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Orthotropic Enriched Extended Isogeometric Analysis for Fracture Analysis of Composites

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Abstract

Fracture analysis of orthotropic cracked plates is investigated by applying the recently developed computational approach, called the extended isogeometric analysis (XIGA) [1]. A signed distance function and orthotropic crack tip enrichment functions are adopted for extrinsically enriching the conventional isogeometric analysis approximation for the representation of significant discontinuities and reproducing the singular field around a crack tip, respectively. For increasing the integration accuracy, the sub-triangles and almost polar techniques are adopted for the cut and crack tip elements, respectively. The interaction integral technique developed by Kim and Paulino [2] is applied for computing the mixed mode stress intensity factors (SIFs). Finally, an orthotropic cracked plate with different orientations of material elastic axes is analyzed by the proposed scheme and the fracture properties (mixed mode SIFs) are compared with those of other methods available in the literature.

Keywords: orthotropic media, crack, extended isogeometric analysis, orthotropic enrichment functions, stress intensity factor, interaction integral.

1 Introduction

In this paper, cracked orthotropic media are analyzed by applying the novel approach of extended isogeometric analysis (XIGA).

Orthotropic materials such as composites have been increasingly applied in many engineering applications e.g. aerospace, automobile and marine structures because of their high strength and stiffness to weight ratios. Considering their strength, they are applied in thin shell forms while crack initiation is probable to take place in them. As a result, fracture analysis of such media has been the center of attention for many researchers in the last few decades.

Analytical solution of stress and displacement fields for an orthotropic plate with a crack has already been obtained by Sih et al. [3]. As the analytical methods cannot be simply employed for complex problems which are common in structural engineering, numerical methods are better alternatives. Asadpoure and Mohammadi [4] succeeded in developing orthotropic enrichment functions from the analytical solutions and applied them in the extended finite element method (XFEM) for analysis of cracked orthotropic plates. Also, similarproblems have been recently solved by the enriched element free Galerkin (EFG) method [5].

Complexity of engineering problems and enormous growing of technology in computers have led to the development of several numerical methods. Among them, XFEM [6, 7] has proved to be a promising powerful tool in modeling fracture problems because it enables improved approximations of non-smooth solutions such as those including jumps and singularities. In this approach, for modeling a crack, classical finite element approximation is enriched by discontinuous function and asymptotic crack-tip displacement fields using the framework of partition of unity (PU). In XFEM, the finite element mesh is not required to conform to the cracks boundaries, and hence a single mesh suffices for modeling the crack stability and capturing its evolution.

On the other hand, isogeometric analysis (IGA) is a promising computational scheme, developed by Hughes et al. [8], that takes advantage of using non-uniform rational B-splines (NURBS) functions for both geometric description and solution field approximation to exactly represent complex geometries, to increase the order of continuities between elements, to simplify the refinement process and to improve solution accuracy. Isogeometric analysis has been effectively applied to a large variety of problems [9].

The two powerful approaches of XFEM and IGA have recently been combined to include the benefits of both [10, 1]. This method which is also called extended isogeometric analysis (XIGA) has been successfully applied for simulation of stationary and propagating cracks in 2D linear-elastic isotropic media.

In this contribution, XIGA is further extended for fracture analysis of cracked linear-elastic orthotropic materials. For this purpose, the orthotropic enrichment functions applied in XFEM [4] are adopted. The Lagrange multiplier method is utilized to impose essential boundary conditions. The Gauss quadrature rule is applied for integration alongside the "sub-triangles approach" and the "almost polar technique" for split and crack tip elements, respectively [1]. In order to compare the results with those avalibable in the literature, mixed mode stress intensity factors are calculated by adopting the interaction integral technique. Finally, an orthotropic cracked plate considering several orientations of material elastic axes is analyzed to demonstrate the accuracy and efficiency of the proposed approach.

