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**Paper 41**

# Dynamic Aspects of Damped Newton's Method

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## Abstract

In this work we study the dynamics of damped Newton's method defined by

$$N_{\lambda,p}(z) = z - \lambda \frac{p(z)}{p'(z)},\ \lambda \in \mathbb{C}$$

applied to different polynomial equations.

For polynomials with two different roots we analyze the influence of the damping factor $\lambda$ in the character of the fixed points as well as in the behavior of the critical points. It is shown that, for some choices of the damping factor, it introduces new free critical points and changes the character of the fixed points.

We also study the influence of $\lambda$ in the form of the related universal Julia sets and their fractal dimensions. Finally we prove a theorem characterizing the Julia sets of the damped Newton's method applied to polynomials with two different roots with the same multiplicity.

**Keywords:** damped Newton's method, dynamics, universal Julia set.

# 1 Introduction

One of the most common problems that appear in many areas related to mathematics is finding a root of a nonlinear equation $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$. There exists lots of iterative methods that allow us solving this kind of equations, but the most used and studied is the Newton's method:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},\ n \geq 0.$$

The dynamics of the Newton's method has been studied in depth (see [3], [5], [6] and [7]).

In this paper, instead of studying the Newton's method, the dynamics of a modification of this method, has been studied. The modification studied consists on introducing a damping factor, $\lambda \in \mathbb{C}$, to control the step size which is taken in the Newton direction:

$$z_{n+1} = z_n - \lambda \frac{f(z_n)}{f'(z_n)}, \quad \lambda \in \mathbb{C}. \tag{1}$$

This method is called damped Newton's method. A way of introducing it is from the continuous Newton's method [8] :

$$z'(t) = \frac{f(z)}{f'(z)}, \tag{2}$$

which comes from solving the following initial value problem:

$$\begin{cases} z'(t) = -\frac{f(z)}{f'(z)} \\ z(0) = z_0 \end{cases}$$

applying the explicit Euler's method with step size $\lambda$:

$$\frac{z_{n+1} - z_n}{\Delta t} = -\frac{f(z_n)}{f'(z_n)}.$$

Finally, rearranging terms, we obtain:

$$z_{n+1} = z_n - \Delta t \frac{f(z_n)}{f'(z_n)}, \quad z_n \in \mathbb{C}.$$

And now taking $\lambda = \Delta t$, we obtain the damped Newton's method defined in (1).

In this paper we will focus our attention in this method applied to polynomials with two different roots $p(z) = (z-a)^n(z-b)^m$ and we will study the dynamics of the rational function that we obtain. The paper is organized as follows. In section 2 we will see some dynamical aspects and some basic features of the damped Newton's method. In section 3 we will see the study of the fixed and critical points of the rational function that we obtain by means of applying damped Newton's method to a polynomials with two different roots as well as some theoretical results. Moreover we will see some pictures of the Universal Julia sets for different values of $\lambda$. Finally in section 4 we give the main conclusions drawn from the study.

# 2 Basic concepts

In this section we present some basic concepts that we use along the paper. If we apply the damped Newton's method to a polynomial $p(z) = (z-a)^n(z-b)^m,\ a, b \in \mathbb{C},\ m, n \in \mathbb{N}$ we obtain the following iteration function:

$$N_{\lambda,p}(z) = z - \lambda \frac{(z-a)(z-b)}{(n+m)z - ma - nb} \tag{3}$$

However, after conjugation with an adequate homeomorphism, the iteration function (3) becomes a rational function depending only on the damping factor $\lambda$ and the multiplicities $m$ and $n$. In particular, we consider the following homeomorphisms that map the roots $a$ and $b$ into 0 and $\infty$ or 1 and $-1$ respectively:

$$M_1(z) = \frac{z-a}{z-b}, \tag{4}$$

$$M_2(z) = 1 + \frac{2(z-a)}{a-b}. \tag{5}$$

We obtain two new rational maps:

$$S_\lambda(z) = M_1 \circ N_{\lambda,p} \circ M_1^{-1}(z) = -\frac{z(-\lambda + n + mz)}{-n + (\lambda - m)z}, \tag{6}$$

$$R_\lambda(z) = M_2 \circ N_{\lambda,p} \circ M_2^{-1}(z) = \frac{\lambda + (-m+n)z + (-\lambda + m + n)z^2}{-m + n + (m+n)z}. \tag{7}$$

The map (4) has been used by several authors, as for instance [4], whereas $M_2$ was introduced in [9] and received the name of "Scaling".

So our main goal in this paper is to study the dynamics of the rational functions given in (6) and (7) for particular choices of $m$ and $n$.

## 2.1 Dynamical concepts

In the study of the dynamics of an iterative method, there are some basic concepts that are needed.

