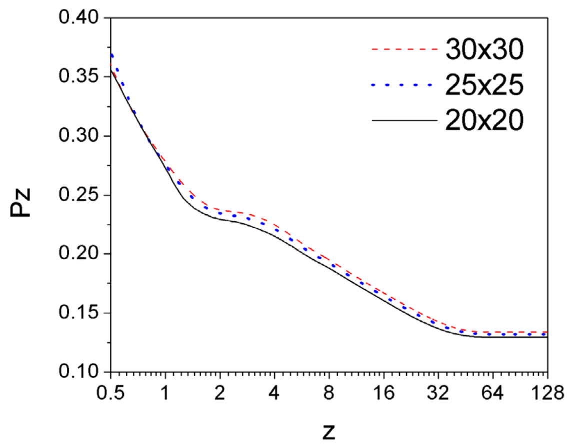



Figure 1: Comparison of the predictions for the axial variation of the absolute value of the axial pressure gradient Pz for three different grid sizes. The results were obtained with the auxiliary potential method.

The next comparison concerns the prediction of the development of the flow and its dependence on the axial step size. Figure 2 shows the axial variation axial pressure gradient for three axial step sizes, 0.05, 0.01 and 0.005. It is seen that the solution is independent from the axial step size. The results of Fig.2 correspond to the Auxiliary Potential method and they are identical to the results obtained with the CVP and the IPOT method.

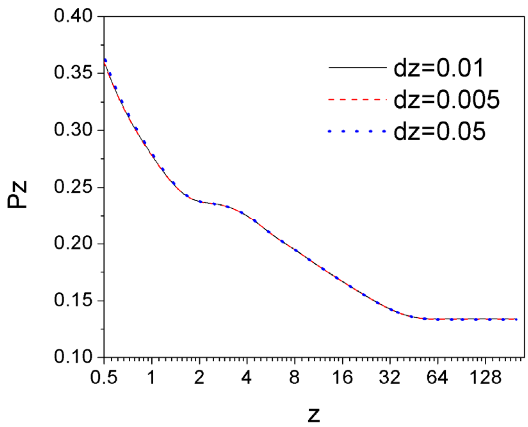



Figure 2: Comparison of the predictions for the axial variation of the absolute value of the axial pressure gradient Pz for three axial step sizes. The results were obtained with the auxiliary potential method.

# 5 Result and discussion

The comparison of the three methodologies concerns the results for the hydrodynamic and thermal entrance lengths and the flow field predictions. In order to examine the thermal results, we must first introduce the Nusselt number, which is defined as

$$Nu = \frac{\overline{h} D_h}{k} \tag{19}$$

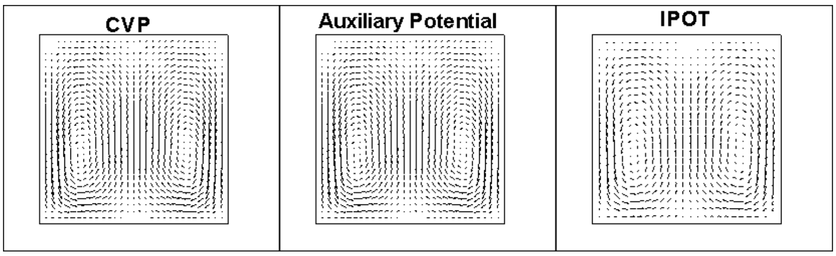
where $\overline{h}$ is the mean heat transfer coefficient averaged over the cross-section and $k$ is the thermal conductivity. Using the non-dimensional variables, the Nusselt number becomes

$$Nu = \frac{1}{1-\theta_b} \left. \frac{\partial \theta}{\partial n} \right|_w \tag{20}$$

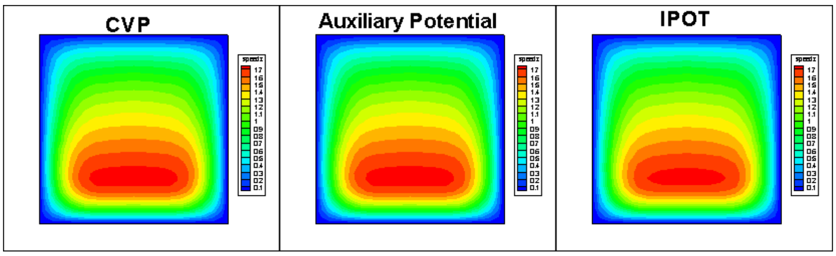
where $\theta_b$ is the bulk temperature of the cross-section and $\hat{\mathbf{n}}$ is the unit outward normal.

