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Domain-Decomposition based $H^{1/2}$ Seminorm Preconditioners for Frictional Contact Problems

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Abstract

The bilateral or unilateral contact problem with Coulomb friction between two elastic bodies is considered [1, 12]. An algorithm is introduced to solve the resulting finite element system using a non-overlapping domain decomposition method. This technique enables the transformation of the solution of the global problem to the solution of the elasticity equations for each body separately and the solution of a smaller problem for the contact surface. The solution is obtained by using a successive approximation method, in each step of this algorithm two intermediate problems are solved, the first with prescribed tangential pressure, and the second with prescribed normal pressure[11].

Keywords: frictional contact problem, domain decomposition, Schur complement, interface preconditioner.

1 Introduction

The purpose of this paper is to study, the quasistatic two-body contact problem for small strains with friction. The mechanical interaction between the bodies is modelled, under the assumption of small displacement, by the bilateral or unilateral contact condition, and Coulomb´s friction law relating the contact force and the displacement. The main difficulties of contact problems are: the non-penetration of the bodies, the friction effect, and the contact surface are unknown in the problem. An algorithm is introduced to solve the resulting finite element system by a non-overlapping domain decomposition method, which consists of a suitable iterations based on the solutions of the elasticity equations for each body separately and the solution of a smaller problem on the contact surface. The central aspect of this work is the adaptation of a preconditioner construction developed in [2, 3, 4] for non-overlapping Dirichlet type domain decomposition method to the contact problem. The circulant matrix representations of the $H^{\frac{1}{2}}$ seminorm has been proved to be spectrally equivalent to the Schur Complement in [8]. Using this equivalence, the interface problem is transformed to an equivalent problem which is solved by a two-stage iterative technique consisting of solving consecutively a problem with prescribed tangential force and a problem with prescribed normal force. Each problem is solved with adequate mathematical programming methods.

The paper is organized in the following way: in Section 2, the contact problem with friction is discussed. In Section 3, the variational formulation of the problem is presented. The Finite Element Method is used to construct approximation spaces and an algorithm based on domain decomposition is presented in Section 4. In Section 5, a preconditioning technique for the resulting interface problem based on the circulant matrix representations of the $H^{\frac{1}{2}}$ seminorm is suggested. Some numerical examples are presented in the last section.

2 Continuous problem

Let us suppose that an elastic body, subject to external forces, occupies a union Ω of two bounded domains Ω^1 and Ω^2, in R^n, $n = 2, 3$, with Lipschitz boundaries $\partial\Omega^i$. Let the boundary $\Gamma = \partial\Omega^1 \cup \partial\Omega^2$ consist of three disjoint parts:

•Γ_d on which we impose displacements.

•Γ_f on which we impose stresses.

•Γ_c the potential contact surface between the two solids. On Γ_c, we impose on Γ_c unilateral contact condition with Coulomb friction.

In the following, the summation convention is used. The general frictional contact problem for small deformations in elastic media is governed by :

(i) equations of equilibrium:

$$\sigma^l_{ij}(u^l)_{,j} + f^l_i = 0, \quad in \quad \Omega^l, \quad l = 1, 2 \tag{1}$$

(ii) material constitutive: For elastic materials, the stress-strain relation is given by the generalized Hooke's law, i.e.

$$\sigma^l_{ij}(u^l) = a^l_{ijkl}\varepsilon_{k,l}(u^l), \quad \text{with} \quad \varepsilon(u^l) = \frac{1}{2}(\nabla u^l + (\nabla u^l)^T). \tag{2}$$

(iii) boundary conditions:

$$u^l_i = u^{l\circ}_i \qquad on \qquad \Gamma^l_d, \tag{3}$$

$$\sigma^l_{ij}n_j = g^l_i \qquad on \quad \Gamma^l_f, \tag{4}$$

where σ^l_{ij} is the Cauchy stress component, f^l_i is the ith component of the body force vector f, t_i is the ith component of the surface traction t and a_{ijkl} are the components of the elasticity tensor, satisfying the symmetry and ellipticity conditions:

$$a_{ijkl} = a_{jikl} = a_{klij} = a_{ijlk}, \quad 1 \leq i, j, k, l \leq n, \tag{5}$$

$$a_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \geq \alpha\varepsilon_{ij}\varepsilon_{ij}, \quad \alpha > 0 \quad const., \forall\varepsilon_{ij}; \tag{6}$$

(iv) the appropriate contact conditions and friction law: Contact conditions should ensure that normal stress on the contact surface is always compressive, and that the displacement on the contact surface satisfies a kinematic contact constraint to prevent interpenetration.

