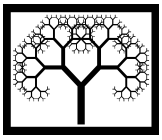
Paper 106



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A Multi-Parameter Perturbation Solution for the Inverse Eigenproblem of Nearly-Resonant N-Dimensional Hamiltonian Systems

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Abstract

The dynamic behavior of structural systems may be strongly characterized by the occurrence of multiple internal resonances for particular combinations of the mechanical parameters. The linear models governing these resonant or nearly-resonant systems tend to exhibit high sensitivity of the eigenvalues and eigenvectors to small parameter modifications. This pathologic condition is recognized as a source of relevant phenomena, such as frequency veering and mode localization or hybridization. The paper presents the generalization of uniformly valid perturbation methods to perform eigensolution sensitivity analyses in N-dimensional Hamiltonian systems with a generic number of close eigenvalues. The leading idea is to systematically treat nearly-resonant systems as multi-parameter perturbations of a perfectly-resonant, non-defective – though a priori unknown – reference system. Given a single nearly-resonant system, a multi-parameter perturbation method is presented to achieve a twofold objective: first, identify a close resonant system suited to serve as a starting point for sensitivity analyses (*inverse problem*); second, asymptotically approximate the eigensolution of all the nearly-resonant systems which may arise from its generic perturbation (*direct problem*). The direct problem solution is analyzed with a focus on the eigensolution sensitivity to parameter perturbations with different physical meanings, such as a slight geometric disorder or weak elastic coupling in periodic structures. Besides the particular class of periodic systems, the work findings apply to a number of internally-resonant engineering structures in which components with different stiffness properties are assembled together, as may happen when a rigid main structure is joined with a set of flexible identical sub-structures. Typical examples in the civil and mechanical engineering fields are cable-stayed bridges, made of a rigid deck supported by several flexible cable stays, and bladed disks, in which several flexible radial blades are attached to a rigid rotor-disk.

Keywords: perturbation methods, eigensolution sensitivity, internal resonances.

1 Introduction

The analytical models of several structural systems show that eigenvalues and eigenvectors may possess high sensitivity to the physical parameters. Since the eigenproperties strongly characterize the dynamic behavior, tracking the eigensolution loci against one or more significant parameters represents a matter of theoretical and practical interest in several fields, including stability analysis, design optimization, model updating, structural identification and vibration control. Within the parameter space, the regions corresponding to the intersection, or closeness, between two or more eigenvalue loci are worth particular attention. Internal resonance or nearly-resonance conditions may activate significant phenomena, such as frequency crossing or veering, vibration localization and modal hybridization [1, 2]. At the same time, energy transfers among the resonant modes may feature the system dynamic response [3].

In principle each structural system, even if made of dissimilar components with strongly different stiffness and mass properties [4], may exhibit close natural frequencies for particular parameter combinations. Nonetheless, in the civil and mechanical engineering context, the nearly-resonance condition tends traditionally to be considered a frequency *mistuning*, and nearly-resonant systems are typically regarded as *imperfect* structures. Such a dominant idea is motivated by recurrent observations that, in conservative structures, the exact coincidence of two or more frequencies occurs in symmetric systems (horizontal cables, circular disks, cylindric shafts), or in periodic assemblies of identical subsystems (pendulum chains, multi-span beams). Small frequency shifts, leading to nearly-resonance conditions, may appear when slight imperfections, typically geometric defects, are introduced to destroy the nominal structural symmetry or periodicity [5, 6]. Even when the symmetry or periodicity is preserved, similar mistuning effects can equally be originated by weak internal symmetric or skew-symmetric interactions, introduced for instance by soft elastic links among periodic subsystems, or by piezoelectric connections in electromechanical systems [7].

In the well-established literature on periodic and symmetric systems, weak linear interactions and small imperfections are denoted as *coupling* and *disorder* terms, respectively, and are both regarded as slight perturbations of a perfect (*uncoupled* and *ordered*) system. In tackling the eigensolution sensitivity problem for these systems, perturbation methods represent a valid alternative to the common, time-consuming, numerical techniques of eigensolution continuation, which are prone to fail in resolving and following a cluster of close and rapidly-evolving solutions. According to the perturbation strategy, known an initial reference system, the eigensolution of close *coupled* and *disordered* systems can be approximated by constructing asymptotic expansions. To this end, the traditional approach consists in including the disorder in the unperturbed reference system, and then performing a single-parameter analysis considering the coupling as a perturbation [1, 5]. However, such a perturbation scheme presents some drawbacks, essentially related to the small validity range of the achievable solutions. To overcome these shortcomings, a refined approach consists instead in retaining both the disorder and the coupling as independent perturbations of the uncoupled and ordered system, and then performing a multi-parameter analysis [8].

When dealing with complex engineering systems (Figure 1), characterized by several degrees of freedom, governed by a large set of mechanical parameters and exhibiting multiple internal resonances, the application of perturbation methods tends to become increasingly involved. In these cases, a systematization and generalization effort, driven by mechanical considerations, may be useful to face different technical operations, as for instance recognizing and separating the resonant degrees of freedom, identifying or defining a suitable reference system (Figure 1a), and finally classifying and properly ordering the significant coupling and disorder parameters (Figure 1b,c).

The leading idea of the present paper is to systematize the multi-parameter approach to achieve a completely general purpose, that is, perform eigensolution sensitivity analyses of N-dimensional Hamiltonian systems in the parameter regions characterized by nearly-resonance conditions. No limitations on the system dimension, nor on the number of nearly-resonant frequencies, are imposed a priori. The proposed algorithm requires minimal sufficient information consisting of the eigensolution (not necessarily complete) of a single nearly-resonant system, as could be derived for instance from experimental measures. As a working hypothesis, the nearly-resonant system is postulated to arise from the unknown perturbation of a perfectly-resonant, thought itself unknown, system. In the statement of the general problem, two different and complementary tasks are distinguished. First, the eigensolution sensitivity of the perfect system, with respect to a generic multi-parameter perturbation, is analyzed (task I: *Direct problem*). The direct problem solution gives the approximated eigenpairs of each nearly-resonant system which could arise from the perturbation. The eigensolution sensitivity to disorder and coupling terms is separately discussed. Second, the unknown perfect system to be perturbed is identified, starting from the knowledge of the experimental system (task II: *Inverse problem*). Questions regarding the existence and uniqueness of the inverse problem solution are discussed. Finally, the entire procedure is successfully applied to a prototypal structural system.

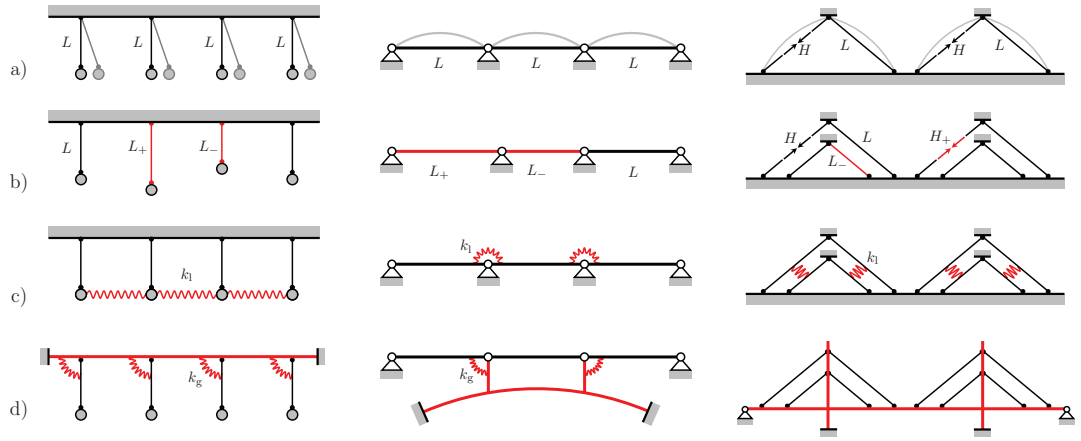

Figure 1: Example nearly-resonant structural systems: pendulum chain (left), multi-span beam (center), cable-stayed bridge (right), with distinction of (a) ordered and uncoupled perfectly-resonant system, (b) disorder, (c) local and (d) global coupling.

