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Surface Meshing with Metric Gradation Control

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Abstract

Scientific computing requires the automatic generation of high quality meshes, in particular isotropic or anisotropic meshes of surfaces defined by a CAD modeller. For this purpose, two major approaches are called direct and indirect. Direct methods (octree, advancing-front or paving) work directly in the tridimensional space, while indirect methods consist in meshing each parametric domain and mapping the resulting mesh onto the composite surface. Using the latter approach, we propose a general scheme for generating 'geometric" (or geometry-preserving) meshes by means of metrics. In addition, we introduce a new methodology for controlling the metric gradation in order to improve the shape quality. Application examples are given to show the capabilities of this approach.

Keywords: CAD surface, parametric surface meshing, curve discretization, anisotropic meshing, mesh gradation, geometric mesh, conforming mesh, discontinuous metrics.

1 Introduction

To obtain numerical solutions of partial differential equations using the finite element method or variants, high quality meshes are necessary. A mesh is a discretization of a geometric domain and can be either isotropic or anisotropic: in a nutshell, isotropic meshes are used in solid mechanics, whereas anisotropic meshes are preferred in fluid mechanics as directional fields must be captured. For defining the domain boundaries, CAD modelers are generally used, in which surfaces are represented by an assembly of parametric patches. In this case, meshes can be automatically generated using an indirect approach, which consists in meshing the planar parametric domain and mapping the resulting mesh onto the surface. This paper concerns the automatic generation of "geometric" (or geometry-preserving) meshes of composite parametric surfaces, following this indirect approach, with the aim of solving scientific computing problems.

Despite its simplicity, the problem with the indirect approach is the generation of a mesh which complies with the metric of the surface. Historically, people were only interested in surface visualization using this indirect approach [1, 2, 3, 4]. In fact, they aimed to minimize the error in the polyhedral approximation of the surface indirectly in the parametric space without paying attention to the quality of the resulting mesh. For people in finite element computation, however, the problem is to generate an anisotropic mesh in the parametric domain, taking into account the metric deformation from the surface to its parametric domain. To this end, various algorithms are proposed [5, 6, 7, 8]. In this paper, we present a general scheme of an indirect approach for generating isotropic and anisotropic geometric meshes of a surface constituted by a conforming assembly of parametric patches, based on the concept of metric.

A "geometric mesh" must satisfy two properties: every mesh element must be close to the surface, and also close to the tangent planes at its vertices. The first property ensures that the gap between the elements and the surface is bounded (this gap defined as the largest distance between any point of the element and the surface). The second property ensures that the surface is locally of order G^1 in terms of continuity. Both properties result in the definition of a mesh metric field depending of surface curvatures, or "geometric metrics". It then remains to generate a "unit mesh", where all the elements are of unit size with respect to these metrics.

A difficult problem with isotropic or anisotropic geometric metrics is that they can produce significant size variations in some areas of the surface and can even be discontinuous along the interface curves. The larger the rate of the mesh size variation, the worse is the shape quality of the resulting mesh. To control this size variation, various methodologies based on metric reduction have been proposed [9] in the case of a continuous isotropic metric. We introduce a novel approach of iterative mesh gradation for discontinuous metrics. This approach uses a specific metric reduction procedure in order to ensure the convergence of the gradation process. In particular, we show that in the worst case the anisotropic discontinuous geometric metric field is reduced to an isotropic continuous geometric metric field for which the gradation is controlled.

In Section 2, we recall the definition of a conforming composite parametric surface and introduce some notations. The general scheme for meshing composite parametric surfaces using an indirect approach is given in Section 3. The definition of geometric metrics is defined in Section 4, and our new scheme for metric gradation control is presented in Section 5. Several application examples are provided in Section 6 to illustrate the capabilities of the proposed method. Finally, in the last section, we conclude with a few words and indicate future prospects.

2 Conforming composite parametric surfaces

A composite parametric surface Σ is defined by an assembly of jointing patches $\{\Sigma_i\}$, where each patch is the image of a parametric domain Ω_i of $\mathbb{R}^2$ by an application σ_i which is assumed to be C^1-continuous:

$$\Sigma = \bigcup_i \Sigma_i \,, \quad \Sigma_i = \sigma_i(\Omega_i) \tag{1}$$

$$\sigma_i : \Omega_i \subset \mathbb{R}^2 \rightarrow \Sigma_i \subset \mathbb{R}^3 \,, \quad \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \sigma_i(u,v) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2}$$

Similarly, each domain Ω_i is defined by its contour, closed and non self-intersecting, constituted by an assembly of contiguous curve segments $\{\gamma_{ij}\}$, where each curve segment is the image of an interval of $\mathbb{R}$ by a C^1-continuous application ω_{ij}:

$$\overline{\Omega_i} = \bigcup_j \gamma_{ij} \,, \quad \gamma_{ij} = \omega_{ij}([a_{ij}, b_{ij}]) \tag{3}$$

$$\omega_{ij} : [a_{ij}, b_{ij}] \subset \mathbb{R} \rightarrow \gamma_{ij} \subset \mathbb{R}^2 \,, \quad t \mapsto \omega_{ij}(t) = \begin{pmatrix} u \\ v \end{pmatrix} \tag{4}$$

Since the contour is non self-intersecting, we have for each pair of curve segments γ_{ij} and γ_{ik}:

$$\gamma_{ij} \cap \gamma_{ik} \quad = \quad \emptyset \quad \text{or} \quad e_{il} \tag{5}$$

where $\emptyset$ denotes the empty set and e_{il} a common extremity of γ_{ij} and γ_{ik}.

We suppose in the following that surface Σ is conforming (see [10] for setting the conformity of patches). By definition, surface Σ is called conforming if and only if, for each pair of patches Σ_i and Σ_j, we have:

$$\Sigma_i \cap \Sigma_j \quad = \quad \emptyset \quad \text{or} \quad \bigcup_k E_{ij,k} \quad \text{or} \quad \bigcup_k \Gamma_{ij,k} \tag{6}$$

where $\exists\, l, m$ such that $E_{ij,k} = \sigma_i(e_{il}) = \sigma_j(e_{jm})$
and $\exists\, l, m$ such that $\Gamma_{ij,k} = \sigma_i(\gamma_{il}) = \sigma_j(\gamma_{jm})$.

Therefore, $\Gamma_{ij,k}$ is a boundary curve segment shared by Σ_i and Σ_j, image of two boundary curve segments γ_{il} of Ω_i and γ_{jm} of Ω_j. Thus, by considering common curve segments only once, we obtain:

$$\bigcup_i \overline{\Sigma_i} \quad = \quad \bigcup_j \Gamma_j \tag{7}$$

where

$$\Gamma_j \cap \Gamma_k \quad = \quad \emptyset \quad \text{or} \quad E_{jk} \tag{8}$$

and there exists a set of indices (i, k) such that each Γ_j is equal to $\sigma_i(\gamma_{ik})$.