2 Fracture mechanics in orthotropic media

The stress-strain law in an arbitrary linear elastic material can be written as

$$\boldsymbol{\varepsilon} = \mathbf{c}\boldsymbol{\sigma} \tag{1}$$

where $\boldsymbol{\varepsilon}$and $\boldsymbol{\sigma}$ are strain and stress vectors, respectively, and $\mathbf{c}$ is the compliance matrix,

$$\mathbf{c}^{3D} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{2}$$

where E, ν and G are Young's modulus, Poisson's ratio and shear modulus, respectively. For a plane stress case, the compliance matrix is reduced to the following form:

$$\mathbf{c}^{2D} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{3}$$

and for a plane strain state,

$$c_{ij}^{2D} = c_{ij}^{3D} - \frac{c_{i3}^{3D}.c_{j3}^{3D}}{c_{33}^{3D}} \quad \text{for } i, j = 1, 2, 6 \tag{4}$$

Now assume an anisotropic body subjected to arbitrary forces with general boundary conditions and a crack. Global Cartesian coordinate (X_1, X_2), local Cartesian coordinate (x, y) and local polar coordinate (r, θ), defined on the crack tip, are illustrated in Fig. 2. A fourth-order partial differential equation with the following characteristic equation can be obtained using equilibrium and compatibility conditions [11].

$$c_{11}s^4 - 2c_{16}s^3 + (2c_{12} + c_{66})\, s^2 - 2c_{26}s + c_{22}0 = 0 \tag{5}$$

where $c_{ij} \quad (i, j = 1, 2, 6)$ are the components of $\mathbf{c}^{2D}$. According to [11], the roots of eq. 5 are always complex or purely imaginary $\big(\big(\; s_k = s_{kx} + is_{ky}, \quad k = 1, 2 \; \big)\big)$ and occur in conjugate pairs as s_1, $\bar{s_1}$ and s_2, $\bar{s_2}$. The two-dimensional displacement and stress fields in the vicinity of the crack-tip have been derived as [3]

- Mode I

$$\begin{aligned} u^I &= K_I\sqrt{\tfrac{2r}{\pi}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(s_1p_2\sqrt{\cos\theta + s_2\sin\theta} - s_2p_1\sqrt{\cos\theta + s_1\sin\theta}\right)\right] \\ v^I &= K_I\sqrt{\tfrac{2r}{\pi}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(s_1q_2\sqrt{\cos\theta + s_2\sin\theta} - s_2q_1\sqrt{\cos\theta + s_1\sin\theta}\right)\right] \\ w^I &= 0 \end{aligned} \tag{6}$$

$$\begin{aligned}\sigma_{xx}^{I}&=\tfrac{K_I}{\sqrt{2\pi r}}\mathrm{Re}\left[\tfrac{s_1 s_2}{s_1-s_2}\left(\tfrac{s_2}{\sqrt{\cos\theta+s_2\sin\theta}}-\tfrac{s_1}{\sqrt{\cos\theta+s_1\sin\theta}}\right)\right]\\ \sigma_{yy}^{I}&=\tfrac{K_I}{\sqrt{2\pi r}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(\tfrac{s_1}{\sqrt{\cos\theta+s_2\sin\theta}}-\tfrac{s_2}{\sqrt{\cos\theta+s_1\sin\theta}}\right)\right]\\ \sigma_{xy}^{I}&=\tfrac{K_I}{\sqrt{2\pi r}}\mathrm{Re}\left[\tfrac{s_1 s_2}{s_1-s_2}\left(\tfrac{1}{\sqrt{\cos\theta+s_1\sin\theta}}-\tfrac{1}{\sqrt{\cos\theta+s_2\sin\theta}}\right)\right]\end{aligned}\tag{7}$$

- Mode II

$$\begin{aligned}u^{II}&=K_{II}\sqrt{\tfrac{2r}{\pi}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(p_2\sqrt{\cos\theta+s_2\sin\theta}-p_1\sqrt{\cos\theta+s_1\sin\theta}\right)\right]\\ v^{II}&=K_{II}\sqrt{\tfrac{2r}{\pi}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(q_2\sqrt{\cos\theta+s_2\sin\theta}-q_1\sqrt{\cos\theta+s_1\sin\theta}\right)\right]\\ w^{II}&=0\end{aligned}\tag{8}$$