2 Eigensolution of nearly-resonant systems

In structural mechanics, the linear dynamic model of a Hamiltonian system $\mathcal{S}$ with N-degrees-of-freedom (*dofs*) is governed by a pair of real-valued symmetric $N \times N$ matrices, the mass $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$. The stiffness matrix is considered depending on a set $\mathbf{p}$ of M independent physical parameters p_{h} ($h = 1, ..., M$), while the mass matrix can be assumed to be unitary ($\mathbf{M} = \mathbf{I}$) without loss of generality. Each parameter set defines a different structural system in the parameter space. The system natural frequencies and modes arise from the solution of the eigenproblem related to the matrix $\mathbf{K}(\mathbf{p})$, consisting of N real (and generally distinct) eigenvalues λ_{i} ($i = 1, ..., N$), listed in ascending order in the diagonal matrix $\mathbf{\Lambda}$, and a complete set of eigenvectors $\boldsymbol{\phi}_{\mathrm{i}}$, collected columnwise in the modal matrix $\mathbf{\Phi}$.

In the general case, the perturbation methods for sensitivity analysis of the eigensolution require to fix an initial point P_0 in the parameter space, corresponding to the reference system $\mathcal{S}_0$, with governing matrix $\mathbf{K}_0(\mathbf{p}_0)$ and known eigensolution $\mathbf{\Lambda}_0(\mathbf{p}_0)$, $\mathbf{\Phi}_0(\mathbf{p}_0)$. In the neighborhood of P_0, each point P corresponds to a new system $\mathcal{S}$, with governing matrix $\mathbf{K}(\mathbf{p})$, and can be treated as a P_0-modification, associated to a perturbed parameter set $\mathbf{p} = \mathbf{p}_0 + \varepsilon \mathbf{p}_1$, where $\mathbf{p}_1$ is a generic multi-parameter perturbation of $\mathbf{p}_0$, corresponding to a generic direction vector in the parameter space, whose (small) amplitude is regulated by the scaling parameter $\varepsilon \ll 1$. Perturbation methods allow the construction of approximations for the exact eigensolution $\mathbf{\Lambda}(\mathbf{p})$, $\mathbf{\Phi}(\mathbf{p})$ of the modified system, in the form of asymptotic ε-power series

$$\mathbf{\Lambda}(\mathbf{p}) \simeq \mathbf{\Lambda}_0(\mathbf{p}_0) + \sum_{\mathrm{j}=1}^{\mathrm{k}} \varepsilon^{\mathrm{j}} \mathbf{\Lambda}_{\mathrm{j}}(\mathbf{p}_0, \mathbf{p}_1), \qquad \mathbf{\Phi}(\mathbf{p}) \simeq \mathbf{\Phi}_0(\mathbf{p}_0) + \sum_{\mathrm{j}=1}^{\mathrm{k}} \varepsilon^{\mathrm{j}} \mathbf{\Phi}_{\mathrm{j}}(\mathbf{p}_0, \mathbf{p}_1) \tag{1}$$

where the expansion coefficients at the j-th order are called *j-th eigensensitivities*.

Nearly-resonant systems feature rapid variation of the eigenproperties, that is, high eigensensitivities in the critical parameter regions. For such systems, traditional perturbation methods, employing an arbitrary initial point to start the asymptotic expansion, can be proved to give approximated solutions with only a narrow, and sometimes not uniform, validity in the parameter space [8]. The proper strategy to overcome such shortcomings consists in considering all the nearly-resonant systems as perturbations of a single, perfectly-resonant, non-defective system, which may be either known or unknown, but can anyway be supposed to exist somewhere in the neighborhood.

Following this mainstream idea to deal with nearly-resonant systems, the perturbation analysis must be distinguished into two mirror tasks, depending on the initial information actually available (Figure 2). If the perfectly-resonant system $\mathcal{S}_0$ is known, (i) the *direct problem* consists in determining the approximated eigensolution of each nearly-resonant system $\mathcal{S}$ by employing a generic multi-parameter perturbation of the $\mathcal{S}_0$ eigensolution. Otherwise, given a nearly-resonant system $\tilde{\mathcal{S}}$, (ii) the *inverse problem* consists in identifying the perfectly-resonant system $\mathcal{S}_0$ by forcing the asymptotic eigensolution of a nearly-resonant system $\mathcal{S}$, born from the multi-parameter $\mathcal{S}_0$ perturbation, to exactly reproduce the $\tilde{\mathcal{S}}$ eigensolution.

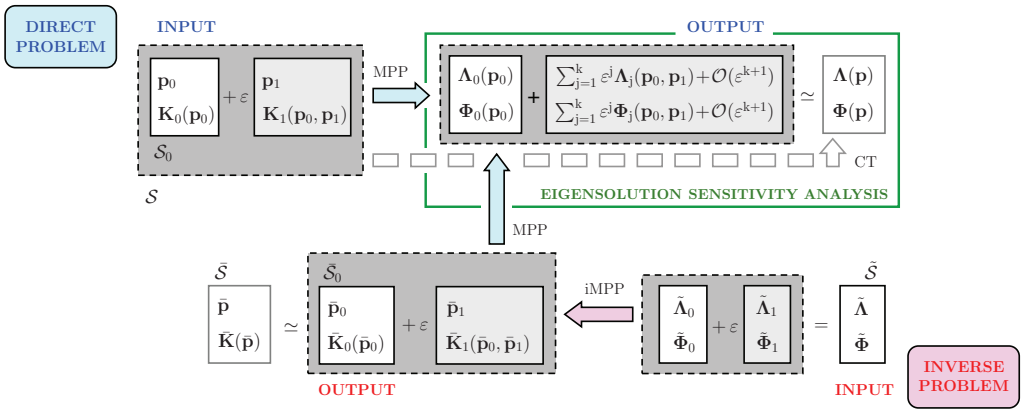

Figure 2: Multiparameter eigensolution sensitivity analysis of nearly-resonant systems: conceptual schemes of the direct problem (MPP) and inverse problem (iMPP).

With regard to engineering practice, many applications deal with a nearly-resonant system $\tilde{\mathcal{S}}$ with known eigenproperties $\tilde{\mathbf{\Lambda}}$,$\tilde{\mathbf{\Phi}}$, furnished for instance by finite element analyses or experimental dynamic measures. Therefore, a typical issue consists in determining the sensitivities of the nearly-coincident eigenvalues (and the associated eigenvectors) with respect to small modifications of the significant parameters. According to this recurrent scenario the nearly-resonant system $\tilde{\mathcal{S}}$ can be denoted as the *experimental system*. Since the experimental system is not suitable to serve as a starting point for the sensitivity analysis, the inverse problem has to be solved first. In fact the inverse problem solution, if it exists, allows the identification of the perfectly-resonant system $\bar{\mathcal{S}}_0$, locating its parameter set $\bar{\mathbf{p}}_0$ in the parameter space (or equivalently reconstructing the matrix $\bar{\mathbf{K}}_0$). Moreover, the solution also identifies the multi-parameter perturbation $\bar{\mathbf{p}}_1$ (or $\bar{\mathbf{K}}_1$), separating $\bar{\mathcal{S}}_0$ from the assigned experimental system $\tilde{\mathcal{S}}$. The identified perfectly-resonant system $\bar{\mathcal{S}}_0$ can be denoted as the *ideal system*, since it should be regarded as a theoretical abstraction and, in principle, might even correspond to a parameter set without physical meaning. Next, the direct problem can be approached, employing the ideal system $\bar{\mathcal{S}}_0$ as a suited starting point for the asymptotic sensitivity analysis of the eigensolution, extended to all the nearly-resonant systems $\mathcal{S}$ which could arise from a generic multi-parameter perturbation, including – as a special case – the experimental system $\tilde{\mathcal{S}}$. In distinction from the ideal perfectly-resonant system, the nearly-resonant systems $\mathcal{S}$ are referred to as the *real systems*. It is worth noting that the eigensolution of the experimental nearly-resonant system plays a double role, since it simultaneously constitutes the only input of the inverse problem and a particular output of the direct problem.

Although common applications may require solving the inverse problem first, because its solution is necessary as a starting point for the subsequent eigensolution sensitivity analysis, from a theoretical viewpoint it is preferable to approach the direct problem first. However, to preserve the required coherence between the two mirror problems, the input data and output results of the inverse problem must be considered as working hypotheses and solution constraints of the direct problem, respectively.