3 General scheme for meshing composite surfaces

The generation of a geometric mesh of a composite parametric surface Σ is given by the following general scheme:

1. Specification of the initial geometric metric field associated with all the points of Σ.
2. Meshing of Σ complying with the specified metric field.
3. Definition of a new metric field resulting from mesh gradation with respect to a given ratio threshold using the current mesh of Σ.
4. If the new metric field is close to the previous one then exit, else repeat the process from step 2.

The initial geometric field is defined on the whole surface. The graded metric field is defined by interpolating the graded map from the vertices of the current mesh. The purpose of the repetition of steps 2, 3 and 4 is not only to control the mesh gradation but also to accurately capture the surface geometry. In practice, the above procedure converges after two or three iterations and it is not necessary to compare the new metric to the previous one. Step 2, meshing a surface with respect to a given metric field, in the present case of a composite parametric surface and using an indirect approach, includes the following substeps:

2.1 Discretization of each Γ_j.

2.2 Transfer of the discretization of each Γ_j onto corresponding segments γ_{ik}.

2.3 Mesh generation of each Ω_i from the discretization of its boundary (obtained in the previous step).

2.4 Mapping the mesh of each Ω_i onto Σ_i.

2.5 Construction of the mesh of Σ from meshes of Σ_i.

Note that curves are discretized in $\mathbb{R}^3$ while patches are meshed in $\mathbb{R}^2$ via the parametric domains, hence the term of "indirect approach". In the following, we detail in three different sections the specification of a geometric metric field, the new mesh gradation scheme for a general anisotropic and discontinuous metric field, and the meshing strategy.

4 Geometric metric field

Within a classical framework, mainly two categories of size fields or metric fields can be considered. The first category concerns uniform meshes with a given constant size h or a given constant metric $\mathcal{M} = \frac{1}{h^2}\,\mathcal{I}_3$ (the size specification results in a given metric and a mesh complying with this size is a mesh whose edge length equals unity in this metric). The advantage of this kind of meshing is that it provides, in general, equilateral meshes. On the other hand, it cannot guarantee a good representation of the geometry of the domain for a given size. The second category concerns meshes referred to as geometric, adapted to the geometry of the patches composing the surface. To define the size or the metric at a given point of the surface, three cases are discussed hereafter, depending on the position of the point on the patches: internal point, interface or boundary point and extremity point.

Internal point. An internal point P is a point belonging to the interior of a patch Σ_i. In an isotropic framework, it can be demonstrated that the local geometric size at P must be proportional to the minimal radius of curvature $\rho_1(P)$ of patch Σ_i [11]:

$$\mathcal{M}_{iso}(\Sigma_i, P) = \frac{1}{h_1^2(P)}\,\mathcal{I}_3 \quad \text{with} \quad h_1(P) = \lambda_1\,\rho_1(P) \tag{9}$$

where $\lambda_1 = 2\,\sin\theta$, θ being the maximum angle between an element and the tangent planes at its vertices, or equivalently $\lambda_1 = 2\,\sqrt{\varepsilon\,(2-\varepsilon)}$, ε being the maximum relative distance between an element and the surface. In an anisotropic framework, the metric can also be deduced from the principal radii of curvature ($\rho_1(P)$ and $\rho_2(P)$, assuming $\rho_1(P) < \rho_2(P)$) and the principal directions of curvature (defined by two orthogonal unit vectors $\boldsymbol{e_1}(P)$ and $\boldsymbol{e_2}(P)$) of patch Σ_i [12]:

$$\mathcal{M}_{aniso}(\Sigma_i, P) = \begin{pmatrix} \boldsymbol{e_1}(P) & \boldsymbol{e_2}(P) \end{pmatrix} \begin{pmatrix} \frac{1}{h_1^2(P)} & 0 \\ 0 & \frac{1}{h_2^2(P)} \end{pmatrix} \begin{pmatrix} \boldsymbol{e_1}(P)^T \\ \boldsymbol{e_2}(P)^T \end{pmatrix} \tag{10}$$

with $h_1(P) = \lambda_1\,\rho_1(P)$ and $h_2(P) = \lambda_2\,\rho_2(P)$, where λ_1 can be defined again by $\lambda_1 = 2\,\sqrt{\varepsilon\,(2-\varepsilon)}$ and λ_2 is a smaller coefficient given by $\lambda_2 = 2\,\sqrt{\varepsilon\,\frac{\rho_1}{\rho_2}\,(2-\varepsilon\,\frac{\rho_1}{\rho_2})}$. The above anisotropic geometric metric is degenerate since the size is not defined in the direction perpendicular to both of the vectors $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. In order to obtain a well-defined metric consistent with the isotropic case, we redefine the anisotropic geometric as:

$$\mathcal{M}_{aniso}(\Sigma_i, P) = \begin{pmatrix} \boldsymbol{e_1}(P) & \boldsymbol{e_2}(P) & \boldsymbol{n}(P) \end{pmatrix} \begin{pmatrix} \frac{1}{h_1^2(P)} & 0 & 0 \\ 0 & \frac{1}{h_2^2(P)} & 0 \\ 0 & 0 & \frac{1}{h_1^2(P)} \end{pmatrix} \begin{pmatrix} \boldsymbol{e_1}(P)^T \\ \boldsymbol{e_2}(P)^T \\ \boldsymbol{n}(P)^T \end{pmatrix} \tag{11}$$

where $\boldsymbol{n}(P)$ is the unit normal to the surface at point P. In practice, the sizes in the above metrics are bounded by specified minimal and maximal size values and thus these metrics are always well defined.

Interface or boundary point. An interface or boundary point C is a point belonging to the interior of a curve segment Γ_j. For an interface point, curve Γ_j is shared by at least two patches while for an boundary point, curve Γ_j belongs to only one patch. Let us denote by $\{\Sigma_{ij}\}$ the set of patches containing Γ_j. The geometric size at C depends on the geometric size of each Σ_{ij} and also the geometric size of curve Γ_j. If $\rho(C)$ is the radius of curvature of curve Γ_j at C, the geometric size of curve Γ_j is defined by:

$$\mathcal{M}(\Gamma_j, C) = \frac{1}{h^2(C)}\,\mathcal{I}_3 \quad \text{with} \quad h(C) = \lambda_1\,\rho(C)\,. \tag{12}$$

Hence, at an interface or boundary point C, several geometric metrics are defined ($\mathcal{M}_{iso}(\Sigma_{ij}, C)$ or $\mathcal{M}_{aniso}(\Sigma_{ij}, C)$ and $\mathcal{M}(\Gamma_j, C)$).

Extremity point. An extremity point E is a common extremity of a set of curves $\{\Gamma_j\}$. Each $\{\Gamma_j\}$ belongs to a set of patches $\{\Sigma_{ij}\}$. Therefore, the geometric size at E depends on the geometric size of each curve Γ_j and the geometric size of corresponding patches Σ_{ij}. Similarly, at an extremity point E, several geometric metrics ($\mathcal{M}_{iso}(\Sigma_{ij}, E)$ or $\mathcal{M}_{aniso}(\Sigma_{ij}, E)$ for all i, j such that Σ_{ij} contains a curve $\{\Gamma_j\}$ with E as extremity and $\mathcal{M}(\Gamma_j, E)$ for all j such that E is an extremity of Γ_j) are defined.