Afterwards we adopt the following notation for any displacement field and for any density of surface forces defined on Γ_c:($\mathbf{n} = \{n_i\}$ is the outward unit normal of Ω)

$$u = u_n\mathbf{n} + \mathbf{u}_t, \quad \text{and} \quad \sigma(u)\mathbf{n} = \sigma_n(u)\mathbf{n} + \sigma_{\mathbf{t}}(u),$$

$$\text{where} \quad u_n = u_i n_i, \quad u_{t_i} = u_i - u_n n_i, \quad \sigma_n(u) = \sigma_{ij}(u)n_i n_j,$$
$$\text{and} \quad \sigma_{t_i}(u) = \sigma_{ij}(u)n_j - \sigma_n(u)n_i.$$

We define the relative normal, tangential displacement w_n and $\mathbf{w}_t$ such that:

$$w_n = u_n^1 + u_n^2, \quad \text{and} \quad \mathbf{w}_t = \mathbf{u}_t^1 - \mathbf{u}_t^2,$$

$$\text{if } w_n < d \quad then \quad p = 0 \quad (no \quad contact) \tag{7}$$

$$\text{if } w_n = d \quad then \quad p_n < 0 \quad (contact) \tag{8}$$

$$\text{if } \mid \mathbf{p_t} \mid < \nu \mid p_n \mid \quad then \;\; \mathbf{w}_t = 0 : \;\; (sticking) \tag{9}$$

$$\text{if } \mid \mathbf{p_t} \mid = \nu \mid p_n \mid \;\; then \; \exists\lambda \; such \; \mathbf{w}_t = -\lambda\mathbf{p_t} : \;\; (sliding) \tag{10}$$

p_n and $\mathbf{p_t}$ are the normal, and the tangential stress on Γ_c and d is the initial gap between the two solids. The conditions (7)-(8) express unilateral contact between the two bodies, finally conditions (9)-(10) define a form of Coulomb's law of friction for elastostactic problems and ν is the coefficient of friction.

3 Variational problem

Let us introduce the following functional spaces $V(\Omega^l), (l = 1, 2)$:

$$V(\Omega^l) = \{v \in (H^1(\Omega^l))^n, v = 0 \quad \text{on} \quad \Gamma_d^l\}$$

and a vector field $v \in V = V(\Omega^1) \times V(\Omega^2)$ is denoted $v = (v^1, v^2)$. Supplying V with the standard inner product and norm, respectively:

$$(u, v) = (u^1, v^1)_{(H^1(\Omega^1))^n} + (u^2, v^2)_{(H^1(\Omega^2))^n}, \qquad \| u \| = (u, u)^{\frac{1}{2}}, \forall u, v \in V$$

V is a Hilbert space. We further assume, that:

$$a_{ijkl}^l \in L^\infty(\Omega^l), \quad \nu \in L^\infty(\Gamma_c), \quad f^l \in (L^2(\Omega^l))^n, \text{ and } g^l \in (L^2(\Gamma_f^l))^n.$$

We define the bilinear form:

$$a(u,v)=\sum_{l=1}^{2}a^l(u^l,v^l)=\sum_{l=1}^{2}\int_{\Omega^l}a^l_{ijkl}\varepsilon_{kl}(u^l)\varepsilon_{ij}(v^l)d\Omega,$$

for all $u, v \in V$. Next, we denote $L(.)$ the linear form which corresponds to the external loads:

$$L(v)=\sum_{l=1}^{2}L^l(v^l)=\sum_{l=1}^{2}(\int_{\Omega^l}f^l.v^l d\Omega+\int_{\Gamma^l_f}g^l.v^l d\Gamma).$$