3 Direct problem

The ideal $\mathcal{S}_0$ is selected as the starting reference for the perturbation expansion. A generic real system $\mathcal{S}$, associated to the perturbed parameter set $\mathbf{p} = \mathbf{p}_0 + \varepsilon\,\mathbf{p}_1$, is governed by the stiffness matrix $\mathbf{K}(\mathbf{p})$ which, at first order approximation, is

$$\mathbf{K}(\mathbf{p}) \simeq \mathbf{K}_0(\mathbf{p}_0) + \varepsilon \mathbf{K}_1(\mathbf{p}_0, \mathbf{p}_1) \tag{2}$$

where $\mathbf{K}_1$ can be referred to as the stiffness perturbation matrix. Since $\mathbf{p}_0$ is a priori unknown in the direct problem, the unperturbed stiffness matrix $\mathbf{K}_0$ can be artfully constructed, provided only that the corresponding eigensolution presents a multiple eigenvalue with algebraic (and geometric) multiplicity n. The remaining $m = N - n$ non-resonant eigenvalues can be considered sufficiently far from the multiple eigenvalue and are assumed distinct from each other. Driven by these constraints on the eigensolution, a suitable partition of the perturbed stiffness matrix can be introduced

$$\mathbf{K}(\mathbf{p}) = \left[\begin{array}{c|c} \mathbf{K}^{\mathrm{nn}} & \mathbf{K}^{\mathrm{nm}} \\ \hline \mathbf{K}^{\mathrm{mn}} & \mathbf{K}^{\mathrm{mm}} \end{array}\right] \tag{3}$$

where $\mathbf{K}^{\mathrm{nn}}$, $\mathbf{K}^{\mathrm{mm}}$ are symmetric $n \times n$ and $m \times m$ square blocks, respectively, $\mathbf{K}^{\mathrm{nm}}$ is a $n \times m$ rectangular block, and finally $\mathbf{K}^{\mathrm{mn}} = (\mathbf{K}^{\mathrm{nm}})^{\top}$ for the symmetry. Therefore, the first order approximation in Eq.(2) is obtained introducing the matrix ordering

$$\mathbf{K}^{\mathrm{nn}} \simeq \mathbf{K}_0^{\mathrm{nn}} + \varepsilon\,\mathbf{K}_1^{\mathrm{nn}} \qquad \mathbf{K}^{\mathrm{nm}} \simeq \varepsilon\,\mathbf{K}_1^{\mathrm{nm}}, \qquad \mathbf{K}^{\mathrm{mm}} \simeq \mathbf{K}_0^{\mathrm{mm}} + \varepsilon\,\mathbf{K}_1^{\mathrm{mm}} \tag{4}$$

where it is also assumed that $\mathbf{K}_0^{\mathrm{nn}}$ is a diagonal matrix, as a working hypothesis.

From a structural viewpoint, the architecture of the stiffness matrices amounts to artificially realizing the ideal system $\mathcal{S}_0$ joining two uncoupled subsystems

- a resonant subsystem $\mathcal{S}_0^{\mathrm{r}}$, made of n simple harmonic oscillators independent from each other, with frequencies stored in the diagonal of the sub-matrix $\mathbf{K}_0^{\mathrm{nn}}$,
- a non-resonant subsystem $\mathcal{S}_0^{\mathrm{s}}$, composed of m generically coupled degrees-of-freedom (dofs), and governed by the non-diagonal sub-matrix $\mathbf{K}_0^{\mathrm{mm}}$.

The real system $\mathcal{S}$ arises from small perturbations of the ideal system, represented by the ε-order matrices $\mathbf{K}_1^{\mathrm{nn}}$ and $\mathbf{K}_1^{\mathrm{mm}}$, which independently affect each subsystem. The ε-order terms $\mathbf{K}_1^{\mathrm{nm}}$ introduces instead a weak coupling between the two subsystems.

The formulated model is representative of a number of engineering structures in which components with different stiffness properties are assembled together, as may happen when a rigid main structure (the non-resonant subsystem) is assembled with a set of flexible identical sub-structures (the resonant subsystem). Typical examples in the civil engineering field are cable-stayed bridges, made of a rigid deck supported by several flexible cable stays. Typical examples in the mechanical engineering field are bladed disks, in which several flexible radial blades are attached to a rigid rotor-disk.

3.1 Multi-parameter perturbation solution of the eigenproblem

Given the eigenproblem of the real system $\mathcal{S}$, a classical perturbation scheme can be applied expanding the modal equation up to the second order and collecting the terms of the same ε-power, the following hierarchy of equation pairs is obtained

$$\begin{aligned}
\varepsilon^0: \quad & \left(\mathbf{K}_0^{\mathrm{nn}} - \lambda_0 \mathbf{I}\right) \boldsymbol{\phi}_0^{\mathrm{n}} = \mathbf{0} \\
& \left(\mathbf{K}_0^{\mathrm{mm}} - \lambda_0 \mathbf{I}\right) \boldsymbol{\phi}_0^{\mathrm{m}} = \mathbf{0}
\end{aligned} \tag{5}$$

$$\begin{aligned}
\varepsilon^1: \quad & \left(\mathbf{K}_0^{\mathrm{nn}} - \lambda_0 \mathbf{I}\right) \boldsymbol{\phi}_1^{\mathrm{n}} = -\mathbf{K}_1^{\mathrm{nn}} \boldsymbol{\phi}_0^{\mathrm{n}} - \mathbf{K}_1^{\mathrm{nm}} \boldsymbol{\phi}_0^{\mathrm{m}} + \lambda_1 \boldsymbol{\phi}_0^{\mathrm{n}} \\
& \left(\mathbf{K}_0^{\mathrm{mm}} - \lambda_0 \mathbf{I}\right) \boldsymbol{\phi}_1^{\mathrm{m}} = -\mathbf{K}_1^{\mathrm{mn}} \boldsymbol{\phi}_0^{\mathrm{n}} - \mathbf{K}_1^{\mathrm{mm}} \boldsymbol{\phi}_0^{\mathrm{m}} + \lambda_1 \boldsymbol{\phi}_0^{\mathrm{m}}
\end{aligned} \tag{6}$$

$$\begin{aligned}
\varepsilon^2: \quad & \left(\mathbf{K}_0^{\mathrm{nn}} - \lambda_0 \mathbf{I}\right) \boldsymbol{\phi}_2^{\mathrm{n}} = -\mathbf{K}_1^{\mathrm{nn}} \boldsymbol{\phi}_1^{\mathrm{n}} - \mathbf{K}_1^{\mathrm{nm}} \boldsymbol{\phi}_1^{\mathrm{m}} + \lambda_1 \boldsymbol{\phi}_1^{\mathrm{n}} + \lambda_2 \boldsymbol{\phi}_0^{\mathrm{n}} \\
& \left(\mathbf{K}_0^{\mathrm{mm}} - \lambda_0 \mathbf{I}\right) \boldsymbol{\phi}_2^{\mathrm{m}} = -\mathbf{K}_1^{\mathrm{mn}} \boldsymbol{\phi}_1^{\mathrm{n}} - \mathbf{K}_1^{\mathrm{mm}} \boldsymbol{\phi}_1^{\mathrm{m}} + \lambda_1 \boldsymbol{\phi}_1^{\mathrm{m}} + \lambda_2 \boldsymbol{\phi}_0^{\mathrm{m}}
\end{aligned} \tag{7}$$

where the j^{th} order eigenvector has been decomposed into an upper ($n\times1$) and a lower ($m\times1$) part, that is $\boldsymbol{\phi}_{\mathrm{j}} = \{\boldsymbol{\phi}_{\mathrm{j}}^{\mathrm{n}}, \boldsymbol{\phi}_{\mathrm{j}}^{\mathrm{m}}\}^{\top}$, and $\mathbf{I}$ denotes the identity matrix.