First of all we denote $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We say that $z_0 \in \bar{\mathbb{C}}$ is a fixed point of a rational function $R(z) : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ if it satisfies: $R(z_0) = z_0$. A point $z^*$ is a critical point of a rational function $R$ if $R$ fails to be injective in any neighborhood of $z^*$. For instance if $R'(z^*) = 0$, $z^*$ is a critical point of $R$.

There exists different types of fixed points depending on the multiplier $\mu = (R)'(z_0)$.

**Definition 1** *A fixed point $z_0$ of a rational function can be:*

- *Super-Attractor if $\mu = 0$.*
- *Attractor if $|\mu| < 1$.*
- *Parabolic if there exists a $q \in \mathbb{Z}$ such that $\mu^q = 1$*

- *Neutral or indifferent if* $|\mu| = 1$ *and* $\mu^q \neq 1$ *for every value* $q \in \mathbb{Z}$

- *Repulsor if* $|\mu| > 1$.

The basin of attraction of an attractor, $z^*$, is the set:

$$\mathbb{A}(z^*) = \{z_0 \in \mathbb{C} : z_{n+1} = F(z_n) \xrightarrow[n\to\infty]{} z^*\}$$

A point $z$ is a periodic point if there exists $p \in \mathbb{Z}$ such that $N^p(z) = z$ which is called period. The orbit of a $z$:

$$\mathcal{O}(z) = \{z, R(z), R^2(z), \cdots, R^{p-1}(z)\}$$

is called $p$-cycle. We say that a $p$-cycle is attractor if $|(R^p)'(z)| < 1$, indifferent if $|(R^p)'(z)| = 1$ and repulsor if $|(R^p)'(z) > 1|$.

The Julia set associated to a rational function $R$, and denoted as $\mathcal{J}(R)$, is the closure of the set of repelling periodic points or the boundary of the basin of attraction of every fixed point or attracting cycle.

The Fatou set associated to a rational function $R$ is the complementary set of the Julia set $\mathcal{J}(R)$. We denote the Fatou set as $\mathcal{F}(R)$.

Finally we need the following definition, related to the Julia sets introduced in [4]:

**Definition 2** *We say an iterative root-finding algorithm* $f \to T_f$ *has an universal Julia set (for polynomials of degree* $d$*) if there exists a rational map* $R$ *such that for every degree* $d$ *polynomial* $f$*,* $\mathcal{J}(T_f)$ *is conjugate by a Möbius transformation to* $\mathcal{J}(R)$.

## 2.2 Basic features of the damped Newton's method

Some of the basic features of damped Newton's method are the following (see [9]):

- The roots of the equation $p(z) = 0$ are the fixed points of $N_{\lambda,p}(z)$.

- The derivative of the damped Newton's method has the following form:

  $$N'_{\lambda,p}(z) = (1-\lambda) + \lambda L_p(z),$$

  where

  $$L_p(z_n) = \frac{p(z_n)p''(z_n)}{p'(z_n)^2}. \tag{8}$$

- If $r$ is a simple root, then $p(r) = 0$ and $p'(r) \neq 0$ we have:

  $$N'_{\lambda,p}(r) = 1 - \lambda.$$

- If $r$ is a root of multiplicity $m \geq 2$, $p(z) = (z-r)^m q(z)$, $q(r) \neq 0$ we have:

$$N_{\lambda,p}(r) = r,$$

$$N'_{\lambda,p}(r) = 1 - \frac{\lambda}{m}.$$

- The critical points of $N_{\lambda,p}$ are the solutions of the equation

$$L_p(z) = \frac{\lambda - 1}{\lambda},$$

  where $L_p$ is given in (8).

- $\infty$ is a fixed point of $N_{\lambda,p}$ if and only if 0 is a fixed point of $Q(z) = \frac{1}{N_{\lambda,p}(\frac{1}{z})}$.

- If $p$ is a polynomial of degree $n$: $Q'(0) = \frac{n}{n-\lambda}$.

# 3 Study of the dynamics

In this section we will study the influence of the damping factor $\lambda$ and the multiplicity of the roots in the dynamics of the damped Newton's method applied to a polynomial $p(z) = (z-a)^n (z-b)^m,\ a,\ b \in \mathbb{C},\ n,\ m \in \mathbb{N}$. The study has been divided in two different cases depending on the multiplicity of the roots. First of all we will see the case when both roots has the same multiplicity. Afterwards we will study the case when the multiplicity of one root is greater than the multiplicity of the other root. Some pictures will illustrate this study.