Using the above metrics for meshing, a difficult problem is that a high variation of the curvature implies a high variation of the prescribed size, hence a deterioration in the quality of the elements. To remedy this, it is sufficient to modify the metric field according to the desired size variation. Indeed, the latter can be controlled by methods of size smoothing or mesh gradation. This issue is detailed in the next section.

5 Metric gradation control: a new scheme

The metrics defined in the previous section can locally produce important size variations, in particular in the present context of geometric mesh generation. These size variations entail a generation of elements having a poor shape quality. To remedy this, metrics can be modified while accounting for the size constraints at best and while controlling the underlying gradation, which measures the size variation in the vicinity of a vertex [9]. In the following, for each case of isotropic or anisotropic geometric metric fields (defined by interpolating a metric map associated with the vertices of a mesh), the metric gradation strategy is detailed and the corresponding algorithm is given.

5.1 Isotropic geometric metrics

In the isotropic case, the geometric metric at each vertex of the mesh is defined as follows, depending on the position of the vertex on the patches:

- **Internal vertex.** If the vertex is a point P belonging to the interior of a patch Σ_i, its metric is unique and defined by $\mathcal{M}_{iso}(\Sigma_i, P)$.
- **Interface or boundary vertex.** If the vertex is a point C belonging to the interior of a curve segment Γ_j, several metrics $\overline{\mathcal{M}}_{iso}(\Sigma_{ij}, C)$ are defined. However, in the case of isotropic geometric metrics, these metrics are identical for all patches $\Sigma_{i,j}$ because all these isotropic metrics give the same length in the direction of the tangent to Γ_j. This common metric is then denoted by $\overline{\mathcal{M}}_{iso}(\Sigma_{*j}, C)$.
- **Extremity vertex.** If the vertex is a point E being a common extremity of a set of curves $\{\Gamma_j\}$, several metrics $\overline{\mathcal{M}}_{iso}(\Sigma_{*j}, E)$ corresponding to every $\{\Gamma_j\}$ are

defined. Among these metrics, there exists a metric denoted by $\overline{\mathcal{M}}_{iso}(\Sigma_{**}, E)$ which gives the smallest length in all directions. The latter metric is taken into account in the gradation control methodology.

General algorithm for metric gradation. The modification of the geometric metrics consists in locally modifying these metrics by considering the size variation on each edge of the mesh. For each edge, the modification includes two successive steps: the calculation of the shock and, if the shock is too strong, a metric update. Using the above notations, the gradation algorithm for isotropic metrics can be written in simplified pseudo-code as shown on Figure 1. Its inputs are a metric field (a mesh and geometric metrics $\mathcal{M}_{iso}$ at the mesh vertices) and a threshold c_{goal}. The outer loop runs until $c_{max} \leq c_{goal}$, where c_{max} is the maximum shock on all the edges. Consequently, in output, metrics are modified so that the gradation is bounded by the given threshold c_{goal}. The two procedures called by this algorithm to compute the shock and to update the metric are detailed in the following.

Calculation of the shock. Let PQ be an edge, and let $\mathcal{M}(P)$ and $\mathcal{M}(Q)$ be the metrics at its extremities. If $h(P)$ and $h(Q)$ respectively represent the sizes specified by these metrics (in all directions and in particular in the direction of vector $\boldsymbol{e} = \overrightarrow{PQ}$), let us assume without loss of generality that $h(P) \leq h(Q)$. The shock (also called H-shock) $c(PQ)$ related to the edge PQ is the value:

$$c(PQ) = \left(\frac{h(Q)}{h(P)}\right)^{1/l(PQ)} \tag{13}$$

where $l(PQ)$ is the length of edge PQ in a metric interpolating the size given by the two extremity metrics $\mathcal{M}(P)$ and $\mathcal{M}(Q)$ in direction $\boldsymbol{e} = \overrightarrow{PQ}$:

$$l(PQ) = ||\boldsymbol{e}|| \int_0^1 \frac{1}{h(P + t\,\boldsymbol{e})}\, dt \tag{14}$$

```
Input: mesh, M_iso, c_goal
Repeat {
    c_max = 0
    For each edge PQ of the mesh {
        Compute c(PQ), the shock on PQ
        If (c(PQ) > c_goal) update M_iso(Q)
        c_max = max(c_max, c(PQ))
    }
} until (c_max ≤ c_goal)
Output: M_iso,gra = M_iso
```

Figure 1: Gradation algorithm in the isotropic case.

Metric update. If the shock $c(PQ)$ is greater than the given threshold c_{goal}, then the size $h(Q)$ is multiplied by η, or equivalently the metric $\mathcal{M}(Q)$ is divided by η^2, where η is a size reduction factor given by:

$$\eta = \left(\frac{c_{goal}}{c(PQ)}\right)^{l(PQ)} \quad < \quad 1 \tag{15}$$

Actually, the shock $c(PQ)$ expressed by Equation (13) can have a value which is greater than necessary, so the algorithm can be improved by a penalization limiting η. In practise, this lower bound of η is equal to $1/c_{goal}$.

5.2 Anisotropic geometric metrics

In the anisotropic case, the geometric metrics at each vertex of the mesh are defined as follows:

- **Internal vertex.** If the vertex is a point P belonging to the interior of a patch Σ_i, its metric is unique and defined by $\mathcal{M}_{aniso}(\Sigma_i, P)$.
- **Interface or boundary vertex.** If the vertex is a point C belonging to the interior of a curve segment Γ_j, several metrics $\overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, C)$ are defined. Indeed, the anisotropic metric is discontinuous at C.
- **Extremity vertex.** If the vertex is a point E being a common extremity of a set of curves $\{\Gamma_j\}$, several metrics $\overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, E)$ corresponding to every $\{\Gamma_j\}$ are defined. In addition, for all patch Σ_{ij} containing E, the metrics $\mathcal{M}_{aniso}(\Sigma_{ij}, E)$ are also considered. Notice also that the metric is discontinuous at E.

General algorithm for metric gradation. The gradation algorithm in the anisotropic case is written in simplified pseudo-code on Figure 2 with inputs and outputs similar to the isotropic case. The two procedures concerning the calculation of the shock and the metric update are clarified hereafter.

Calculation of the shock. Let PQ be an edge of a mesh of a patch Σ_i. For each extremity, for instance point P, are defined a metric $\mathcal{M}(P)$ and a direction $\boldsymbol{v}(P)$ as follows:

- If PQ is an internal edge of the mesh, $\boldsymbol{v}(P) = \overrightarrow{PQ}$ and there are three possibilities for metric $\mathcal{M}(P)$: if P belongs to the interior of Σ_i, $\mathcal{M}(P) = \mathcal{M}_{aniso}(\Sigma_i, P)$; if P belongs to the interior of a curve Γ_j, $\mathcal{M}(P) = \overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, P)$; if P is an extremity, $\mathcal{M}(P) = \mathcal{M}_{aniso}(\Sigma_i, P)$ independently from any curve Γ_j.
- Otherwise, PQ belongs to the discretization of a curve Γ_j. Direction $\boldsymbol{v}(P)$ is given by the tangent to Γ_j at P (see Section 6.2) and metric $\mathcal{M}(P)$ is defined by $\overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, P)$.