At the zeroth-order, the equation pair (5) governs the eigenproblem of the unperturbed system $\mathcal{S}_0$. Due to the absence of coupling terms, each equation actually states an independent sub-eigenproblem, related to one or the other of the component subsystems $\mathcal{S}_0^{\mathrm{r}}$ and $\mathcal{S}_0^{\mathrm{s}}$. Consequently, the complete set of N eigenvalues $\boldsymbol{\Lambda}_0 = \mathrm{diag}\left(\lambda_{01}, ..., \lambda_{0\mathrm{N}}\right)$ is the union of two distinct subsets

- $\boldsymbol{\Lambda}_0^{\mathrm{n}} = \mathrm{diag}\left(\lambda_{01}^{\mathrm{r}}, ..., \lambda_{0\mathrm{n}}^{\mathrm{r}}\right)$, with n resonant eigenvalues of the subsystem $\mathcal{S}_0^{\mathrm{r}}$
- $\boldsymbol{\Lambda}_0^{\mathrm{m}} = \mathrm{diag}\left(\lambda_{01}^{\mathrm{s}}, ..., \lambda_{0\mathrm{m}}^{\mathrm{s}}\right)$, with m non-resonant eigenvalues of the subsystem $\mathcal{S}_0^{\mathrm{s}}$

satisfying the singularity condition imposed on the left-hand algebraic operator in equation (5a) and (5b), respectively. However, once a particular zeroth-order eigenvalue $\lambda_{0\mathrm{i}}$ is determined, no matter wherefrom, the corresponding eigenvector $\boldsymbol{\phi}_{0\mathrm{i}}$ requires the simultaneous solution of both the first equation (to determine the upper part $\boldsymbol{\phi}_{0\mathrm{i}}^{\mathrm{n}}$) and the second equation (to determine the lower part $\boldsymbol{\phi}_{0\mathrm{i}}^{\mathrm{m}}$).

Consistently, the multi-parameter perturbation of the perfectly-resonant system $\mathcal{S}_0$ determines, in the perturbed nearly-resonant system $\mathcal{S}$, the co-existence of

- a *nearly-resonant* eigensolution $\left(\lambda^{\mathrm{r}}, \boldsymbol{\phi}^{\mathrm{r}}\right)$ generated by the eigenvalues λ_0^{r},
- a *non-resonant* eigensolution $\left(\lambda^{\mathrm{s}}, \boldsymbol{\phi}^{\mathrm{s}}\right)$ generated by the eigenvalues λ_0^{s}

the former, referred to as the *resonant* eigensolution to simplify the nomenclature, is sufficient to the aims of the present work, and is presented in the following.

3.1.1 Resonant eigensolution

Known a resonant eigenvalue $\lambda_{0\mathrm{i}}^{\mathrm{r}}$, the equation pair (5) gives (i) the *essential* (upper) part $\boldsymbol{\phi}_{0\mathrm{i}}^{\mathrm{rn}}$ of the eigenvector, collecting the *resonant dofs* (the n dofs of the resonant subsystem $\mathcal{S}_0^{\mathrm{r}}$), and (ii) the *complementary* (lower) part $\boldsymbol{\phi}_{0\mathrm{i}}^{\mathrm{rm}}$ of the eigenvector, collecting the *non resonant dofs* (the m dofs of the non-resonant subsystem $\mathcal{S}_0^{\mathrm{s}}$).

$\mathcal{O}(\varepsilon)$	Eigenvalues	Eigenvectors
0th	$\lambda_0 = \mathrm{diag}\big(\mathbf{K}_0^{\mathrm{nn}}\big)$	$\boldsymbol{\phi}_0^{\mathrm{n}} = \mathbf{a}$ (undetermined) $\boldsymbol{\phi}_0^{\mathrm{m}} = \mathbf{0}$
1st	$\lambda_1 =$ eigenvalues of $\mathbf{K}_1^{\mathrm{nn}}$	$\mathbf{a} =$ eigenvectors of $\mathbf{K}_1^{\mathrm{nn}}$ $\boldsymbol{\phi}_1^{\mathrm{m}} = -\big(\mathbf{L}_0^{\mathrm{m}}\big)^{-1}\mathbf{K}_1^{\mathrm{mn}}\,\mathbf{a}$
2nd	$\lambda_2 = \alpha\,\mathbf{a}^{\top}\mathbf{K}_1^{\mathrm{nm}}\,\boldsymbol{\phi}_1^{\mathrm{m}}$	$\boldsymbol{\phi}_1^{\mathrm{n}} = -\mathbf{A}\big[\,\mathbf{A}^{\top}\mathbf{L}_1^{\mathrm{n}}\mathbf{A}\,\big]^{-1}\mathbf{A}^{\top}\big(\mathbf{K}_1^{\mathrm{nm}}\,\boldsymbol{\phi}_1^{\mathrm{m}} + \lambda_2\,\mathbf{a}\big)$ $\boldsymbol{\phi}_2^{\mathrm{m}} = -\big(\mathbf{L}_0^{\mathrm{m}}\big)^{-1}\big(\mathbf{K}_1^{\mathrm{mn}}\,\boldsymbol{\phi}_1^{\mathrm{n}} + \mathbf{L}_1^{\mathrm{m}}\boldsymbol{\phi}_1^{\mathrm{m}}\big)$

Note: apex r and index i are omitted for $\alpha_{\mathrm{i}}^{\mathrm{r}}$,$\mathbf{A}_{\mathrm{i}}^{\mathrm{r}}$,$\lambda_{\mathrm{ki}}^{\mathrm{r}}$,$\boldsymbol{\phi}_{\mathrm{ki}}^{\mathrm{rn}}$,$\boldsymbol{\phi}_{\mathrm{ki}}^{\mathrm{rm}}$,$\mathbf{L}_{\mathrm{ki}}^{\mathrm{rn}}$,$\mathbf{L}_{\mathrm{ki}}^{\mathrm{rm}}$ (i=1...n, k=0,1,2).

Table 1: Summary of the resonant eigensolution at different approximation orders.

Therefore, the first and second eigensensitivities (λ_1^{r},$\boldsymbol{\phi}_1^{\mathrm{r}}$ and λ_2^{r},$\boldsymbol{\phi}_2^{\mathrm{r}}$) can be determined from the equation pairs (6) and (7), respectively, when lower-order eigensolutions are introduced. The contributions to the resonant eigensolution are presented in Table 1, where $\alpha_{\mathrm{i}}^{\mathrm{r}} = \big(\mathbf{a}_{\mathrm{i}}{}^{\top}\mathbf{a}_{\mathrm{i}}\big)^{-1}$, $\mathbf{L}_{\mathrm{ki}}^{\mathrm{rj}} = \big(\,\mathbf{K}_{\mathrm{ki}}^{\mathrm{ij}} - \lambda_{\mathrm{ki}}^{\mathrm{r}}\mathbf{I}\,\big)$ and $\mathbf{A}_{\mathrm{i}}^{\mathrm{r}} = \big[\,\mathbf{a}_1, \ldots, \mathbf{a}_{\mathrm{j}\neq\mathrm{i}}, \ldots, \mathbf{a}_{\mathrm{n}}\big]$. The perturbation scheme is qualitatively illustrated in Figure 3, where the surface Σ represents the locus of perfectly-resonant systems in the parameter space Π, including $\mathcal{S}_0$, located by the parameter set $\mathbf{p}_0$. Therefrom, a small parameter perturbation $\mathbf{p}_1$ generates the nearly-resonant system $\mathcal{S}$ (Figure 3a). Consequently, the multiple eigenvalue λ_0^{r} splits into a cluster of close eigenvalues $\lambda_{\mathrm{i}}^{\mathrm{r}}$ (i=1,...,n), each well-approximated by the perturbation solution reconstructed up to the second order λ_0^{r}+$\lambda_{1\mathrm{i}}^{\mathrm{r}}$+$\lambda_{2\mathrm{i}}^{\mathrm{r}}$ (Figure 3b).

The analytical results are worthy of a synthetic physical interpretation. Focusing on the zeroth-order, the simplest – although not unique – physical realization of the perfectly-resonant subsystem $\mathcal{S}_0^{\mathrm{r}}$ is represented by a set of identical and independent oscillators, with the same stiffness and mass in order to have equal natural frequencies. In other words, the nominal periodicity (*order*), in absence of internal interactions (*coupling*), is sufficient to realize the perfect internal resonance (*tuning*).

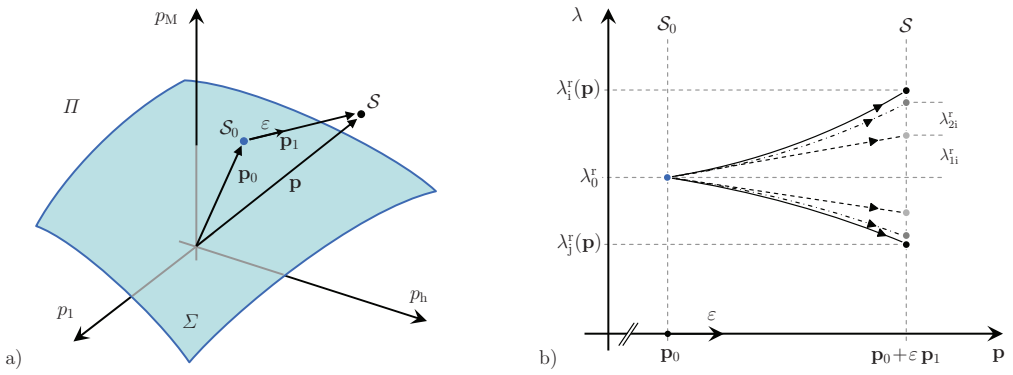

Figure 3: Direct problem: (a) generic perturbation of the resonant system in the parameter space, (b) multi-parameter perturbation scheme for the resonant eigensolution.