From the study of this kind of polynomials we have proof the following result, which is a generalization of the theorem given by Yang in [11]:

**Theorem 3.1** *Let $p(z) = (z-a)^n (z-b)^n$ with $a \neq b,\ n \geq 1$ and let $N_{\lambda,p}$ be the damped Newton's method defined by:*

$$N_{\lambda,p}(z) = z - \lambda \frac{p(z)}{p'(z)},\ 0 < \lambda < 2n.$$

*Then $\mathcal{J}(N_{\lambda,p}(z))$ is a straight line. This straight line is the bisector of the segment joining $a$ and $b$.*

**Proof** We begin supposing that $n = m$, so $p(z) = (z-a)^n (z-b)^n$, and we want to see that $\mathcal{J}(N_{\lambda,p}(z))$ is a straight line. First of all, the method has the following form:

$$N_{\lambda,p}(z) = z - \frac{\lambda}{n} \frac{(z-a)(z-b)}{(z-a)+(z-b)}$$

And now applying the Möbius transformation (4) we have :

$$\widetilde{N}_{\lambda,p}(z) = R_\lambda(z) \circ N_{\lambda,p} \circ R_\lambda^{-1}(z) = z - \frac{\lambda}{n}\frac{(z-1)(z+1)}{(z-1)+(z+1)}$$

Let be:

$$\mathcal{A} = \{z | Re(z) > 0\}$$
$$\mathcal{B} = \{z | Re(z) < 0\}$$
$$\mathcal{C} = \{z | Re(z) = 0\}$$

and we want to proof that $\mathcal{F} = \mathcal{A} \bigcup \mathcal{B}$ and $\mathcal{J} = \mathcal{C}$.

- $\mathcal{A} \subseteq \mathcal{F}$

  If $z \in \mathcal{A}$, $z = a + bi,\ a > 0$ then $\widetilde{N}_{\lambda,p}(z) \in \mathcal{A}$. Indeed,

  $$\widetilde{N}_{\lambda,p}(z) = \frac{z^2(2n-\lambda)+\lambda}{2nz} = \frac{(a+bi)^2(2n-\lambda)+\lambda}{2n(a+bi)} = \frac{(a^2+b^2)(2n-\lambda)(a+bi)+\lambda(a-bi)}{2n(a^2+b^2)}.$$

  And so, for all $\lambda < 2n$, we have:

  $$Re(\widetilde{N}_{\lambda,p}(z)) = \frac{a((a^2+b^2)(2n-\lambda)+\lambda)}{2n(a^2+b^2)} > 0$$

  and, therefore

  $$\widetilde{N}_{\lambda,p}(z) \in \mathcal{A}.$$

  Moreover we have that from the fixed point theorem $f$ is contractive and $\widetilde{N}_{\lambda,p}(z) \subset \mathcal{A}$, hence, there exists only una fixed point $z = 1 \in \mathcal{A}$ and $\lim_{n\to\infty} \widetilde{N}^n_{\lambda,p}(z) = 1$:

  $$|\widetilde{N}_{\lambda,p}(z) - 1| = \left|(z-1)\frac{n(z-1)+n(z+1)-\lambda(z+1)}{n(z+1)+n(z-1)}\right|$$
  $$= \left|(z-1)\frac{n-\lambda+n\frac{z-1}{z+1}}{n+n\frac{z-1}{z+1}}\right| < |z-1|$$

- $\mathcal{B} \subseteq \mathcal{F}$

  If $z \in \mathcal{B}$, $z = a + bi,\ a < 0$, then $\widetilde{N}_{\lambda,p}(z) \in \mathcal{B}$. Since function $\widetilde{N}_{\lambda,p}(z)$ is symmetric:

  $$\widetilde{N}_{\lambda,p}(-z) = \frac{z^2(2n-\lambda)+\lambda}{-2nz} = -\widetilde{N}_{\lambda,p}(z)$$

  we can ensure that $\mathcal{B} \subseteq \mathcal{F}$ and

  $$|\widetilde{N}_{\lambda,p}(z) + 1| < |z+1|$$

- $\mathcal{C}$ is invariant, $\widetilde{N}_{\lambda,p}(\mathcal{C}) = \mathcal{C} = \widetilde{N}^{-1}_{\lambda,p}(\mathcal{C})$.

  $\mathcal{C} = \{z \in \mathbb{C} | z = bi,\ b \in \mathbb{R}\}$

- $z \in \mathcal{C}$, then $\widetilde{N}_{\lambda,p}(z) = \frac{-b^2(2n-\lambda)+\lambda}{2bni} = (\frac{b^2(2n-\lambda)-\lambda}{2b^2n})i$. So $Re(\widetilde{N}_{\lambda,p}(z)) = 0$ and $\widetilde{N}_{\lambda,p}(z) \in \mathcal{C}$
- $\widetilde{N}_{\lambda,p}(z) \in \mathcal{C}$, as a consequence $\widetilde{N}_{\lambda,p}(z) = \frac{z^2(2n-\lambda)+\lambda)}{2nz} = Bi$. Now rearranging terms we have that:

$$z^2(2n-\lambda) - 2Bniz + \lambda = 0$$

and so $z \in \mathcal{C}$.