$$\begin{aligned}\sigma_{xx}^{II}&=\tfrac{K_{II}}{\sqrt{2\pi r}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(\tfrac{s_2^2}{\sqrt{\cos\theta+s_2\sin\theta}}-\tfrac{s_1^2}{\sqrt{\cos\theta+s_1\sin\theta}}\right)\right]\\ \sigma_{yy}^{II}&=\tfrac{K_{II}}{\sqrt{2\pi r}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(\tfrac{1}{\sqrt{\cos\theta+s_2\sin\theta}}-\tfrac{1}{\sqrt{\cos\theta+s_1\sin\theta}}\right)\right]\\ \sigma_{xy}^{II}&=\tfrac{K_{II}}{\sqrt{2\pi r}}\mathrm{Re}\left[\tfrac{1}{s_1-s_2}\left(\tfrac{s_1}{\sqrt{\cos\theta+s_1\sin\theta}}-\tfrac{s_2}{\sqrt{\cos\theta+s_2\sin\theta}}\right)\right]\end{aligned}\tag{9}$$

where Re denotes the real part of the statement and K_1 and K_{II} are stress intensity factors for mode I and mode II, respectively. p_i and q_i can be defined by

$$p_i = c_{11}s_i^2 + c_{12} - c_{16}s_i, \quad (i = 1, 2) \tag{10}$$

$$q_i = c_{12}s_i + \tfrac{c_{22}}{s_i} - c_{26}, \quad (i = 1, 2) \tag{11}$$

3 Isogeometric analysis

Isogeometric analysis is an isoparametric finite element method where the non-uniform rational B-spline (NURBS) functions are applied as the basis functions. So, for both geometry description and solution field approximation, NURBS functions are utilized,

$$\mathbf{X}\left(\xi^1,\xi^2\right)=\sum_{k=1}^{n_{cp}} R_k^{p,q}\left(\xi^1,\xi^2\right)\mathbf{P}_k \tag{12}$$

$$\mathbf{u}^h\left(\xi^1,\xi^2\right)=\sum_{k=1}^{n_{cp}} R_k^{p,q}\left(\xi^1,\xi^2\right)\mathbf{u}_k \tag{13}$$

where $\mathbf{X}$ and $\mathbf{u}^h$ are vectors of physical coordinates and solution field approximation of the parametric coordinate (ξ^1, ξ^2), respectively. $\{R_k^{p,q}\}$ are the NURBS functions of order p in ξ^1 direction and order q in ξ^2 direction (see Section (3.1)). n_{cp} is the number of control points and basis functions, $\{\mathbf{P}_i\}$ are the physical coordinates of

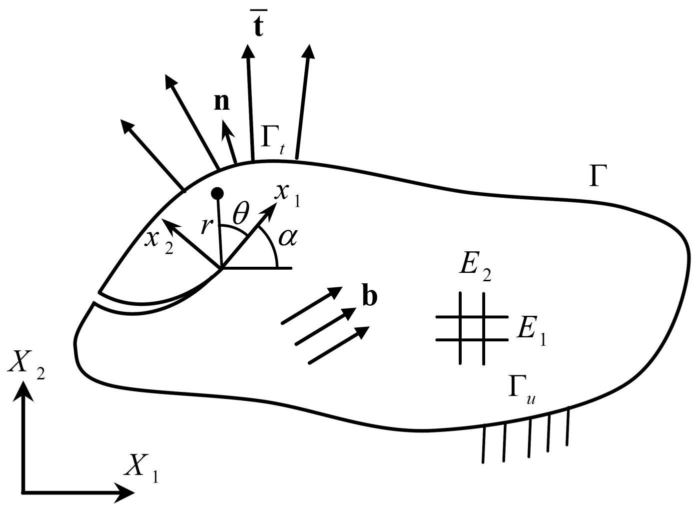


Figure 1: An arbitrary orthotropic cracked body subjected to body force $\mathbf{b}$ and traction $\bar{\mathbf{t}}$.

control points, and $\{\mathbf{u}_i\}$ are the control variables. It is noted that the control points are not necessarily located in the constructed physical geometry.

NURBS shape funtions and the linear elasticity problem formulations are defined in the following sub-sections.

3.1 NURBS functions

NURBS shape functions of order p in ξ^1 direction and order q in ξ^2 direction is defined using tensor product feature as follow,

$$R_{i,j}^{p,q}\left(\xi^1,\xi^2\right) = \frac{N_i^p\left(\xi^1\right)N_j^q\left(\xi^2\right)w_{i,j}}{\sum_{\hat{i}=1}^{n}\sum_{\hat{j}=1}^{m}N_{\hat{i}}^p\left(\xi^1\right)N_{\hat{j}}^q\left(\xi^2\right)w_{\hat{i},\hat{j}}} \tag{14}$$

where $\{w_{i,j}\}$ are the weights corresponding to each control point. $\{N_i^p\left(\xi^1\right)\}$ and $\left\{N_j^q\left(\xi^2\right)\right\}$ are the B-spline basis functions of order p in ξ^1 direction and order q in ξ^2 direction, respectively, which are defined in a parametric space $\left[\Xi^1 \times \Xi^2\right]$. Definitions of B-spline shape functions of both directions are the same. In the following, they are defined in one direction.