Focusing on higher orders and looking first at the eigenvalue sensitivity, some qualitative and quantitative considerations can be made

- the first eigenvalue sensitivity λ_1^{r}, depending only on the multi-parameter stiffness perturbation $\mathbf{K}_1^{\mathrm{nn}}$, splits the resonant subsystem $\mathcal{S}_0^{\mathrm{r}}$ in a cluster of n distinct, but close eigenvalues. This analytical result describes the nearly-resonance condition characterizing quasi-periodic structures, when small periodicity-breaking disturbs cause slight frequency shifts (*mistuning*). Disturbs can be classified as *disorder* and *local coupling* terms, corresponding to diagonal and out-of-diagonal terms of the perturbation matrix $\mathbf{K}_1^{\mathrm{nn}}$, respectively (see Figure 1b,c).
- the second eigenvalue sensitivity λ_2^{r} depends also on the multi-parameter stiffness perturbation $\mathbf{K}_1^{\mathrm{mn}}$, describing the *global coupling* between the resonant and the non-resonant subsystems (Figure 1d). Therefore, even in the absence of disorder or local coupling, a global coupling may cause the splitting of the multiple eigenvalue, but this effect can be captured only by a second-order analysis.

Looking at the eigenvectors, the essential and the complementary parts possess different sensitivities to the perturbating stiffness terms

- the *essential* part remains undetermined at each order until the higher-order equations are solved. This result means that natural modes of perfectly-periodic structures remain undistinguished until a small perturbation $\mathbf{K}_1^{\mathrm{nn}}$ is introduced (at zero-th order), and – more generally – that natural modes of resonant structures are highly-sensitive to small parameter perturbations (at higher orders).
- a non-null *complementary* part is originated only by the global coupling $\mathbf{K}_1^{\mathrm{mn}}$ (at the first order) or the local coupling and disorder $\mathbf{K}_1^{\mathrm{nn}}$ (at the second-order), meaning that it is less sensitive than the essential part to parameter perturbations. This result evidences how the non-resonant degrees-of-freedom of the non-resonant subsystem weakly participate to the resonant natural modes.

Numerical example The multi-parameter perturbation method is applied to a 7-dofs nearly-resonant system $\mathcal{S}$ with three close eigenvalues ($N = 7$, $n = 3$). The resonant $\mathcal{S}_0^{\mathrm{r}}$ (3-dofs) and non-resonant subsystem $\mathcal{S}_0^{\mathrm{s}}$ (4-dofs) are governed by the unitary sub-matrix $\mathbf{K}_0^{\mathrm{nn}}$ and a random sub-matrix $\mathbf{K}_0^{\mathrm{mm}}$, respectively. Zeroing parts of the random perturbation matrix $\mathbf{K}_1$ (as shown in Table 2), different particular cases can be considered separately, as disorder ($M = 3$), local coupling ($M = 3$), global coupling ($M = 12$), and their combinations. The effectiveness in approximating the exact solution can be appreciated in figure 4, where the first and second eigenvalue sensitivities are followed with fine agreement for increasing perturbation amplitudes.

$\mathbf{K}_1^{\mathrm{nn}}$ (disorder)			$\mathbf{K}_1^{\mathrm{nn}}$ (local coupling)			$\mathbf{K}_1^{\mathrm{nm}}$ (global coupling)			
0.3031	-	-	-	0.4578	-0.1871	-0.6990	0.3321	-0.1932	-0.6833
-	0.0233	-		-	-0.3144	-0.0718	0.0750	0.0897	-0.3094
-	-	-0.4550	sym		-	-0.3564	-0.3609	0.0189	0.1763

Table 2: Perturbation matrix $\mathbf{K}_1$ for the example 7-dofs nearly-resonant system

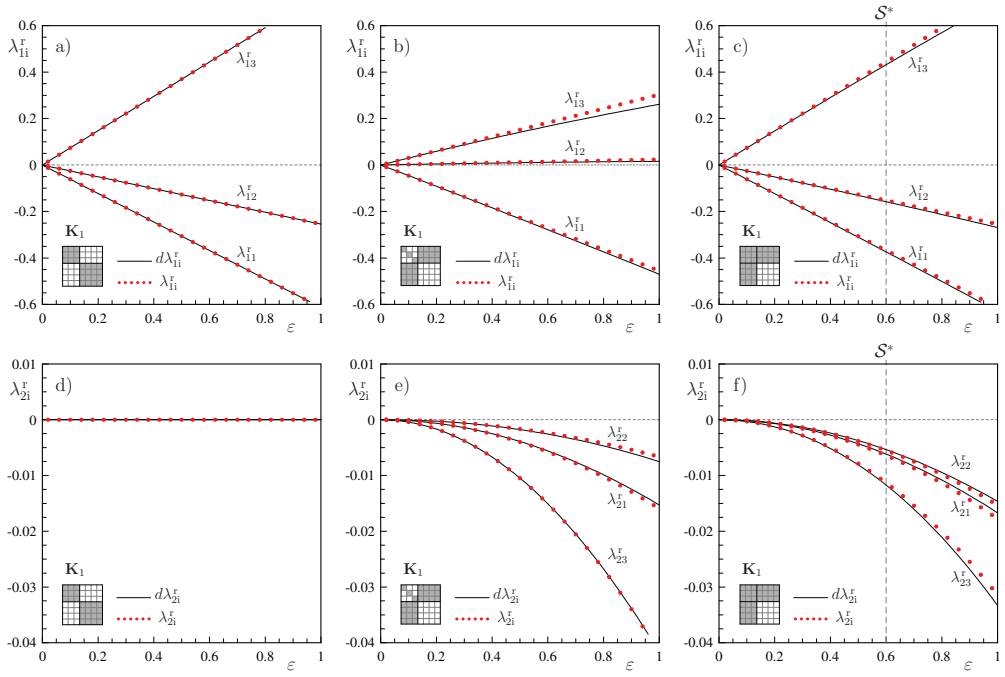

Figure 4: Exact (lines) and approximate (dots) first and second (upper and lower row) sensitivity of the three resonant eigenvalues to (a),(d) disorder and local coupling; (b),(e) disorder and global coupling; (c),(f) disorder, local and global coupling.

4 Inverse problem

The inverse problem can be first formulated for a generic system, and then specialized for nearly-resonant systems. The available eigensolution is supposed incomplete; the minimum sufficient information includes L eigenvalues of a single experimental system $\tilde{\mathcal{S}}$, collected in the $L\times L$ diagonal matrix $\tilde{\mathbf{\Lambda}}$, and the associated complete eigenvectors, cast column-wise in the $N\times L$ rectangular matrix $\tilde{\mathbf{\Phi}}$. The sought solution is the real-valued, positive stiffness matrix $\bar{\mathbf{K}}$ of the experimental system.

Imposing the matrix symmetry ($\mathbf{K}=\mathbf{K}^\top$), omitting the tilde and the bar, the inverse eigenproblem for incomplete data can be formulated in alternative forms

$$\mathbf{K}\mathbf{\Phi} = \mathbf{\Phi}\mathbf{\Lambda}, \qquad \mathbf{\Phi}^\top\mathbf{K} = \mathbf{\Lambda}\mathbf{\Phi}^\top \tag{8}$$

which constitute a pair of so-called *simultaneous linear matrix equations* [9] in the matrix unknown $\mathbf{K}$. Under certain solvability conditions, the two equations can be proved to have a common solution. Since the matrix $\mathbf{\Phi}$ is composed of real-valued linearly-independent column vectors, the solution assumes the simplified form

$$\mathbf{K} = \mathbf{\Phi}\,\mathbf{\Lambda}\,(\mathbf{\Phi}^\top\mathbf{\Phi})^{-1}\mathbf{\Phi}^\top + \left[\,\mathbf{I} - \mathbf{\Phi}\,(\mathbf{\Phi}^\top\mathbf{\Phi})^{-1}\mathbf{\Phi}^\top\right]\mathbf{Z}\left[\,\mathbf{I} - \mathbf{\Phi}\,(\mathbf{\Phi}^\top\mathbf{\Phi})^{-1}\mathbf{\Phi}^\top\,\right] \tag{9}$$

where $\mathbf{Z}$ is a real-valued indeterminate $N\times N$ matrix, symmetric and semi-positive. The first term reconstructs the essential part of the system stiffness, that is, the minimal matrix still possessing the given incomplete modal properties, while the second term adds a stiffness complement, which determines the unassigned modal data.