In addition $0 \in \mathcal{C}$, and 0 belongs to the inverse orbit of $\infty$ and so $0 \in \mathcal{J}$, then $\mathcal{C} = \mathcal{J}$. This is due to a theorem given in [2] and ensures that there is no exists any subset of the Julia set totally invariant.

On the other hand let's suppose that $\mathcal{J}(N_{\lambda,p}(z))$ is a straight , then its complementary $\mathcal{F}(N_{\lambda,p}(z))$ has exactly two components. If $p(z)$ had only one root, $p(z) = (z-a)^q$, $q > 1$. From Yang's theorem, $\mathcal{J}(N_{\lambda,p}(z)) = \infty$ and so, $\mathcal{F}(N_{\lambda,p}(z))$ wouldn't be a straight line.

Supposing that $m < n$, then:

$$N_{\lambda,p}(z) = z - \lambda \frac{(z-a)(z-b)}{m(z-a)+n(z-b)}$$

And now applying the Möbius transformation (4) we obtain:

$$\widetilde{N}_{\lambda,p}(z) = z - \lambda \frac{(z-1)(z+1)}{m(z-1)+n(z+1)}$$

We will proof it by contradiction.

$$|\widetilde{N}_{\lambda,p}(z) - 1| = \left|(z-1)\frac{n(z+1)+m(z-1)-\lambda(z+1)}{n(z+1)+m(z-1)}\right|$$

$$= \left|(z-1)\frac{n-\lambda+m\frac{z-1}{z+1}}{n+m\frac{z-1}{z+1}}\right| < |z-1|$$

- Let be $\bar{\mathcal{A}} = \{z | Re(z) \geq 0\}$ then:

  $$\left|\frac{(n-\lambda)+m\frac{z-1}{z+1}}{n+m\frac{z-1}{z+1}}\right| < 1 \Rightarrow \left|\widetilde{N}_{\lambda,p}(z) - 1\right| < |z-1|$$

  Then the closed set $\bar{\mathcal{A}}$ is a subset of the Fatou component that contains the fixed point $z = 1$.

- Let be $\bar{\mathcal{B}} = \{z | Re(z) \leq 0\}$. Using symmetry we obtain that it is a subset of the Fatou component that contains the fixed point $z = -1$. Taking into account that $\bar{\mathcal{A}} \cup \bar{\mathcal{B}} = \mathbb{C}$ and $\mathcal{F}(N_{\lambda,f}(z))$ wouldn't be a straight line. So this is a contradiction.

As the only critical points are $z \pm \sqrt{\frac{\lambda}{2n-\lambda}}$. It is well-known that there exist al least a critical point associated to each invariant Fatou component. As $z = 1$, y $z = -1$ are super-attractor fixed points of $\tilde{N}_{\lambda,p}$ each gives a Fatou component. ∎

## 3.1 The case $p(z) = (z-a)^n(z-b)^n,\ a \neq b,\ n \in \mathbb{N}$

If we apply the Newton's method to the above polynomial, we obtain the following expression:

$$N_{\lambda,p}(z) = z + \frac{(a-z)(b-z)\lambda}{n(a+b-2z)}.$$

Now using the transformation $M_1$, we have:

$$S_{\lambda,p}(z) = \frac{z(n+nz-\lambda)}{n+nz-z\lambda}.$$

For the sake of simplicity we will call $\alpha = \frac{n-\lambda}{n}$, and now our rational function turns to:

$$S_\alpha(z) = z\frac{\alpha+z}{1+\alpha z}. \tag{9}$$

Notice that this function only depends on the multiplicity of the roots and the damping factor. If we focus our attention in calculating fixed points, we have that the fixed points are $z = 0$, $z = 1$ and $z = \infty$. The points $z = 0$ and $z = \infty$ are related to the roots of the polynomial $p$ and $z = 1$ is related to $\infty$. The multipliers associated to every fixed point are:

- $\mu_0 = \alpha$.
- $\mu_1 = \frac{2}{1+\alpha}$.
- $\mu_\infty = \alpha$.

So we have that the character of $z = 0$ and $z = \infty$ is the same for every value of $\alpha$. If we take values of $\lambda \in \mathbb{R}$ we can see the changes of character in Table 1, on the other hand if we take $\alpha \in \mathbb{R}$ we see changes in Table 2, where $A$ means attractor, $SA$ super-attractor, $R$ repulsor and $I$ indifferent.