Input: mesh, $\mathcal{M}_{aniso}$, c_{goal}
Run the gradation algorithm in the isotropic case, giving $\mathcal{M}_{iso,gra}$
Repeat {
    $c_{max,1} = 0$
    For each patch Σ_i of the mesh {
        Repeat {
            $c_{max,2} = 0$
            For each edge PQ of patch Σ_i {
                Compute $c(PQ)$, the shock on PQ
                If ($c(PQ) > c_{goal}$) update $\mathcal{M}_{aniso}(Q)$ or else $\mathcal{M}_{aniso}(P)$
                $c_{max,2} = \max(c_{max,2},\ c(PQ))$
            }
            $c_{max,1} = \max(c_{max,1},\ c_{max,2})$
        } until ($c_{max,2} \leq c_{goal}$)
    }
} until ($c_{max,1} \leq c_{goal}$)
Adjust $\mathcal{M}_{aniso}(E)$, the metrics at the extremities
Output: $\mathcal{M}_{aniso,gra} = \mathcal{M}_{aniso}$

Figure 2: Gradation algorithm in the anisotropic case.

Denoting by $h(P)$ the size specified by metric $\mathcal{M}(P)$ in direction $\boldsymbol{v}(P)$, sizes $h(P)$ and $h(Q)$ are defined at both extremities of PQ and the shock $c(PQ)$ is calculated like in the isotropic case using Equations (13) and (14).

Metric update. If the shock $c(PQ)$ is greater than the given threshold c_{goal}, a metric update is necessary. The procedure detailed in Section 5.1 for the isotropic case is rather straightforward: assuming that $h(P) < h(Q)$, the metric associated with Q is divided by a factor η^2. In the anisotropic case, this procedure is more complicated, as explained in the following.

Firstly, the metric field is discontinuous along interface curves and thus, several metrics are associated with a given point. The metric to be updated for a point P is $\mathcal{M}(P)$ whose definition is given above, depending on edge PQ (internal or not) and on point P (internal to a patch, internal to a curve, or extremity). Careful attention must be paid if $\mathcal{M}(P) = \overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, P)$, a metric defined on a curve Γ_j. In this case, the updated metric gives a new metric length in the direction of the tangent to Γ_j at P, and all the patches sharing Γ_j must be updated so that their local metrics give the same metric length at P.

A second point is that a simple homothetic reduction of a metric $\mathcal{M}(P)$ does not guarantee the convergence of the gradation process. Indeed, a reduction on one patch implies other reductions on adjacent patches, which may imply a reduction on the first patch, resulting in an endless loop. To avoid this, the key idea is to run beforehand

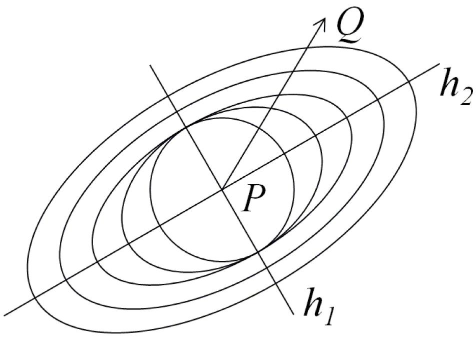

Figure 3: Reduction of a metric complying with a lower bound h_{lim}.

an isotropic gradation defining an isotropic metric at each vertex. The latter is used as a lower limit for the anisotropic gradation. This methodology guarantees the convergence of the process and ensures that a smaller number of elements is generated in the anisotropic case. More precisely, let h_{lim} be the size limit given by the isotropic metric, let $h_1 < h_2$ be the sizes along the principal axes of a metric $\mathcal{M}$, and let η be the size reduction factor. To reduce metric $\mathcal{M}$ with a factor $\eta < 1$, first a homothetic reduction replaces h_1 by $h'_1 = \eta\, h_1$ and h_2 by $h'_2 = \eta\, h_2$. However, if $h'_1 < h_{lim}$, h'_1 is set to h_{lim} and h'_2 is computed so that the size in direction $\boldsymbol{e} = \overrightarrow{PQ}$ is $h'(P) = \eta\, h(P)$. This procedure is illustrated on Figure 3. Metric $\mathcal{M}$ is represented by the outer ellipse and h_{lim} is the radius of the inner circle. If η is near 1 then a homothetic reduction is made, but if η becomes smaller the metric becomes “more isotropic”. The prior isotropic gradation guarantees that it is never necessary to go below the size limit h_{lim}.

Thirdly, a problem may occur during the anisotropic gradation process. If a shock $c(PQ) > c_{goal}$ is detected on an edge PQ such that $h(P) < h(Q)$, a metric update at Q may be impossible because the size limit is reached: $h_1 = h_2 = h_{lim}$. This may happen because the edges are not analyzed in the same order in the isotropic and anisotropic gradations. In this case, it is still possible to update the metric at the other extremity P; an iterative procedure finds a new reduction factor $\eta_P < 1$ such that the shock on edge PQ is less than c_{goal}, and metric $\mathcal{M}(P)$ is reduced with this factor η_P.

6 Meshing strategy using an indirect approach

This section recalls in detail the general meshing scheme introduced in Section 3.

6.1 Discretization of curve segments Γ_j

The discretization of each curve segment Γ_j consists in subdividing Γ_j by curve segments of unit length with respect to a specified isotropic metric function. For each point C of a curve, this metric length is obtained regarding the metric at the point C in the direction of the tangent to the curve. In the geometric case, as mentioned above, several metrics are defined ($\mathcal{M}_{iso}(\Sigma_{ij}, C)$ or $\mathcal{M}_{aniso}(\Sigma_{ij}, C)$ on adjacent patches, and $\mathcal{M}(\Gamma_j, C)$ on the curve). Thus the "metric length" at C is the minimum length specified by these metrics in the direction of the tangent at C to the curve. To compute the length of a curve segment with respect to a metric, a polyline approximating the curve is constructed and the length of this polyline is calculated.

6.2 Inverse mapping of discretized Γ_j into parametric domains

The discretization of Γ_j is defined by a set of vertices ordered by their curvilinear abscissae. This discretization is mapped back to the corresponding curve segments γ_{ik} in parametric domains [13]. The discretization of all curve segments γ in the parametric domains being well defined, the corresponding metrics in parametric domains must now be provided. These bidimensional metrics will be calculated from metrics in the tridimensional space that are defined in the following.