4.1 Perturbation solution of the inverse eigenproblem

Before focusing on nearly-resonant systems, it may be interesting to tackle the inverse problem in perturbation form, that is, to analyze how the general solution modifies for small perturbations of the input modal information

$$\boldsymbol{\Lambda}_0 + \varepsilon\boldsymbol{\Lambda}_1, \qquad \boldsymbol{\Phi}_0 + \varepsilon\boldsymbol{\Phi}_1 \tag{10}$$

Substituting, employing the algebraic rules for the perturbation of an inverse matrix, and finally collecting the terms of the same ε-order, the perturbed solution reads

$$\mathbf{K}_0 + \varepsilon\,\mathbf{K}_1 + \mathcal{O}(\varepsilon^2) \tag{11}$$

where $\mathbf{K}_0$ is the stiffness matrix associated to the unperturbed modal data $(\boldsymbol{\Lambda}_0, \boldsymbol{\Phi}_0)$, and $\mathbf{K}_1$ is the first-order stiffness modification, due to the data perturbation $(\boldsymbol{\Lambda}_1, \boldsymbol{\Phi}_1)$

$$\mathbf{K}_0 = \boldsymbol{\Xi}_0 + \boldsymbol{\Theta}_0\,\mathbf{Z}\,\boldsymbol{\Theta}_0, \qquad \mathbf{K}_1 = \boldsymbol{\Xi}_1 - \boldsymbol{\Theta}_0\,\mathbf{Z}\,\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_1\mathbf{Z}\,\boldsymbol{\Theta}_0 \tag{12}$$

where the following auxiliary square matrices have been introduced

$$\boldsymbol{\Xi}_0 = \boldsymbol{\Phi}_0\boldsymbol{\Lambda}_0\boldsymbol{\Psi}_0\boldsymbol{\Phi}_0^\top, \quad \boldsymbol{\Xi}_1 = \boldsymbol{\Phi}_0\boldsymbol{\Lambda}_0\boldsymbol{\Psi}_0\boldsymbol{\Phi}_1^\top - \boldsymbol{\Phi}_0\boldsymbol{\Lambda}_0\boldsymbol{\Psi}_1\boldsymbol{\Phi}_0^\top + \boldsymbol{\Phi}_0\boldsymbol{\Lambda}_1\boldsymbol{\Psi}_0\boldsymbol{\Phi}_0^\top + \boldsymbol{\Phi}_1\boldsymbol{\Lambda}_0\boldsymbol{\Psi}_0\boldsymbol{\Phi}_0^\top \tag{13}$$

$$\boldsymbol{\Theta}_0 = \mathbf{I} - \boldsymbol{\Phi}_0\,\boldsymbol{\Psi}_0\boldsymbol{\Phi}_0^\top, \quad \boldsymbol{\Theta}_1 = \boldsymbol{\Phi}_0\,\boldsymbol{\Psi}_0\boldsymbol{\Phi}_1^\top - \boldsymbol{\Phi}_0\,\boldsymbol{\Psi}_1\boldsymbol{\Phi}_0^\top + \boldsymbol{\Phi}_1\boldsymbol{\Psi}_0\boldsymbol{\Phi}_0^\top \tag{14}$$

and the variables $\boldsymbol{\Psi}_0 = \left(\boldsymbol{\Phi}_0^\top\boldsymbol{\Phi}_0\right)^{-1}$, $\boldsymbol{\Psi}_1 = \left(\boldsymbol{\Phi}_0^\top\boldsymbol{\Phi}_0\right)^{-1}\left(\boldsymbol{\Phi}_0^\top\boldsymbol{\Phi}_1 + \boldsymbol{\Phi}_1^\top\boldsymbol{\Phi}_0\right)\left(\boldsymbol{\Phi}_0^\top\boldsymbol{\Phi}_0\right)^{-1}$.

It is worth noting that the data perturbations $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Phi}_1$ cannot be assigned arbitrarily, nor independently of each other. Otherwise, the inverse problem would reconstruct a meaningless stiffness matrix, typically non-symmetric or non positively-defined.

4.1.1 Nearly-resonant systems

The inverse problem must be re-formulated according to the idea that nearly-resonant systems arise from perturbations of a perfectly-resonant system. From this perspective, the nearly-resonant eigenproperties relinquish the usual role of *known data* (input), and perform instead as *known target* (output), to be obtained as perturbed eigenproperties of an unknown perfectly-resonant system undergoing an unknown stiffness perturbation. The main difference is that, while the inverse problem solution typically consists only of the stiffness sensitivity to perturbations of the assigned experimental eigenproperties, for nearly-resonant systems the solution includes: (i) the initial stiffness $\mathbf{K}_0$ of the perfectly-resonant system, and (ii) the stiffness perturbation $\mathbf{K}_1$ which lets the perturbed eigenproperties exactly match the assigned experimental data.

According to the above considerations, the sought unknowns correspond to the inverse problem solution (12) only if the experimental modal matrices $(\boldsymbol{\Lambda}, \boldsymbol{\Phi})$ satisfy particular conditions, that is assume the first-order perturbation form predicted by the direct problem for nearly-resonant systems. Therefore, the inverse problem does not require a specific statement for nearly-resonant systems, provided that the experimental data fulfill these conditions and can consequently be decomposed and ordered.

In virtue of the direct problem solution, it is known that the complete experimental eigensolution ($L=N$) for nearly-resonant systems can be partitioned

$$\mathbf{\Lambda} = \left[\begin{array}{c|c} \mathbf{\Lambda}^{\mathrm{r}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{\Lambda}^{\mathrm{s}} \end{array}\right], \qquad \mathbf{\Phi} = \left[\begin{array}{c|c} \mathbf{\Phi}^{\mathrm{r}} & \mathbf{\Phi}^{\mathrm{s}} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{\Phi}^{\mathrm{rn}} & \mathbf{\Phi}^{\mathrm{sn}} \\ \hline \mathbf{\Phi}^{\mathrm{rm}} & \mathbf{\Phi}^{\mathrm{sm}} \end{array}\right] \tag{15}$$

where the minimum sufficient set of experimental data for the inverse problem consists of the resonant experimental eigensolution only ($L=n$). The experimental resonant eigenvalues $\mathbf{\Lambda}^{\mathrm{r}}$ and eigenvectors $\mathbf{\Phi}^{\mathrm{r}}$ must be liable to the decomposition and ordering

$$\mathbf{\Lambda} = \lambda_0^{\mathrm{r}}\,\mathbf{I} + \varepsilon\,\mathbf{\Lambda}_1, \qquad \left[\begin{array}{c} \mathbf{\Phi}^{\mathrm{n}} \\ \hline \mathbf{\Phi}^{\mathrm{m}} \end{array}\right] = \left[\begin{array}{c} \mathbf{\Phi}_0^{\mathrm{n}} \\ \hline \mathbf{0} \end{array}\right] + \varepsilon \left[\begin{array}{c} \mathbf{\Phi}_1^{\mathrm{n}} \\ \hline \mathbf{\Phi}_1^{\mathrm{m}} \end{array}\right] \tag{16}$$

where the superscript r has been omitted. Qualitatively, all the eigenvalues must be small perturbations of the same multiple eigenvalue, whereas the eigenvectors must be small perturbations of a dominant term, made of the essential (upper) part only. This requirement can be regarded as a compatibility condition which must be fulfilled by the data, confirming the physical expectation that a generic experimental data cannot always be considered the eigensolution of an unknown nearly-resonant system.