If we now study values of $\alpha \in \mathbb{C}$, we have the following result:

**Theorem 3.2** *The rational function $S_\alpha(z)$ defined in (9), with $\alpha \in \mathbb{C}$, has 3 fixed points: $z = 0$, $z = 1$ and $z = \infty$. The character associated to each fixed point is the following:*

*1. If $\alpha \in B(0,1)$*

| Interval of $\lambda$ | 0 | 1 | $\infty$ |
|---|---|---|---|
| $(-\infty, 0)$ | R | A | R |
| $0$ | I | I | I |
| $(0, n)$ | A | R | A |
| $n$ | SA | R | SA |
| $(n, 2n)$ | A | R | A |
| $2n$ | I | R | I |
| $(2n, 4n)$ | R | R | R |
| $4n$ | R | I | R |
| $(4n, \infty)$ | R | A | R |

Table 1: Character of the fixed points for different values of $\lambda \in \mathbb{R}$

| Interval of $\alpha$ | 0 | 1 | $\infty$ |
|---|---|---|---|
| $(-\infty, -3)$ | R | A | R |
| $-3$ | R | I | R |
| $(-3, -1)$ | R | R | R |
| $-1$ | I | R | I |
| $(-1, 0)$ | A | R | A |
| $0$ | SA | R | SA |
| $(0, 1)$ | A | R | A |
| $1$ | I | I | I |
| $(1, \infty)$ | R | A | R |

Table 2: Character of the fixed points for different values of $\alpha \in \mathbb{R}$

- $z = 0$, $z = \infty$ *are attractors.*
- $z = 1$ *is repulsor.*

2. *When* $\alpha \in \partial\overline{B(0,1)}$
   - $z = 0$, $z = \infty$ *are indifferent .*
   - $z = 1$ *is repulsor.*

3. *If* $\alpha \in B(-1,2) - \overline{B(0,1)}$
   - $z = 0$, $z = \infty$, $z = 1$ *are repulsor.*

4. *When* $\alpha \in \partial B(-1,2)$
   - $z = 0$, $z = \infty$ *are repulsor.*
   - $z = 1$ *is indifferent.*

5. *If* $\alpha \notin \overline{B(-1,2)}$

   - $z = 0$, $z = \infty$ *are repulsor.*
   - $z = 1$ *is attractor.*

In Figure 1 we can see the basin of attraction of the damped Newton's method applied to a polynomial with two different roots and the same multiplicity.

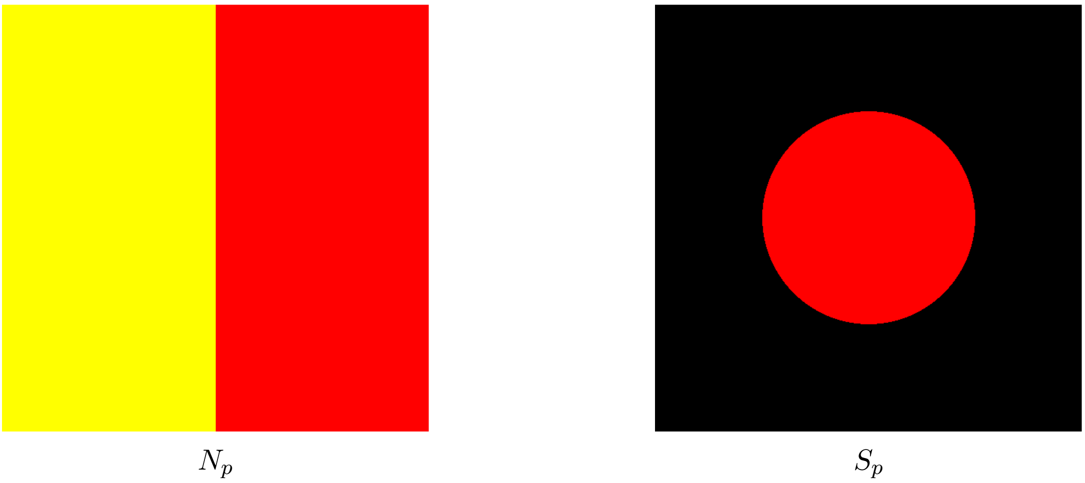


Figure 1: In the right hand basins of attraction of $N_p$ applied to $p(z) = z^2 - 1$ are shown and in the right hand appear the basins of attraction of $S_p$.

Finally, if we calculate the critical points we obtain that when $\alpha \neq 0$ it appears two new free critical points:

- $z_1 = \frac{\sqrt{1-\alpha^2}-1}{\alpha}$.
- $z_2 = \frac{-\sqrt{1-\alpha^2}-1}{\alpha}$.