The vector plots of the transversal velocities and the contours of the axial velocity and the temperature for flow with Re=200, Pr=0.7 and Gr=100 are shown in Figure 3. Each set of plots corresponds to the three methods and they depict the thermal and hydrodynamic flow field at a plane located at z=10. It is observed that due to the buoyancy forces there is a strong recirculation of the secondary velocity and the maximum axial velocity is located in the lower half of the duct. The CVP and Auxiliary potential methods predict similar velocity and temperature distributions and only the results of the IPOT method present a slight difference in the flow field and more specifically in the shape of the contours of the maximum axial velocity and of the minimum temperature. However, the differences are very small and can be considered as negligible.

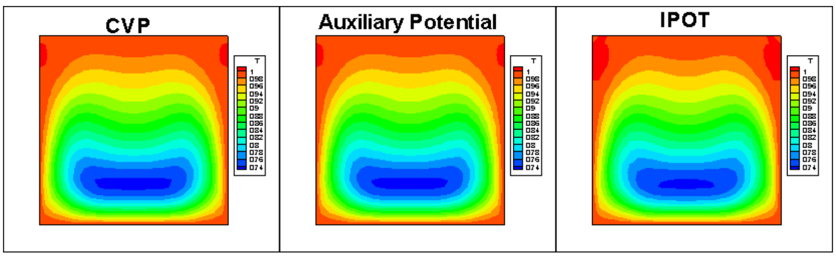
Figure 3 compares the results of the three methods (CVP, Auxiliary Potential and IPOT) concerning the axial pressure gradient, for the aforementioned flow with Re=200, Pr=0.7 and Gr=100. It is observed that the predictions of the three methods are identical. There is a smooth reduction of the axial pressure gradient as the flow moves downstream, until it reaches a fully developed state. The fluctuations in the pressure gradient are due to the buoyancy forces that create significant recirculation of the flow.




(a)




(b)




(c)

Figure 3: Predictions of the three methods for Re=200, Pr=0.7 and Gr=100 at downstream location z=10. The sets of plots correspond to (a) secondary velocity vector plots, (b) contours of the axial velocity and (c) contours of the temperature.

Figure 5 shows the results for the downstream variation of the Nusselt number. It is observed that Nu exhibits an initial sharp drop, while after approximately z=1 it is stabilised and only fluctuates to reach its fully developed value. Concerning the comparison of the three methods, it is observed that the results are identical. Taking into account all the results for the flow case with Re=200, Pr=0.7 and Gr=100, it is concluded that the predictions of the three methods are in very close agreement in the both the hydrodynamic and the thermal flow field.

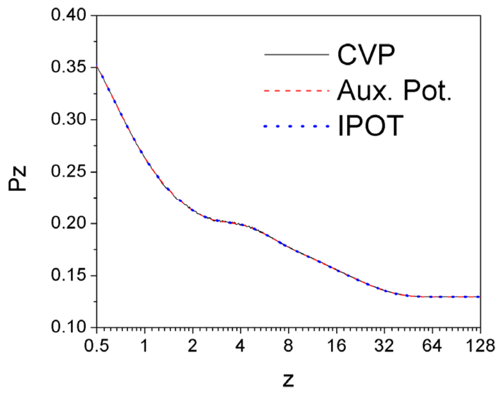



Figure 4: Comparison of the predictions for the axial variation of the absolute value of the axial pressure gradient Pz for the three methods. The case corresponds to with Re=200, Pr=0.7 and Gr=100.