Decomposing and ordering the experimental modal data according to (16), and then substituting into (12), the two terms of the inverse problem solution become

$$\mathbf{K}_0 = \left[\begin{array}{c|c} \lambda_0^{\mathrm{r}}\,\mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{Z}^{\mathrm{m}} \end{array}\right], \qquad \mathbf{K}_1 = \left[\begin{array}{c|c} \mathbf{K}_1^{\mathrm{nn}} & \mathbf{K}_1^{\mathrm{nm}} \\ \hline \mathbf{K}_1^{\mathrm{mn}} & \mathbf{0} \end{array}\right] \tag{17}$$

where the submatrices of the stiffness perturbation $\mathbf{K}_1$ are

$$\mathbf{K}_1^{\mathrm{nn}} = \mathbf{\Phi}_0^{\mathrm{n}}\,\mathbf{\Lambda}_1(\mathbf{\Phi}_0^{\mathrm{n}})^{-1} + 2\,\lambda_0^{\mathrm{r}}\left(\mathbf{G}_1^{\mathrm{n}} + \mathbf{G}_1^{\mathrm{n}\top}\right) \tag{18}$$

$$\mathbf{K}_1^{\mathrm{mn}} = \left(\lambda_0^{\mathrm{r}}\mathbf{I} - \mathbf{Z}^{\mathrm{m}}\right)\mathbf{G}_1^{\mathrm{mn}} - 2\,\mathbf{Z}^{\mathrm{mn}}\left(\mathbf{G}_1^{\mathrm{n}} + \mathbf{G}_1^{\mathrm{n}\top}\right) \tag{19}$$

where $\mathbf{G}_1^{\mathrm{n}} = \mathbf{\Phi}_1^{\mathrm{n}}\,(\mathbf{\Phi}_0^{\mathrm{n}})^{-1}$ and $\mathbf{G}_1^{\mathrm{mn}} = \mathbf{\Phi}_1^{\mathrm{m}}\,(\mathbf{\Phi}_0^{\mathrm{n}})^{-1}$. The real-valued $m\times m$ and $m\times n$ matrices $\mathbf{Z}^{\mathrm{m}}$ and $\mathbf{Z}^{\mathrm{mn}}$ are indeterminate (with $\mathbf{Z}^{\mathrm{m}}$ symmetric and positive).

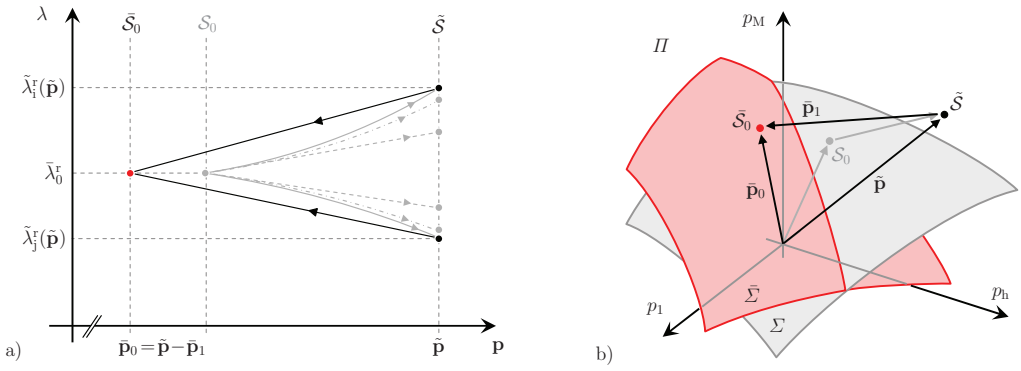

Figure 5: Inverse problem: (a) inverse multi-parameter perturbation scheme to identify the ideal system; (b) search for the ideal system in the parameter space.

4.1.2 Existence and uniqueness of the solution

Since solution (17) has *general* validity, given compatible modal data, a physically-acceptable (symmetric, positive definite) realization of the stiffness matrices governing the ideal and the experimental systems can be always reconstructed. Nonetheless, the subsequent identification of the parameter set $\mathbf{p}_0$ and $\mathbf{p}_1$ realizing the stiffness matrices, that is, the actual possibility of locating the ideal system in a physically-meaningful region of the parameter space, depends instead on the *particular* structural system, since it involves the invertibility of the stiffness-to-parameter relations (2). Moreover, accepting the first-order approximation of the backward perturbation approach, the ideal system $\bar{\mathcal{S}}_0$ solving the inverse problem is necessarily slightly different from the perfect system $\mathcal{S}_0$ (Figure 5a). Consequently, in the parameter space the locus of ideal systems differ from that of perfect systems ($\bar{\Sigma}$ and Σ in Figure 5b).

The solution non-uniqueness follows from the indeterminacy inherent to: (i) the incompleteness of the experimental data, which allow only a partial reconstruction of the stiffness matrix, while the remaining part is arbitrarily assessed by assigning the $\mathbf{Z}^{\mathrm{m}}$ matrix, and (ii) the particular perturbation scheme formulated for nearly-resonant systems, since the perfectly-resonant system is not univocally determined, depending on the arbitrary choice of the multiple eigenvalue λ_0^{r} in the preliminary decomposition of the nearly-resonant experimental eigensolution (16).

The first source of indeterminacy can be reduced only gathering additional experimental information. With respect to the second issue, instead, a convincing criterion to eliminate the problem indeterminacy is to improve the asymptotic approximation by reducing the perturbation amplitude. In the parameter space, this idea corresponds to the search for an ideal system as close as possible to the experimental system. In mathematical terms, adopting the euclidean norm $||\operatorname{diag}(\mathbf{\Lambda}_1)||$ as measure, the perturbation amplitude attains a minimum if the multiple eigenvalue is chosen as the arithmetic mean of the experimental eigenvalues, that is, $\lambda_0^{\mathrm{r}} = \frac{1}{n}\operatorname{tr}(\mathbf{\Lambda})$.

An additional element of arbitrariness regards the decomposition of their essential part in equation (16), which appears to be not unique. However, requiring that the unperturbed essential part of the eigenvector matrix $\mathbf{\Phi}_0^{\mathrm{n}}$ commutes the eigenvalue perturbation matrix $\mathbf{\Lambda}_1$, which is a necessary and sufficient condition to respect the symmetry of the perturbation stiffness sub-matrix $\mathbf{K}_1^{\mathrm{nn}}$, the decomposition rule reads

$$\mathbf{\Phi}_0^{\mathrm{n}} = \mathbf{\Phi}^{\mathrm{n}}\big(\mathbf{I} + 1/2\,\mathbf{D}^{-1}\mathbf{R}\big)^{-1}, \qquad \mathbf{\Phi}_1^{\mathrm{n}} = 1/2\,\mathbf{D}^{-1}\mathbf{R} \tag{20}$$

where the $n\times n$ matrices $\mathbf{D}$ and $\mathbf{R}$ collect the (dominant) diagonal and the (small) out-of-diagonal terms of the quasi-diagonal matrix $\mathbf{N} = \mathbf{\Phi}^{\mathrm{n}\top}\mathbf{\Phi}^{\mathrm{n}}$, respectively.

As final remark, since the ideal system $\bar{\mathcal{S}}_0$ is definitely a mathematical abstraction, which serves only to the particular perturbation strategy, no matter if it corresponds to a parameter set $\bar{\mathbf{p}}_0$ with a meaningless physical interpretation, or an unrealistic structural counterpart [8]. In this respect, inverse problem solutions may exist somewhere outside of the M-dimensional space $\mathit{\Pi}$ of the physical parameters, which – if necessary or convenient – can be extended to include auxiliary non-physical parameters.

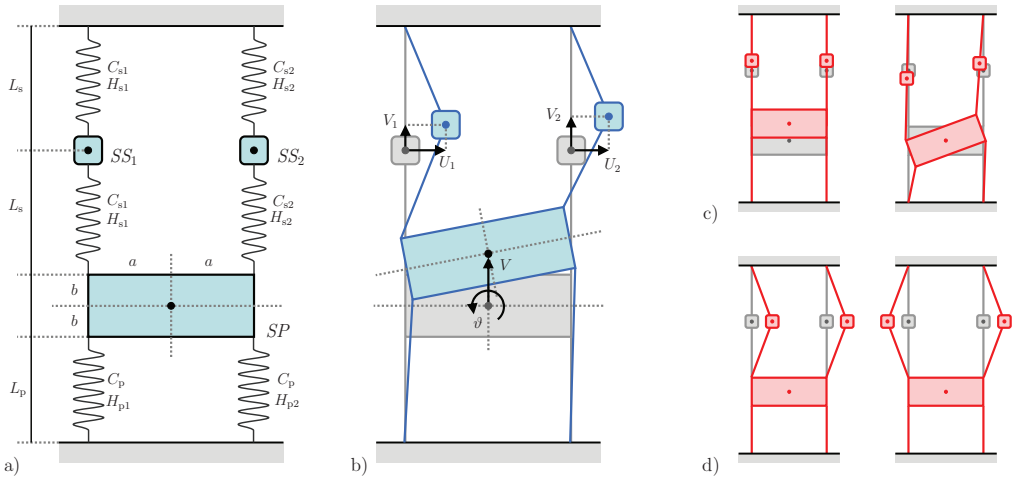

Figure 6: Multibody sectional model of a suspended bridge: (a) geometric sketch; (b) actual configuration and dynamic variables of motion; (c),(d) global and local modes.