Notice that when $\alpha = 0$

$$S_1'(z) = 2z$$

and so $z = 0$ is the unique possible critical point, but it is related to the root of the function. So in this case we can conclude that the damping factor introduces two free critical points.

### 3.2 The case $p(z) = (z-a)^n(z-b)^m,\ a \neq b,\ n > m$

Using the same ideas as in the previous section we will make a similar study but changing the multiplicities. In concrete the polynomial will be $p(z) = (z-a)^2(z-b)$ with $a \neq b$. We begin by using the transformation $M_1$:

$$S_\lambda(z) = z\frac{-2 - z + \lambda}{-2 + z(-1+\lambda)}. \tag{10}$$

The new rational function $S_\lambda(z)$ has three fixed points $z = 0$, $z = 1$ and $z = \infty$. The character of these fixed points depends on the damping factor:

- $\mu_0 = 1 - \frac{\lambda}{2}$.
- $\mu_1 = \frac{3}{3-\lambda}$.
- $\mu_\infty = 1 - \lambda$.

Table 3 shows the character of the fixed points for different values of $\lambda \in \mathbb{R}$. We observe that there exists problems with some values of the parameter. When $\lambda = 0$ the iterative function is the identity so every point is a fixed point. When $\lambda = 4$ the iterative function $S_4(z) = -z$ and so the iterations of every point, except 0 and $\infty$, goes to a 2−cycle. When $\lambda = 3$ we have that:

$$S_3 = \frac{-z}{2},$$

and $z = 1$ is not a fixed point of that function.

| $\lambda$-Interval | 0 | 1 | $\infty$ |
|---|---|---|---|
| $(-\infty, 0)$ | R | A | R |
| 0 | - | - | - |
| $(0,1)$ | A | R | A |
| 1 | A | R | SA |
| $(1,2)$ | A | R | A |
| 2 | SA | R | I |
| $(2,3)$ | A | R | R |
| 3 | A | - | R |
| $(3,4)$ | A | R | R |
| 4 | I | R | R |
| $(4,6)$ | R | R | R |
| 6 | R | I | R |
| $(6,\infty)$ | R | A | R |

Table 3: Character of the fixed points of $S_\lambda$ for different values of $\lambda$.

If we now focus our attention in values of $\lambda \in \mathbb{C}$ we obtain the following result:

**Theorem 3.3** *The rational function $S_\lambda(z)$ defined in (10), with $\lambda \in \mathbb{C}$, has 3 fixed points: $z = 0$, $z = 1$ and $z = \infty$. The character associated to each fixed point is the following:*

1. *If* $\lambda \in B(1,1)$
   - $z = 0$*,* $z = \infty$ *are attractors.*
   - $z = 1$ *is repulsor.*
2. *If* $\lambda \in \partial\overline{B(1,1)}$
   - $z = 0$ *is indifferent.*
   - $z = 1$ *is repulsor.*
   - $z = \infty$ *is attractor.*
3. *If* $\lambda \in B(2,2) - \overline{B(1,1)}$
   - $z = \infty$ *is attractor.*
   - $z = 0$*,* $z = 1$ *are repulsors.*
4. *If* $\lambda \in \partial B(2,2)$
   - $z = 0$*,* $z = 1$ *are repulsors.*
   - $z = \infty$ *is indifferent.*
5. *If* $\lambda \in B(3,3) - \overline{B(2,2)}$
   - $z = 0$*,* $z = 1$*,* $z = \infty$ *are repulsors.*
6. *If* $\lambda \in \partial B(3,3)$
   - $z = 0$*,* $z = \infty$ *are repulsors.*
   - $z = 1$ *is indifferent.*
7. *If* $\lambda \notin \overline{B(3,3)}$
   - $z = 0$*,* $z = \infty$ *are repulsors.*
   - $z = 1$ *is attractor.*

In Figure 2 it is shown the regions of the plane where fixed points changes its characters.

Finally we study the existence of free critical points of $S_\lambda$ and we obtain the following:

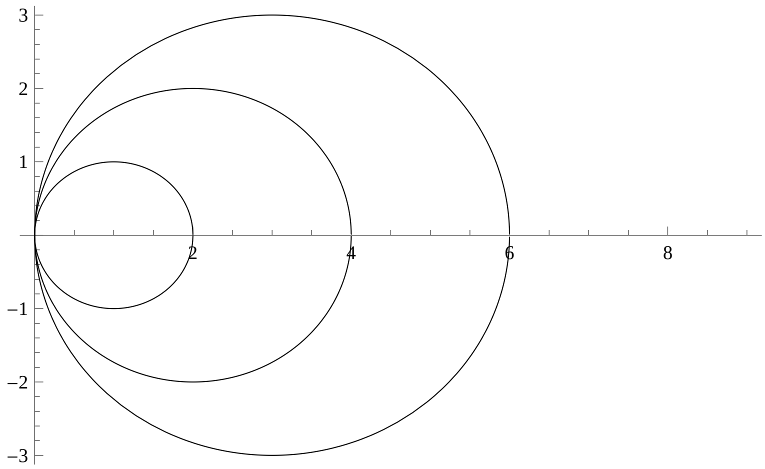



Figure 2: Regions of the plane where the character of the fixed points of $S_\lambda$ change.