Ξ is called the knot vector and has the following form,

$$\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\} \quad \xi_i \leq \xi_{i+1},\ i = 1, 2, \ldots n+p \tag{15}$$

where n is the number of basis functions and the knots $\{\xi_i\}$ are real numbers representing the coordinates in the parametric space [0,1]. In order to satisfy the Kronecker delta property at the boundary points, the so-called open knot vectors are utilized where the first and last knots are repeated $p+1$ times.

B-spline basis functions are defined as:

$$N_{i,0} = \begin{cases} 1 & \xi_i \leq \xi \leq \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$N_{i,p} = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad \text{for} \quad p = 1, 2, 3, \ldots \tag{17}$$

Readers are referred to [12] for more information about the NURBS.

3.2 Linear elasticity problem

Strong form of a linear elasticity problem and the boundary conditions are defined in the following forms:

$$\nabla . \boldsymbol{\sigma} + \boldsymbol{b} = 0 \tag{18}$$

$$\boldsymbol{\sigma}.\mathbf{n} = \bar{\mathbf{t}} \quad \text{on} \quad \Gamma_t \tag{19}$$

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on} \quad \Gamma_u \tag{20}$$

where $\boldsymbol{\sigma}$ and b is stress tensor and body force vector, respectively. Γ_t and Γ_u are parts of the problem boundary where prescribed traction $\bar{\mathbf{t}}$ and displacement $\bar{\mathbf{u}}$ are imposed respectively.

4 Extended isogeometric analysis

Isogeometric analysis method has been recently developed for fracture analysis of isotropic cracked bodies using the concept of extended finite element method (XFEM) [10, 1]. Ghorashi et al. [1] called this promising method "extended isogeometric analysis (XIGA)". In the XIGA approach, cracks can be defined independent of the mesh and can propagates without the necessity of remeshing. This is achieved by extrisically enriching the solution field approximation (eq. (13)),

$$\begin{aligned} \mathbf{u}^h(\xi^1, \xi^2) = \textstyle\sum_{i=1}^{n_{en}} R_i^{p,q}(\xi^1, \xi^2)\, \mathbf{u}_i + \sum_{j=1}^{n_H} R_j^{p,q}(\xi^1, \xi^2)\, H \mathbf{a}_j \\ + \textstyle\sum_{k=1}^{n_Q} R_k^{p,q}(\xi^1, \xi^2) \sum_{\alpha=1}^{4} Q_\alpha \mathbf{b}_k^\alpha \end{aligned} \tag{21}$$

where n_{en} is the number of non-zero basis functions defined at the parametic coordinates (ξ^1, ξ^2). n_H is the number of basis functions whose support domains are cut by crack but do not contain the crack tip and n_Q is the number of basis function whose support domains include the crack tip. $\mathbf{a}_j$ and $\mathbf{b}_k^\alpha$ are vectors of additional DOFs that are related to the modeling of crack faces and crack tips, respectively. H is the generalized Heaviside function,

$$H(\mathbf{X}) = \begin{cases} +1 & \text{if } (\mathbf{X} - \mathbf{X}^*) \,.\mathbf{e}_n > 0 \\ -1 & \text{otherwise} \end{cases} \tag{22}$$

where $\mathbf{e}_n$ is the unit normal vector of crack alignment in point $\mathbf{X}^*$on the crack surface which is the nearest point to $\mathbf{X}(\xi^{\mathbf{1}}, \xi^{\mathbf{2}})$.