For an interface or boundary point C of a curve segment Γ_j belonging to a given patch Σ_{ij}, the metric $\mathcal{M}_{iso}(\Sigma_{ij}, C)$ or $\mathcal{M}_{aniso}(\Sigma_{ij}, C)$ is shrunk to fit the metric length at C in the direction of the tangent to the curve giving the new geometric metric $\overline{\mathcal{M}}_{iso}(\Sigma_{ij}, C)$ or $\overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, C)$. For an extremity point E of a patch Σ_{ij}, the same procedure is applied considering each interface or boundary curve Γ_j of Σ_{ij} such that E is an extremity of Γ_j leading to different geometric metrics $\overline{\mathcal{M}}_{iso}(\Sigma_{ij}, E)$ or $\overline{\mathcal{M}}_{aniso}(\Sigma_{ij}, E)$ and we consider the new geometric metric at E the metric $\overline{\overline{\mathcal{M}}}_{iso}(\Sigma_{ij}, E)$ or $\overline{\overline{\mathcal{M}}}_{aniso}(\Sigma_{ij}, E)$ giving the smallest size along the tangent direction at each curve Γ_j. Thus for an extremity point E, the geometric metric with respect to a patch Σ_{ij} is such that the minimal metric length at E is satisfied.

As an ilustration, Figure 4 (left) shows an interface curve Γ shared by two patches Σ_1 and Σ_2. Using the previous notations, Γ is in fact equal to a curve Γ_j, and Σ_1 (resp. Σ_2) is equal to a patch $\Sigma_{i_1 j}$ (resp. $\Sigma_{i_2 j}$. As explained in Section 6.1, the metric length at a point C belonging to the interior of Γ is the minimum length specified by the three metrics $\mathcal{M}_1 = \mathcal{M}_{aniso}(\Sigma_{i_1 j}, C)$, $\mathcal{M}_2 = \mathcal{M}_{aniso}(\Sigma_{i_2 j}, C)$ and $\mathcal{M}(\Gamma_j, C)$ in the direction of the tangent τ at C to the curve. In this example, the minimum length l_{min} is given by the latter metric. Consequently, the shrunk metrics $\overline{\mathcal{M}}_1 = \overline{\mathcal{M}}_{aniso}(\Sigma_{i_1 j}, C)$ and $\overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{aniso}(\Sigma_{i_2 j}, C)$ are represented by ellipsoids centered at C and passing through a same point of τ at a distance l_{min} of C.

On Figure 4 (right), an extremity E is shared by two curves $\Gamma_1 = \Gamma_{j_1}$ and $\Gamma_2 = \Gamma_{j_2}$ at the boundary of patch $\Sigma = \Sigma_{ij_1} = \Sigma_{ij_2}$. The previous process gives one point on tangent τ_1 to Γ_1 and a second point on tangent τ_2 to Γ_2, and the corresponding metrics $\overline{\mathcal{M}}_1 = \overline{\mathcal{M}}_{aniso}(\Sigma_{ij_1}, E)$ and $\overline{\mathcal{M}}_2 = \overline{\mathcal{M}}_{aniso}(\Sigma_{ij_2}, E)$. In this new example, the minimal metric length is given by the second metric and thus the geometric metric at E with respect to patch Σ is $\overline{\overline{\mathcal{M}}} = \overline{\mathcal{M}}_2$ or $\overline{\overline{\mathcal{M}}}_{aniso}(\Sigma, E) = \overline{\mathcal{M}}_{aniso}(\Sigma_{ij_2}, E)$.

Figure 4: Left: anisotropic metrics at an interface point C belonging to the interior of a curve segment Γ shared by two patches Σ_1 and Σ_2. Right: anisotropic metrics at a common extremity E of two curves Γ_1 and Γ_2 bounding a patch Σ.

6.3 Mesh generation of domains Ω_i

We use an indirect method for meshing general parametric surfaces complying with a pre-specified metric field $\mathcal{M}_3$ in the tridimensional space (for more details, see [11]). Let Σ be such a surface parameterized by:

$$\sigma \,:\, \Omega \longrightarrow \Sigma, \qquad (u,v) \longmapsto \sigma(u,v)\,, \tag{16}$$

where Ω denotes the parametric domain. The Riemannian metric specification $\mathcal{M}_3$ gives the unit measure in any direction. In the geometric case this metric is defined as:

- internal point: $\mathcal{M}_{iso}(\Sigma_i)$ or $\mathcal{M}_{aniso}(\Sigma_i)$.
- interface or boundary point: $\overline{\mathcal{M}}_{iso}(\Sigma_i)$ or $\overline{\mathcal{M}}_{aniso}(\Sigma_i)$.
- extremity point: $\overline{\overline{\mathcal{M}}}_{iso}(\Sigma_i)$ or $\overline{\overline{\mathcal{M}}}_{aniso}(\Sigma_i)$.

The goal is to generate a mesh of Σ such that the edge lengths are equal to one with respect to the related Riemannian space (such meshes being referred to as "unit" meshes). Based on the intrinsic properties of the surface, namely the first fundamental form:

$$\mathcal{M}_\sigma = \begin{pmatrix} \sigma_u^T\,\sigma_u & \sigma_u^T\,\sigma_v \\ \sigma_v^T\,\sigma_u & \sigma_v^T\,\sigma_v \end{pmatrix}, \tag{17}$$

the Riemannian structure $\mathcal{M}_3$ is induced into the parametric space as follows:

$$\widetilde{\mathcal{M}}_2 = \begin{pmatrix} \sigma_u^T \\ \sigma_v^T \end{pmatrix} \mathcal{M}_3 \begin{pmatrix} \sigma_u & \sigma_v \end{pmatrix}. \tag{18}$$

The above equation is the product of three matrices respectively of order 2×3, 3×3 and 3×2, resulting in a metric of order 2×2 in the parametric domain.

Even if the metric specification $\mathcal{M}_3$ is isotropic, the induced metric in parametric space is in general anisotropic, due to the variation of the tangent plane along the surface. Finally, a unit mesh is generated completely inside the parametric space such that it complies with the induced metric $\mathcal{M}_2$. This mesh is constructed using a combined advancing-front – Delaunay approach applied within a Riemannian context: the field points are defined after an advancing front method and are connected using a generalized Delaunay type method.

This method is efficient if the metric $\mathcal{M}_\sigma$ of the first fundamental form of the surface is well defined and its variation is bounded. If this is not the case, one can consider the metric in the vicinity of the degenerated points.

6.4 Mapping back the mesh of each Ω_i onto Σ_i

The mesh of each Σ_i is constituted by vertices, images by σ_i of the vertices of the mesh of Ω_i, keeping the same connectivity. This methodology is functional if the tangent plane metric does not involve strong variations (i.e., the image of an edge of the mesh of the parametric domain is close to the straight segment joining the images of its extremities).

6.5 Construction of the mesh of Σ from meshes of Σ_i

The global mesh of Σ is obtained by gathering all the meshes of patches Σ_i. In this process, vertices of the discretizations of the boundary curves must not be duplicated.