- If $\lambda = 3$:
  - $S_3'(z) = \frac{-8}{(3z+1)^2}$, and so there is no critical points.
- When $\lambda \neq 3$ we obtain two free critical points:
  - $z_1 = \frac{-3+\lambda+2\sqrt{2}\sqrt{3\lambda-\lambda^2}}{3\lambda-9}$.
  - $z_2 = -\frac{-3+\lambda+2\sqrt{2}\sqrt{3\lambda-\lambda^2}}{3\lambda-9}$.

So we can conclude that the damping factor $\lambda$ introduces two new free critical points.

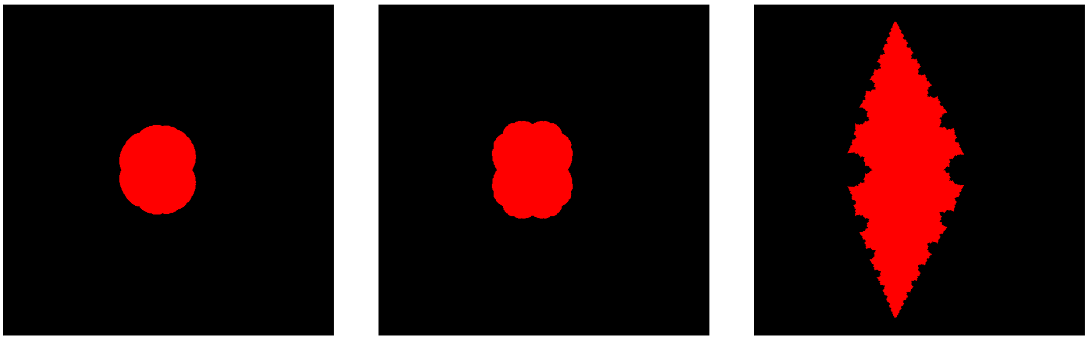
Finally we will see some pictures of the basins of attraction for different values of $\lambda \in \mathbb{R}$ to study the influence of the damping factor in the form of the Julia set. If we use the transformations defined in (4) and (5) we can define two new rational maps:

$$S_\lambda(z) = z\frac{-2-z+\lambda}{-2+z(-1+\lambda)}. \tag{11}$$

$$R_\lambda(z) = z + \frac{\lambda - z^2\lambda}{(1+3z)}. \tag{12}$$

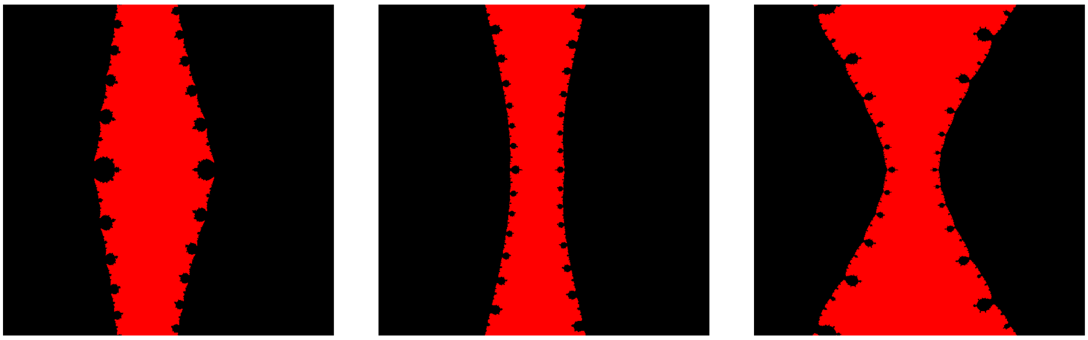
If we observe Figure 3 and Figure 4 we can observe that the damping factor has a clear influence in the Julia set of the function (11). When $\lambda = 0.5$ the Julia set seems to be like a cloud, but when the damping factor is near to 2 the Julia set is a Mandelbrot-like set.