For $v = (v^1, v^2) \in V$, we obtain from (1), through a formal application of the Green-Gauss theorem, that:

$$\sum_{l=1}^{2}\int_{\Omega^l}\sigma^l_{ij}(\frac{\partial v^l_i}{\partial x_j})d\Omega=\sum_{l=1}^{2}(\int_{\Omega^l}f^l_i v^l_i d\Omega+\int_{\Gamma^l_f}g^l_i v^l_i d\Gamma)+$$

$$\int_{\Gamma_c}\sigma^1_{ij}n^1_j v^1_i d\Gamma+\int_{\Gamma_c}\sigma^2_{ij}n^2_j v^2_i d\Gamma,\quad \forall v\in V,$$

$$a(u,v)=L(v)+\int_{\Gamma_c}\sigma^1_{ij}n^1_j v^1_i d\Gamma+\int_{\Gamma_c}\sigma^2_{ij}n^2_j v^2_i d\Gamma,\quad \forall v\in V. \tag{11}$$

We define normal and tangential contact stresses as

$$p_n=\sigma^1_{ij}n^1_i n^1_j=\sigma^2_{ij}n^2_i n^2_j,\qquad \mathbf{p_t}=\sigma^{\mathbf{1}}_{\mathbf{t}}=-\sigma^{\mathbf{2}}_{\mathbf{t}},$$

where $p^l_{t_i} = \sigma^l_{ij}n_j - \sigma_n n^l_i$. The above relations represent the action and reaction principle. The variational formulation of problem (1)-(4) and (7)-(10) in its mixed form consists of finding $(u, \mathbf{p} = (p_n, \mathbf{p}_t)) \in K \times W$ which satisfies:

$$(P)\left\{\begin{array}{c} a(u,v)-\int_{\Gamma_c}(p_n(v^1_n+v^2_n)+\mathbf{p_t}.(\mathbf{v}^1_t-\mathbf{v}^2_t))d\Gamma=L(v),\forall v\in V,\\ K=\{v\in V_{ad},\quad (v^1_i n^1_i+v^2_i n^2_i)\le d \quad \text{on} \quad \Gamma_c\},\\ W=\{\mathbf{p}\in H^{-\frac{1}{2}}(\Gamma_c)\times(H^{-\frac{1}{2}}(\Gamma_c))^n,\quad p_n\le 0,\quad \mid \mathbf{p}_t \mid\le \nu\mid p_n\mid\},\\ V_{ad}=\{v=(v^1,v^2)\in V,\quad u^l_i=u^{l\circ}_i \qquad on \qquad \Gamma^l_d\}, \end{array}\right\} \tag{12}$$

where $H^{-\frac{1}{2}}(\Gamma_c)$ is the dual space of $H^{\frac{1}{2}}(\Gamma_c)$.

To find the solution of the problem (P), we use the method of successive approximations starting from an initial $\Gamma^0_{cr} \subset \Gamma_c$ real contact zone. The kth step of the iteration is given by:

1. Solve the contact problem with given $\Gamma^k_{cr} \subset \Gamma_c$.
2. The solution is used to find the new contact zone $\Gamma^{k+1}_{cr} \subset \Gamma_c$.

The process is stopped if two successive contact zones are the same, $\Gamma^{k+1}_{cr} = \Gamma^k_{cr}$.

4 Finite Element Discretization and Domain Decomposition

4.1 Discretization

In order to obtain an approximation of (12), we will use the method of finite elements. To this end, we associate to each subomain Ω^l a regular family of triangulation $\{T_h^l\}$(triangles or quadrilatere in R^2, tetrahedra in R^3). Moreover, we suppose that the extreme points x_1 and x_2 of the real contact zone Γ_{cr} are common nodes of the meshes on both bodies. The contact zone Γ_{cr} inherits two independent regular discretization associated with $\{T_h^1\}$ and $\{T_h^2\}$.

We associate each $\{T_h\} = \{T_h^1\} \cup \{T_h^2\}$ with a finite-dimensional space V_h of piecewise linear vector functions:

$$V_h = \{v_h = (v_h^1, v_h^2) \in (C(\bar{\Omega}^1) \times C(\bar{\Omega}^2))^n, v_h^l \mid_T \in (P_1(T))^n \forall T \in T_h^l,$$
$$v_h^l = 0 \quad \text{on} \quad \Gamma_d^l, l = 1, 2\},$$

where $C(\bar{\Omega})$ stands for the space of continuous functions on $\bar{\Omega}$ and $P_k(T)$ represents the space of polynomial functions of degree k on T.