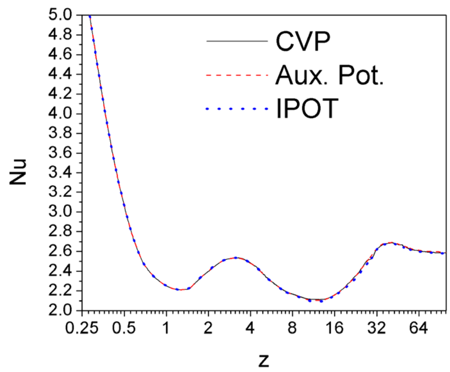
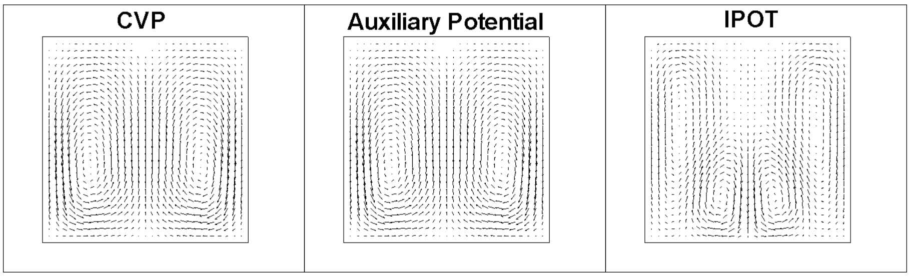
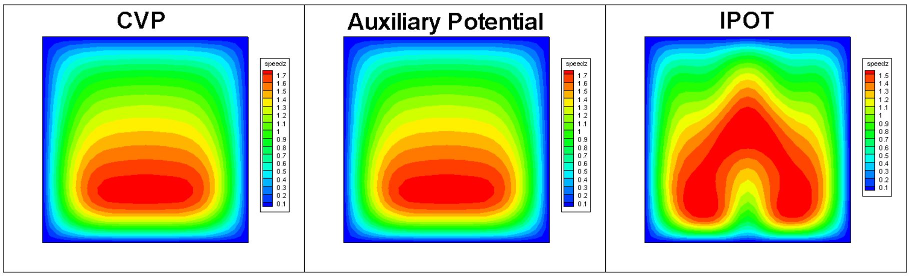



Figure 5: Comparison of the predictions for the axial variation Nusselt number for the three methods. The case corresponds to with Re=200, Pr=0.7 and Gr=100.

The next test case for the three methods is for flow with Re=500, Pr=0.7 and Gr=100. Although the CVP and the Auxiliary Potential methods predict a flow evolution similar to the previous one for Re=200, the IPOT method gives results that reveal a fluctuation in the flow field between one and two pairs of vortices. This can be observed in Fig.6 which shows vector plots of the transversal velocities and contours of the axial velocity and the temperature for the three methods. Due to the intense recirculation of the flow, in the predictions of the IPOT method, the temperature field develops earlier than in the other two methods. This deference in the predictions of the IPOT method is attributed to the insufficiently small axial step, which creates fluctuations in the solution although it eventually converges to the same flow field as the other two methods.




(a)

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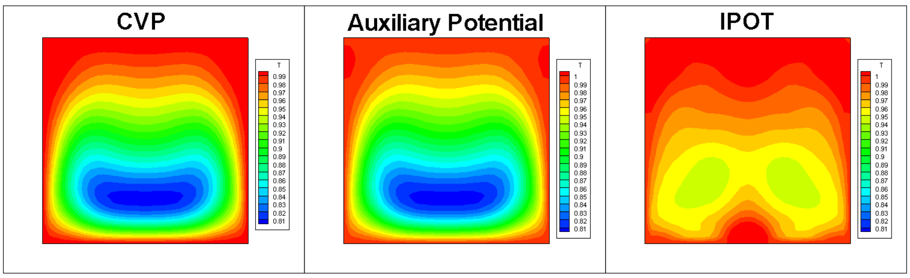


(c)

Figure 6: Predictions of the three methods for Re=500, Pr=0.7 and Gr=100 at downstream location z=30. The sets of plots correspond to (a) secondary velocity vector plots, (b) contours of the axial velocity and (c) contours of the temperature.