In eq. (21), $Q_\alpha \{\alpha = 1, 2, 3, 4\}$ are the crack tip enrchment functions whose roles are reproducing the singular filed around crack tips. In this paper, the following orthotropic crack tip enrichment functions developed by Asadpoure and Mohammadi [4], which were defined based on the analytical solution (equations 6, 7, 8, 9), are adopted,

$$\begin{gathered}\{Q_\alpha\}_{\alpha=1}^4 = \{\sqrt{r}\cos\tfrac{\theta_1}{2}\sqrt{g_1(\theta)}, \sqrt{r}\cos\tfrac{\theta_2}{2}\sqrt{g_2(\theta)}, \ldots \\ \sqrt{r}\sin\tfrac{\theta_1}{2}\sqrt{g_1(\theta)}, \sqrt{r}\sin\tfrac{\theta_2}{2}\sqrt{g_2(\theta)}\}\end{gathered} \tag{23}$$

where

$$\theta_i = \arctan\left[\tfrac{s_{iy}\sin\theta}{\cos\theta + s_{ix}\sin\theta}\right], \quad (i = 1, 2) \tag{24}$$

$$g_i(\theta) = \sqrt{(\cos\theta + s_{ix}\sin\theta)^2 + (s_{iy}\sin\theta)^2}, \quad (i = 1, 2) \tag{25}$$

where s_{ix} and s_{iy} are real and imaginary parts of s_i computed by eq. 5. It is noted that the third and fourth functions in the right-hand side of the equation 23 are discontinuous across the crack faces while the others remain continuous.

5 Numerical example

The proposed method is applied for analysis of a finite rectangular orthotropic plate with an edge crack subjected to uni-axial tension. The plate is considered in the plane stress state and several orientations of material elastic axes are studied. The proportions of width to height and crack length to width are equal to 0.5 (see Fig. 2). The plate is composed of a graphic-epoxy material with orthotropic properties as:

$$E_1 = 114.8 \text{ GPa}, \quad E_2 = 11.7 \text{ GPa}, \quad G_{12} = 9.66 \text{ GPa}, \quad \nu_{12} = 0.21$$

NURBS basis functions of cubic order are applied. 1296 control points and 1089 elements are used for modeling the problem, as illustrated in Figs. 3 and 4. Minimum and maximum sizes of elements are $[w \times h]/27^2$ and $[w \times h]/27$ around the crack tip and far from it, respectively.

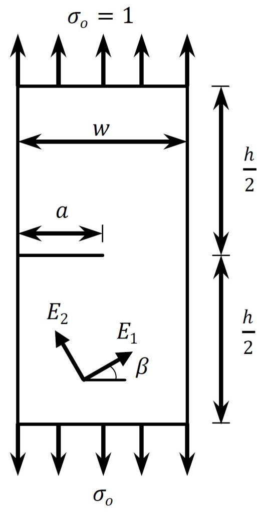


Figure 2: Geometry and loading.

In order to increase the integration accuracy, sub-triangles and almost polar techniques are utilized for integration over elements cut by crack and contain the crack tip, respectively (see [1]). For imopition of essential boundary conditions, the Lagrange multiplier method is adopted (see [1]).

For comparing the obtained results with those available in the literature, the stress intensity factor (SIF), which is among the important parameters of representing the fracture properties of a crack, is calculated. For this purpose, the technique developed by Kim and Paulino [2] is employed. Effects of changing the material elastic angle on mixed mode SIFs in the plate are probed. The comparison of results between the proposed method and the results of the enriched element free Galerkin (EFG) [5], extended finite element method (XFEM) [4] and boundary element method (BEM) [14], is shown in Fig. 5.

It is seen that the results are in good agreement with those obtained by other methods. The results show that the trend of mode I SIF changes around $\beta = 45°$. It has an increasing trend in the span of $\beta = 0°$ to $\beta = 45°$ and then decreases in the span of $\beta = 45°$ to $\beta = 90°$ and reaches a value around its initial value, i.e. when $\beta = 0°$. The turning point for the mode II SIF is about $\beta = 30°$.