Let $\mathbf{W}_n = (W_{ni})^T \in R^k$ and $\mathbf{W}_t = (\mathbf{W}_{ti})^T \in R^{2k}$, $1 \leq i \leq k$ denote the vectors of components of the nodal values on Γ_c of w_{hn} and $\mathbf{w}_{ht}$ respectively. Let R denote the vector of the component of the nodal values on Γ_c of the contact force $\mathbf{p}$ and decomposed on the normal forces $\mathbf{R}_n = (R_{n1}, R_{n2}, ..., R_{nk})^T \in R^k$ and the tangential forces $\mathbf{R}_t = (\mathbf{R}_{t1}, \mathbf{R}_{t2}, ..., \mathbf{R}_{tk})^T \in R^{2k}$.

Using the above notations, the pointwise formulation of the frictional contact conditions are:

$$R_{ni} \leq 0, W_{ni} - d \leq 0, R_{ni}(W_{ni} - d) = 0, \quad 1 \leq i \leq k.$$
$$\mid \mathbf{R}_{ti} \mid < \nu \mid R_{ni} \mid, \ \mathbf{W}_t = 0, \quad 1 \leq i \leq k$$
$$\mid \mathbf{R}_{ti} \mid = \nu \mid R_{ni} \mid, \ \exists \lambda_i > 0 \ \mathbf{W}_{ti} = -\lambda_i \mathbf{R}_{ti}, \ 1 \leq i \leq k$$

This problem is discretized by finite element method and let A^l, U^l, L^l and R^l denote the symmetric positive definite stiffness matrix, the vector of displacements, the vector of prescribed forces and the unknown vector of contact forces on the contact surface associated with the subdomains $\Omega^l, (l = 1, 2)$, respectively. These quantities can be partitioned as follows:

$$A^l = \begin{bmatrix} A_{ii}^l & A_{ic}^l \\ A_{ci}^l & A_{cc}^l \end{bmatrix}, U^l = \begin{bmatrix} U_i^l \\ U_c^l \end{bmatrix}, L^l = \begin{bmatrix} L_i^l \\ L_c^l \end{bmatrix}, R^l = \begin{bmatrix} 0 \\ R_c^l \end{bmatrix},$$

where the subscripts c and i indicate the interface contact boundary and the remainder of degrees of freedom in each subdomain. Using the above notations, the subdomain equations of equilibrium can be written as:

$$\begin{bmatrix} A_{ii}^l & A_{ic}^l \\ A_{ci}^l & A_{cc}^l \end{bmatrix} \begin{bmatrix} U_i^l \\ U_c^l \end{bmatrix} = \begin{bmatrix} L_i^l \\ L_c^l \end{bmatrix} + \begin{bmatrix} 0 \\ R_c^l \end{bmatrix}. \tag{13}$$

4.2 Domain Decomposition Method

As stated before, the presented method is an iterative procedure which is based on a finite element analysis of the contact problem with friction. In each iteration, the contact problem is solved by a domain decomposition method, the solution is used to find the contact interfaces between the elastic bodies, and the modification of the contact interface induce me to remesh each subomain Ω^l. This process is repeated until the stabilization of the contact zone. The re-meshing assure that the nodes of the meshes on both bodies are identical. Each step of the iterative procedure is outlined as follows:

$$\left\{ \begin{array}{c} \text{For a given}(\Gamma_{cr})^k \text{ we apply the DD-method which consists of computing} \\ \text{the solution as follows:} \\ (U^1, U^2) = (U^{P1}, U^{P2}) + (U^{H1}, U^{H2}). \\ \text{1. Compute} \qquad U^{Pl} = (U_i^{Pl}, 0)^T, \\ U_i^{Pl} = (A_{ii}^l)^{-1} L_i^l. \\ \text{2. The remaining part is computed as:} \\ a.) \qquad U_c^{Hl} = S_l^{-1} L_c^l + S_l^{-1} R_c^l - S_l^{-1} A_{ci}^l U_i^{Pl}, \\ b.) \qquad U_i^{Hl} = -(A_{ii}^l)^{-1} A_{ic}^l U_c^{Hl}. \\ \text{where } S_l = A_{cc}^l - A_{ci}^l (A_{ii}^l)^{-1} A_{ic}^l \text{ is the Schur complement of the stiffness} \\ \text{matrix } A^l. \\ \text{3. Assembling of the results :}(U^1, U^2) = (U^{P1}, U^{P2}) + (U^{H1}, U^{H2}), \\ \text{4. The determination of the new real contact surface}(\Gamma_{cr})^{k+1}. \end{array} \right.$$