7 Application examples

The above methods for calculating a geometric metric field on a composite parametric surface, controlling the metric gradation, and meshing the surface w.r.t. the metric field, have all been implemented in the BLSURF software [14]. Two examples are presented in this section to show the capabilities of this approach. In the first example, the input is an IGES file representing a CAD surface of 55 patches modeling a grooved cylinder (file “95147.igs” courtesy of Distene). In the second example, another file represents 2381 patches modeling a combustion engine (file “D.igs” courtesy of Altair). For each test shown in this section, the BLSURF surface mesher reads the input file using the Open Cascade platform, sets the conformity of the patches and generates a geometric mesh which can be isotropic or anisotropic, with or without gradation.

7.1 Grooved cylinder (55 patches)

In this example, a grooved cylinder is modeled by a CAD surface of 55 patches. Several geometric meshes are generated, whose main characteristics are summarized in Table 1.

max. angle (degrees)	mesh type	max. metric gradation	number of vertices	number of triangles	CPU time (seconds)
8	iso	∞	3889	7774	0.249
8	iso	1.5	4741	9478	3.884
8	iso	1.2	6069	12134	3.978
8	aniso	∞	984	1964	0.046
8	aniso	1.5	1459	2914	3.276
8	aniso	1.2	2569	5134	3.369
4	iso	∞	14803	29602	0.920
4	iso	1.5	16751	33498	5.553
4	iso	1.2	19388	38772	5.803
4	aniso	∞	2383	4762	0.124
4	aniso	1.5	3466	6928	3.526
4	aniso	1.2	5531	11058	3.682

Table 1: Grooved cylinder (55 patches) – Meshing statistics.

In the first half of this table, a maximum angle $\theta = 8°$ is specified, where θ bounds the angular gap between each triangle and the tangent planes at its vertices (cf. Section 4). The very first mesh has isotropic metric specifications, without gradation, and is shown at the top of Figure 5. Despite its isotropic specifications, elongated elements can be noticed in the areas where the variations of the surface curvature are sharp. To remedy this, a gradation of 1.5 is applied on the metric field, showing an improvement of the shape quality (Figure 5, Mesh 2). With a smaller threshold of 1.2, the mesh triangles become almost equilateral (Figure 5, Mesh 3). On Figure 6, the first three meshes are now anisotropic meshes with the same gradation thresholds ∞, 1.5 and 1.2. Elements are clearly oriented along the principal directions of curvature. Comparing the number of elements in the isotropic and anisotropic cases, the ratio is about 25% without gradation although the geometric accuracy is the same.

In the second half of Table 1, the maximum angle θ is set to $4°$. Since this angle is divided by 2, the number of elements is approximatively multiplied by 4 for each test. The mesh at the bottom of Figure 5 (resp. 6) corresponds to a maximum angle of $4°$, isotropic (resp. anisotropic) elements, and a maximum metric gradation of 1.2.

Each CPU time of Table 1 is measured on a Dell Precision mobile workstation M6400 at 2.53 GHz. It represents the total time for mesh generation: in the case of graded metrics, this time includes the computation of the initial geometric metric field, the generation of the initial mesh and two adaptations, each consisting of a metric modification and a mesh generation (cf. Section 5).

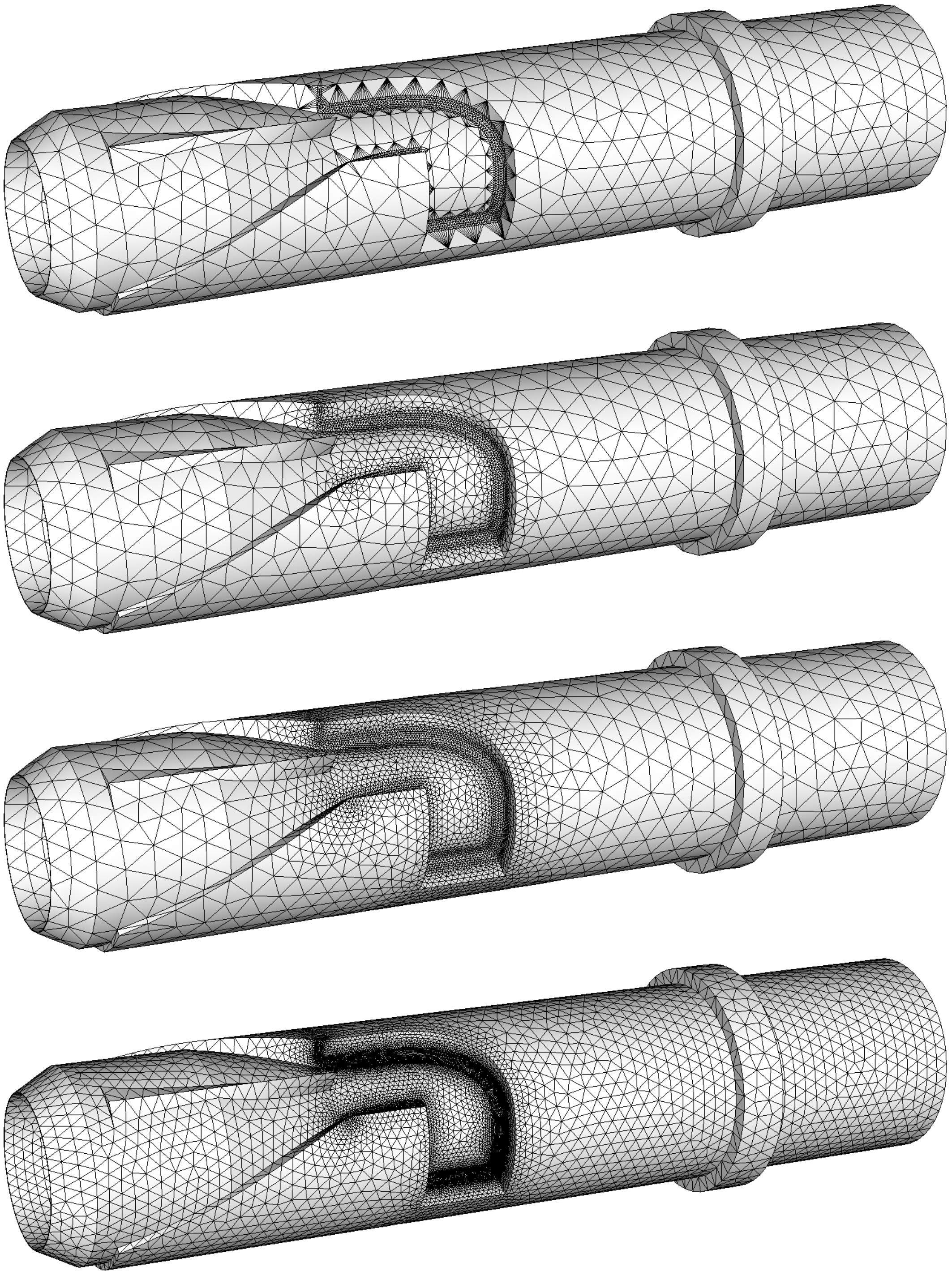

Figure 5: Grooved cylinder: four meshes with **isotropic** metric specifications. Mesh 1, angle 8° and no gradation. Mesh 2, angle 8° and gradation 1.5. Mesh 3, angle 8° and gradation 1.2. Mesh 4, angle 4° and gradation 1.2.