6 Conclusion

In this paper, the newly developed XIGA has been further extended to analysis of cracked orthotropic plates. The recently proposed crack-tip orthotropic enrichment functions have been employed in the XIGA method to increase the approximation accuracy near the crack-tip. For imposition of essential boundary conditions, the Lagrange multiplier method has been utilized and in order to increase the integration

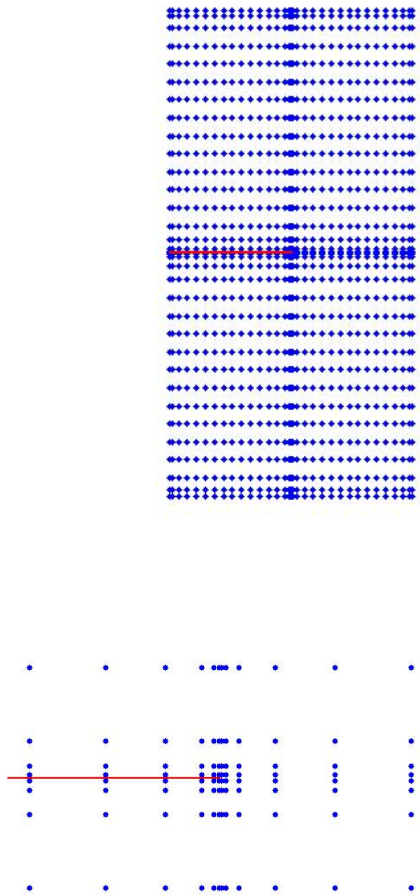


Figure 3: Distribution of control points: whole view and around the crack tip .

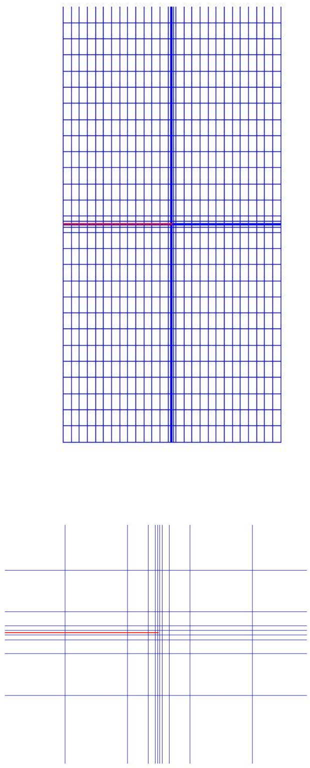
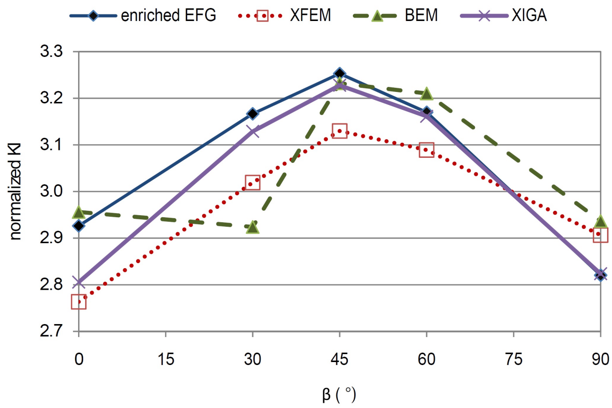
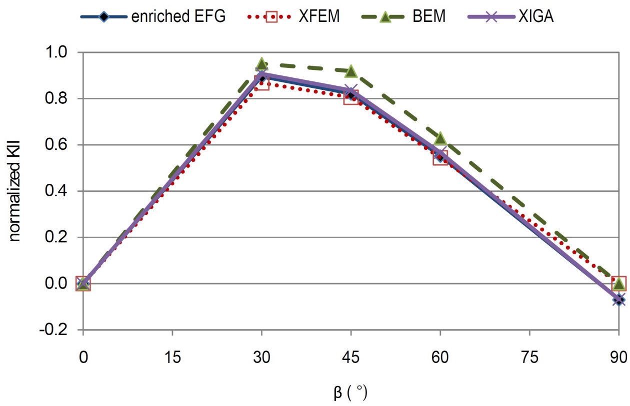


Figure 4: Distribution of elements : whole view and around the crack tip.



(a)



(b)

Figure 5: The effect of various inclinations of elastic material axes on the mixed mode SIFs: a) normalized mode I SIF ($\frac{K_I}{\sigma_o\sqrt{\pi a}}$) , b) normalized mode II SIF ($\frac{K_{II}}{\sigma_o\sqrt{\pi a}}$).

accuracy around the discontinuous and singular fileds, 'sub-triangles' and 'almost polar' techniques are adopted.

A cracked orthotropic plate with different orientations of material elastic axes have been analyzed using the proposed approach. Results of mixed-mode stress intensity factors (SIFs) have been compared with the reference results and proved the accuracy and efficiency of the proposed method.

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