On the other hand in Figure 5 and Figure 6 we observe the influence of $\lambda$ in (12). We observe that when $\lambda = 0.5$ the Julia set is like two rays and the basin of attraction of the double root is greater that the basin of the simple root. When $\lambda = 1.9$ the



$\lambda = 0.5$ $\lambda = 1.0$ $\lambda = 1.9$

Figure 3: Basins of attraction of $S_\lambda$ for different values of $\lambda$



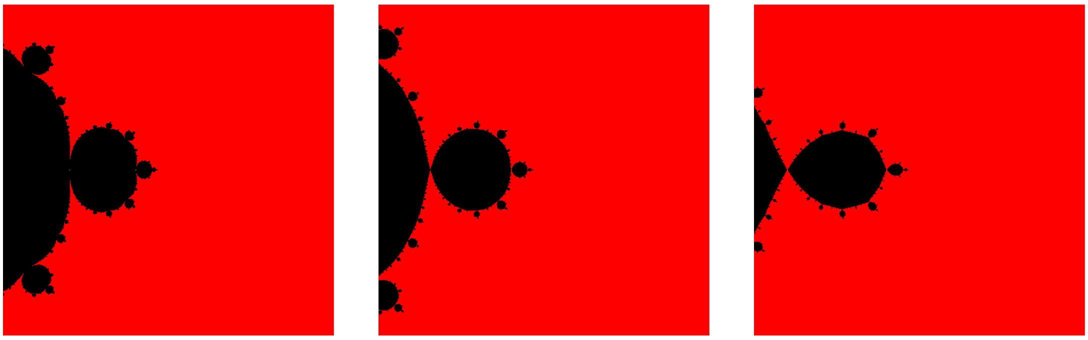
$\lambda = 2.0$ $\lambda = 2.1$ $\lambda = 2.2$

Figure 4: Basins of attraction of $S_\lambda$ for different values of $\lambda$

Julia set becomes a Mandelbrot-like set. And for $\lambda \geq 2$ we have that there is no convergence to the simple root. So we can conclude that the damping factor has an important influence in the dynamics of the damped Newton's method.

$\lambda = 0.5$ $\lambda = 1.0$ $\lambda = 1.9$

Figure 5: Basins of attraction of $R_\lambda$ for different values of $\lambda$



$\lambda = 2.0$ $\lambda = 2.1$ $\lambda = 2.2$

Figure 6: Basins of attraction of $R_\lambda$ for different values of $\lambda$

# 4 Conclusion

From the study the following conclusions are clear:

- If $p(z) = (z-a)^n(z-b)^n$:
  - $\lambda$ does not modify the appearance of the Julia set.
  - $\lambda$ modifies the character of the fixed points.
  - $\lambda$ introduces free critical points.
- If $p(z) = (z-a)^2(z-b)$:
  - The multiplicity modifies the form (symmetry) of the Julia set.
  - In addition $\lambda$ modifies even the form, is a Mandelbrot-like set instead of a set limited by two rays approximately.
  - $\lambda$ also modifies the character of the fixed points.
  - $\lambda$ also introduces free critical points.

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# References

[1] B. I. Epureanu and H. S. Greenside: "Fractal basins on the attraction associated with a damped Newton's method" *SIAM Rev.*, **40** (1998), 102–109.

[2] N. Fagella and X. Jarque: *Iteraciones compleja y fractales*, Vicens Vives, Barcelona, 2007.

[3] W. J. Gilbert: "Generalizations of Newton's method" *Fractals* **9** (2001), n. 3 251–262.
[4] K. Kneisl: "Julia sets for the super-Newton method, Cauchy's method and Halley's method" *Chaos* **11** (2001), no. 2, 359–370.
[5] F. von Haeseler and H. Kriete: "Surgery for relaxed Newton's method" *Complex Variables and Elliptic Equations* **22** (2010),n. 1 129–143.
[6] F. von Haeseler and H.-O. Peitgen: "Newton's method and complex dynamical systems" *Acta Appl. Math.* **13** (1988), 3–358.
[7] J. Hubbard, D. Schleider, S. Sutherland: "How to find all roots of complex polynomials by Newton's method" *Invent. math.* **146** (2001), 1–33.
[8] J. W. Neuberger: "Continuous Newton's Method for polynomials" *Math. Intelligencer*, **21** (1999), 18–23.
[9] S. Plaza and N. Romero: "Attracting cycles for the relaxed Newton's method" *J. Comput. Appl. Math.* **235** (2011), no. 10, 3238–3244.
[10] J. L. Varona: "Graphic and numerical comparison between iterative methods" *Math. Intelligencer* **24** (2002), no. 1, 37–46.
[11] W. Yang: "Symmetries in the Julia sets of Newton's method for multiple root" *Appl. Math. Comput.*, **217** (2010), 2490–2494.