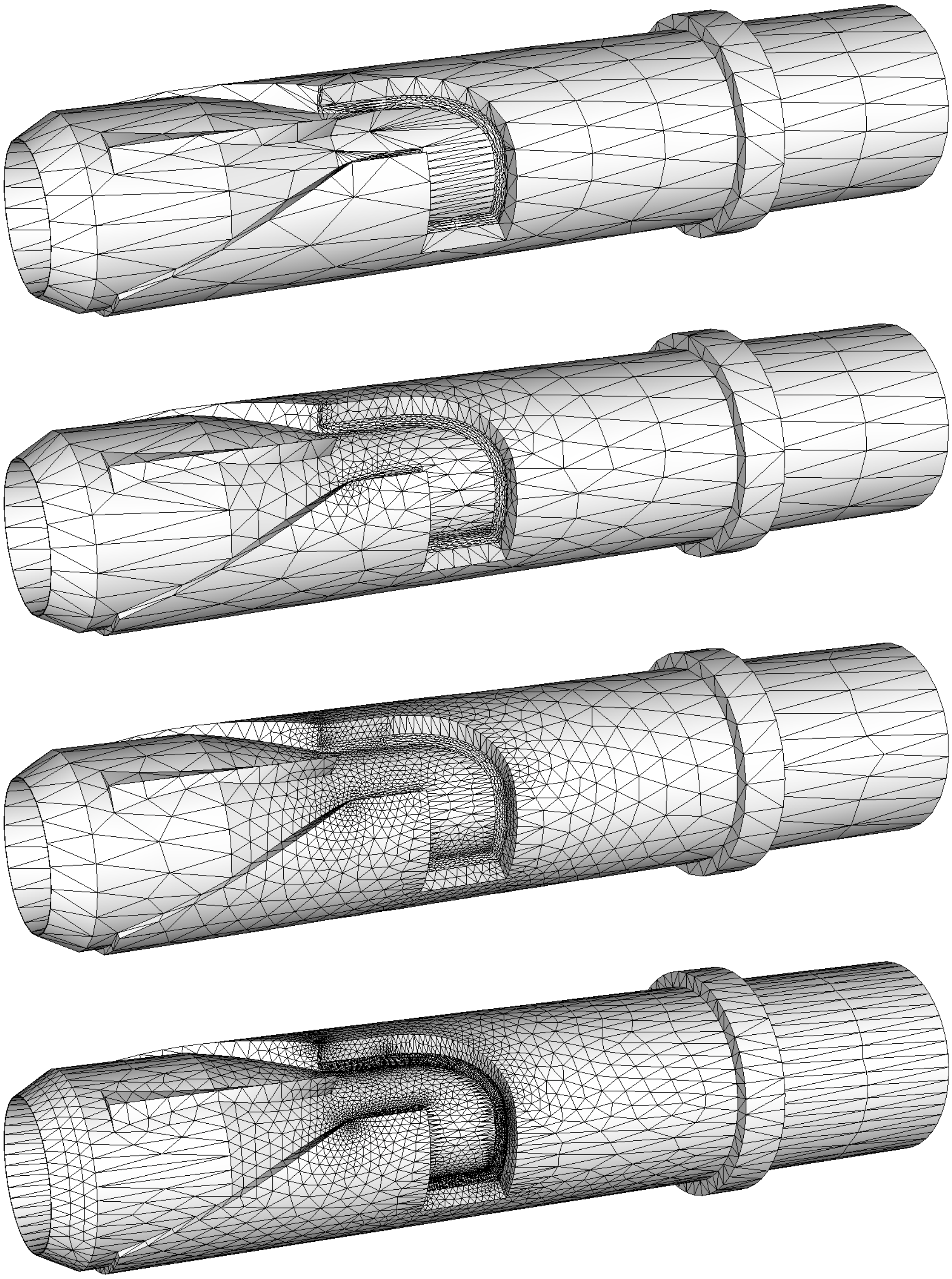

Figure 6: Grooved cylinder: four meshes with **anisotropic** metric specifications. Mesh 1, angle $8°$ and no gradation. Mesh 2, angle $8°$ and gradation 1.5. Mesh 3, angle $8°$ and gradation 1.2. Mesh 4, angle $4°$ and gradation 1.2.

7.2 Combustion engine (2381 patches)

In this second example, a combustion engine is modeled by a CAD surface of 2381 patches. Several geometric meshes are also generated, whose main characteristics are summarized in Table 2.

max. angle (degrees)	mesh type	max. metric gradation	number of vertices	number of triangles	CPU time (seconds)
8	iso	∞	104809	207515	11.481
8	iso	1.5	143536	284955	83.866
8	aniso	∞	28521	55085	2.543
8	aniso	1.5	57594	113188	60.981

Table 2: Combustion engine (2381 surface patches) – Meshing statistics.

For each mesh, a maximum angular gap of is $8°$ is specified (cf. Section 4). The mesh at the top of Figure 7 has isotropic metric specifications without gradation. As previously, elongated elements can be noticed in the presence of high variations of the surface curvature. To remedy this, a gradation of 1.5 is applied on the metric field (Figure 7, bottom). Meshes of Figure 8 have the same specifications except that they are anisotropic, following the principal directions of curvature. With the same geometric accuracy of $8°$, the number of elements without gradation is almost divided by 4. All the CPU times are measured as in Section 7.1.

8 Conclusion

A formal definition of conforming composite parametric surfaces has been given, and a general scheme of an indirect approach for meshing these surfaces has been introduced. Emphasis has been placed on the geometric mesh generation, based on continuous isotropic and discontinuous anisotropic geometric metrics. In addition, a new mesh gradation control strategy for discontinuous anisotropic geometric metrics has been proposed. This strategy can be applied to control the gradation for volume meshing from a 3D continuous metric field. Finally, each step of the general scheme for meshing has been detailed and the proposed methodology has been applied to numerical examples, showing its efficiency.

Future works include patch-independent anisotropic geometric meshing as well as a parallelization of the metric gradation and mesh generation, with the aim of processing large objects with complex geometries.