This fact can be seen from Figure 7 which depicts the downstream variation of the axial pressure gradient. It is observed that the predictions of the IPOT method differ from the ones of the CVP and the Auxiliary potential in the developing section of the flow, but converge to the same solution at the fully developed stage. These results indicate an increased dependence of the IPOT results on the step size, especially for large values of Re and designates smaller robustness compared to the other two methods.

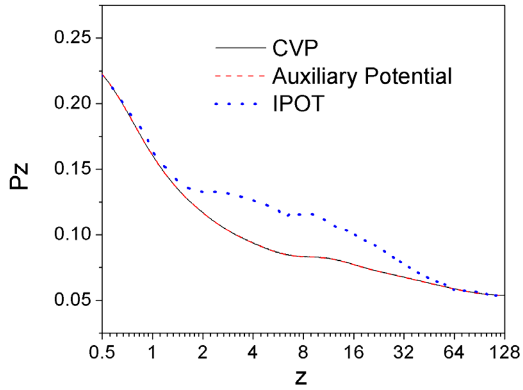


Figure 7: Comparison of the predictions for the axial variation of the absolute value of the axial pressure gradient Pz for the three methods. The case corresponds to Re=500, Pr=0.7 and Gr=100.

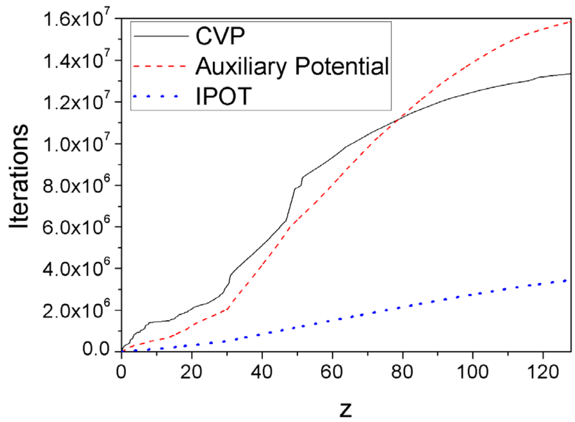


Figure 8: Comparison of total number of iterations of the three methods. The case corresponds to Re=200, Pr=0.7 and Gr=100.

Another comparison that has been conducted in the present research concerns the number of iterations required by each method in order to obtain a converged solution. Figure 8 shows the total number of iterations versus the axial length. It is observed that the IPOT method is by far the most economic as the iterations it requires is one order of magnitude lower than the ones of CVP and Auxiliary Potential. Concerning the last two, it is found that although the Auxiliary Potential converges faster in the initial stage, the CVP requires less iterations in the final stage of the simulation, thus it can more computationally economic in the end.

Another important factor that is investigated it the CPU time required by each method to reach a converged solution. This is different from the iterations' plot as each method requires the solution of different numbers of correction equations. The results are shown in Table 1 where the total CPU time for each method, for the computations up to z=128 is presented. The value of the IPOT method for Re=500 is omitted since the predictions were different from the other methods and the results are not comparable. It is observed that the IPOT method is approximately 16% faster that CVP and 35% faster than Auxiliary Potential. Concerning the comparison between CVP and Auxiliary Potential, it is found that the former is approximately 15% faster than the latter for Re=200, but this difference is diminished to 1% in the computationally more demanding problem of Re=500.

| | Re=200 | Re=500 |
|---|---|---|
| CVP | 44.24 h | 102.75 h |
| Auxiliary Potential | 51.24 h | 112.41 h |
| IPOT | 38 h | - |

Table 1: Comparison of the CPU time of the three methods.

# 6 Conclusions

In the present paper we conducted a critical comparison of the CVP, the auxiliary potential and IPOT methodologies on the solution of laminar incompressible developing flow with buoyancy forces in a straight square duct. The results have shown that although all three methods can predict accurately the development of the flow for low Reynolds numbers, they have their unique properties: the CVP method is the most demanding in terms of the equations that must be solved, but it is also characterized by increased robustness and computational efficiency; the Auxiliary Potential method has great functionality but it can be slower than CVP; the IPOT method is the simplest, fastest and most flexible of the three but its lacks robustness and requires the use of a smaller axial step. These results lead to useful conclusions concerning the characteristics of the three methods and they are crucial to the determination of the most suitable one for the various technological problems.

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