5 Example: sectional model of a suspended bridge

The multibody system in Figure 6a, made of a principal sub-system (the rectangular rigid body *SP*) and two secondary sub-systems (the point bodies SS_1 and SS_2), represents a sectional model of a suspended bridge, in which the vertical/torsional motion of the deck, represented by the principal sub-system, is coupled with the transversal motion of many resonant stay cables, represented by the secondary sub-systems.

The motion of the principal system is described by the centroid vertical displacement V and rotation ϑ (Figure 6b). Two springs, with constant C_p, connect the principal system to the lower ground and simulate the flexural and torsional stiffness of the bridge deck. The motion of each secondary system is described by the vertical V_j and horizontal displacement U_j ($j=1,2$). Two springs connect each secondary system to the principal system and the upper ground, simulating the anchorages to the deck and the (quite rigid) tower. The spring constant C_sj and prestress H_sj account for the axial elastic and transversal geometric stiffness of the cable, respectively. Denoting M_p and J_p the vertical and rotational inertia of the principal system, and M_s the point masses of the secondary systems, nondimensional parameters can be introduced

$$\alpha=\frac{A}{B},\quad \beta=\frac{B}{L_\mathrm{s}},\quad \varrho^2=\frac{M_\mathrm{s}}{M_\mathrm{p}},\quad \chi^2=\frac{J_\mathrm{p}}{M_\mathrm{p}},\quad \mu_\mathrm{kj}=\frac{H_\mathrm{kj}}{H},\quad \gamma_\mathrm{kj}=\frac{C_\mathrm{kj}L_\mathrm{s}}{H} \tag{21}$$

where the subscripts $k=p,s$, while H is a suitable reference force.

Moving from the exact formulation of the model kinematics, the nonlinear equations governing the free undamped oscillations can be linearized around the static prestressed configuration. The initial static equilibrium can always be satisfied imposing constant prestress in each of the two spring chains ($\mu_\mathrm{pj}=\mu_\mathrm{sj}=\mu_\mathrm{j}$). Then, applying a static condensation to the vertical displacements of the secondary systems (under the reasonable assumption that $\gamma_\mathrm{sj}\gg\mu_\mathrm{sj}$), a 4*dofs* linear system is obtained ($N=4$).

For generic parameter combinations, far from the resonance regions, the system exhibits *global modes*, dominated by the vertical/torsional motion of the deck, and *local modes*, dominated by the transversal motion of the cables (Figure 6c,d).

Considering that the deck usually possesses a flat cross-section with small torsional inertia, while the cables are light and flexible, the parameters can be ordered as follows

$$\beta = \varepsilon\beta, \quad \varrho = \varepsilon\varrho, \quad \chi = \varepsilon\chi, \quad \mu_1 = \varepsilon^2\mu, \quad \mu_2 = \varepsilon^2\mu(1+\varepsilon\eta), \quad \gamma_{s2} = \gamma_{s1}(1+\varepsilon\xi) \tag{22}$$

and a triple internal resonance ($n=3$) among the two cable local modes and the deck torsional mode can be enforced by imposing $\varrho = 2\chi$, $\mu = 2\,\alpha^2\beta^2$, so that the only free parameters are $\mathbf{p} = \{\eta, \beta, \xi\}^\top$ $(M=3)$. The differences in the geometric and elastic stiffness of the two cables (η and ξ) play the role of local disorder and global coupling, respectively, while the geometric factor β performs as the cable-deck local coupling.

The multi-parameter perturbation method is applied following the lower scheme in Figure 1. First, a minimal nearly-resonant eigensolution of a pseudo-experimental system is generated and the data decomposition (16),(20) is applied; second, the inverse problem is solved to identify an ideal perfectly-resonant system, using the solution (17)-(19); third, the ideal system is employed in the direct problem to perform the sensitivity analysis of the resonant eigensolution, as described in Table 2. Figures 7a,d show the resonant eigenvalue surfaces in the whole (η-β) parameter space. The disorder η and local coupling β are found to determine a triple frequency veering with increasing amplitudes (Figure 7b,c,e,f), accompanied by a rapid hybridization process of the global and local modes. The first and second-order asymptotic solutions (dot and crosses) match the exact eigenvalue loci (black lines) with fine approximation.

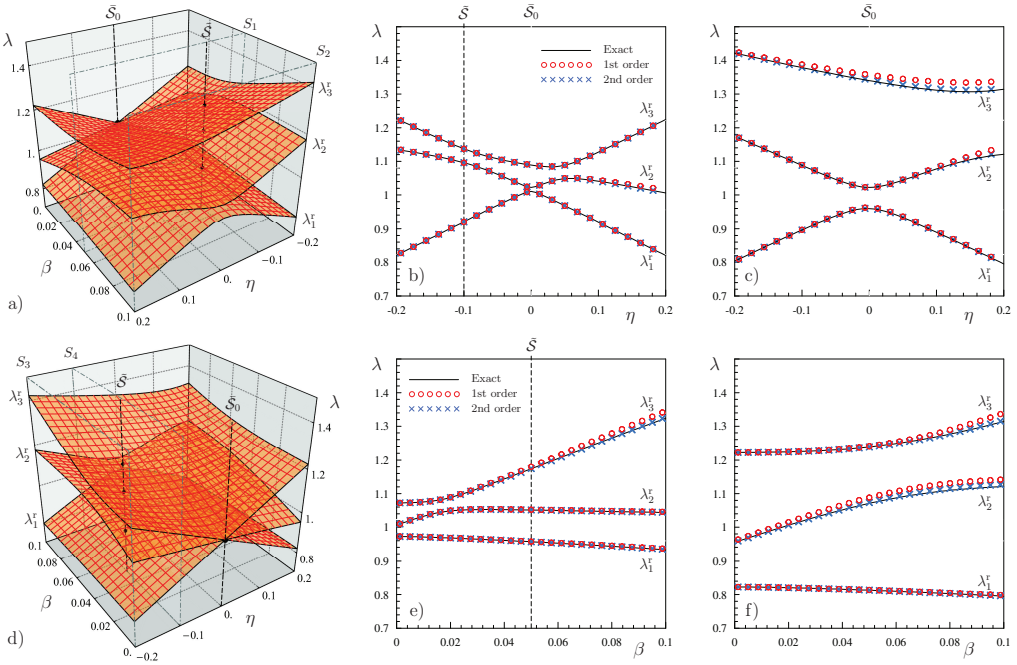

Figure 7: Eigenvalue sensitivity analysis: (a),(d) loci of the resonant eigenvalues in the (η, β)-space; (b),(c),(e),(f) comparison between the exact and asymptotic solution.

6 Conclusion

The eigensolution sensitivity analysis in linear dynamic models of internally-resonant systems has been approached through multi-parameter perturbation methods. Hamiltonian N-dimensional systems with a generic number of close eigenvalues have been considered. Following the leading idea that nearly-resonant systems can be considered as unknown perturbations of a perfectly-resonant – though unknown – reference system, two complementary problems have been tackled. First, given only a single nearly-resonant system (*experimental* system), an inverse problem has been solved to identify, within the neighboring parameter region, a close perfectly-resonant system (*ideal* system), suited to serve as starting point for sensitivity analyses. Second, a direct problem has been solved to asymptotically approximate the eigensolution of all the nearly-resonant systems (*real* systems) originated by a generic multiparameter perturbation of the ideal system, including the experimental one as a particular case. The entire procedure allows uniformly-valid sensitivity analyses for the eigensolution of internally-resonant structures in the critical parameter regions characterized by several close frequencies, corresponding to resonant global and local modes.

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