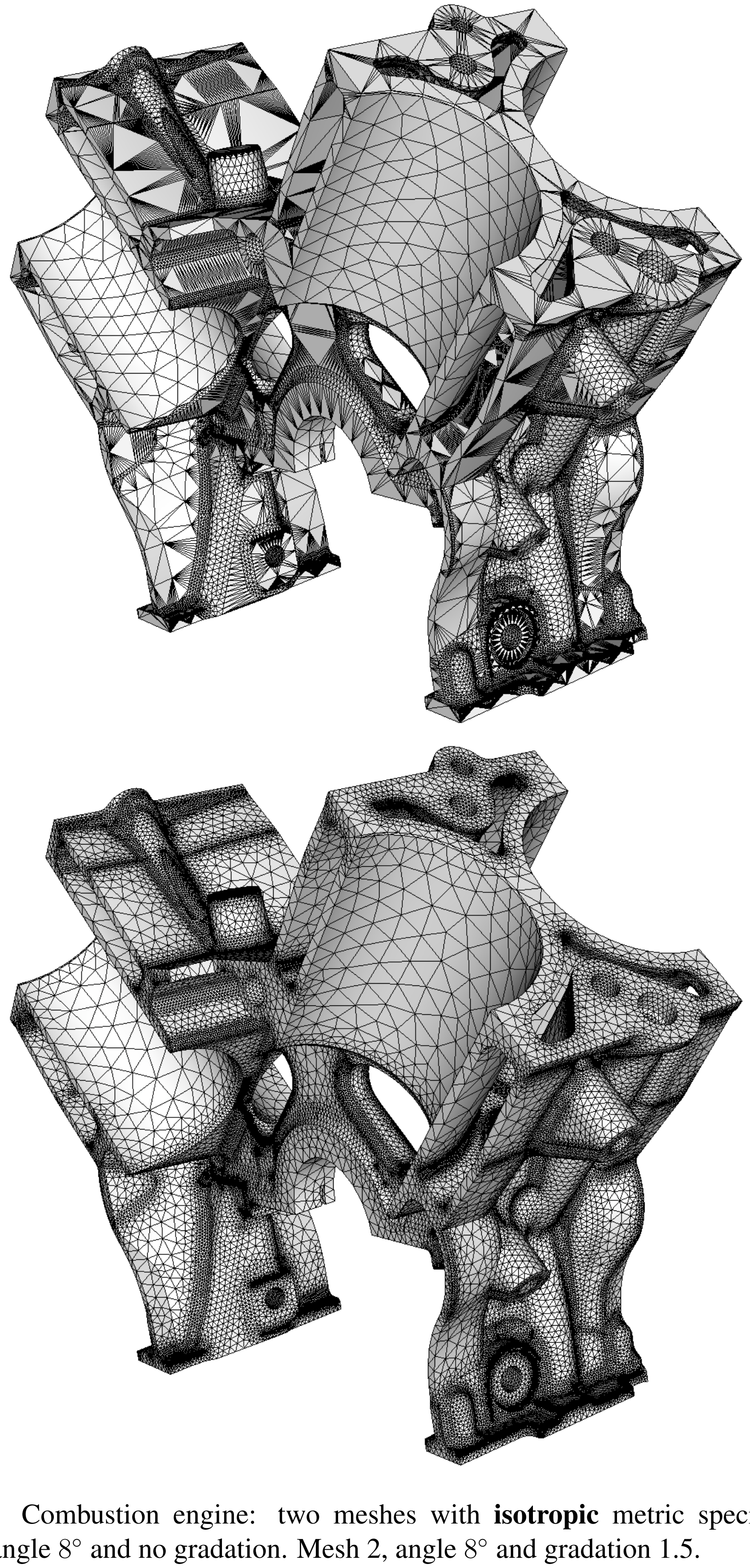

Figure 7: Combustion engine: two meshes with **isotropic** metric specifications. Mesh 1, angle 8° and no gradation. Mesh 2, angle 8° and gradation 1.5.

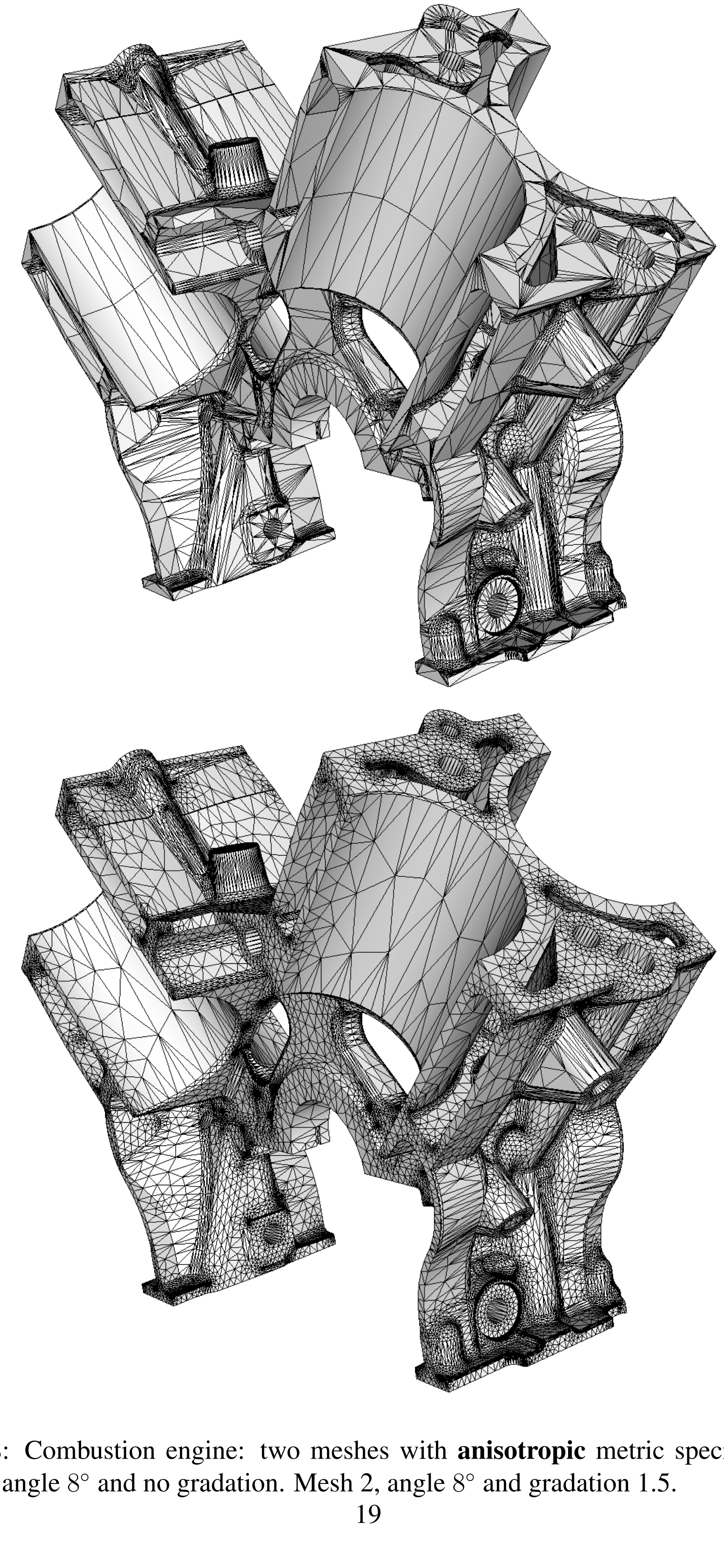

Figure 8: Combustion engine: two meshes with **anisotropic** metric specifications. Mesh 1, angle $8°$ and no gradation. Mesh 2, angle $8°$ and gradation 1.5.

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