The steps 1. and 2. b.) are equivalent to the solutions of an elasticity equations with given boundary tractions on (Γ_c) and the computation of U^{Pl} and U_i^{Hl} can be carried out on each subdomain concurrently. The computation of the displacement U_c and the contact force R_c in (Γ_c) requires the solutions of a nonlinear problem posed in (Γ_c). Using the following notations $U_c^{l0} = S_l^{-1} L_c^l - S_l^{-1} A_{ci}^l U_i^l$ and $R = R_c^1 = -R_c^2$, we obtain the interface problem:

$$(S_1^{-1} + S_2^{-1}) R = (U_c^1 - U_c^2) - (U_c^{10} - U_c^{20}), \tag{14}$$

where R and U_c^l must satisfy the contact conditions:

$$R_{ni} \leq 0, \; W_{ni} = d_{ni}, \; 1 \leq i \leq k, \tag{15}$$

$$\mid \mathbf{R}_{ti} \mid < \nu \mid R_{ni} \mid, \; \mathbf{W}_t = 0, \quad 1 \leq i \leq k, \tag{16}$$

$$\mid \mathbf{R}_{ti} \mid = \nu \mid R_{ni} \mid, \; \exists \lambda_i > 0 \; \mathbf{W}_{ti} = -\lambda_i \mathbf{R}_{ti}, \; 1 \leq i \leq k. \tag{17}$$

5 Solution of the interface problem

5.1 Algorithm

Let P be spectrally equivalent to $S_1^{-1} + S_2^{-1}$. Now, our problem take the following form:

$$PR = W - W^0, \quad + \quad \text{conditions (15)(16)(17).} \tag{18}$$

However, the interface (18) is not equivalent to a minimization problem because the constraint set depends on the solution R_n. Then, we attempt to solve it with a two-stage iterative technique which consists of solving a problem in the normal direction and a problem in the tangential direction consecutively. This technique have been proposed by Panagiotopoulos in [11]. The equation (18) can be written in partitioned form:

$$Q_{nn}R_n + Q_{nt}\mathbf{R}_t = d_n + W_n^0 \tag{19}$$

$$Q_{tn}R_n + Q_{tt}\mathbf{R}_t = \mathbf{W}_t + \mathbf{W}_t^0 \tag{20}$$

where Q_{nn} is a $(k \times k)$ matrix ,Q_{nt} is a $(k \times 2k)$ matrix, Q_{tn} is a $(2k \times k)$ matrix and Q_{tt} is a $(2k \times 2k)$ matrix.

Now two-stage iterative technique of (18) can be written as:

• step Initialization set $\mathbf{R}_t = 0$

• step Normal problem : We fix $\mathbf{R}_t$ and to obtain R_n, we solve the following frictionless problem which is equivalent to the following Quadratic Programming Problem (QPP):

$$\text{Minimize :} \quad J_n(R_n) = \frac{1}{2}R_n^T Q_{nn} R_n + R_n^T F_n \quad \text{with} \quad R_n < 0. \tag{21}$$

• step Tangential problem: We fix R_n and to obtain R_t, we solve a Coulomb friction problem under given normals loads. Coulomb's law connects the tangential forces with the normal forces by the relation:

$$\varphi_i = \nu \mid R_{ni} \mid - \parallel \mathbf{R}_{ti} \parallel, \quad i = 1, .., k.$$

If $\parallel \mathbf{R}_{ti} \parallel < \nu \mid R_{ni} \mid$ (i.e., $\varphi_i > 0$) then we have $\mathbf{W}_t = 0$,

if $\parallel \mathbf{R}_{ti} \parallel = \nu \mid R_{ni} \mid$ (i.e., $\varphi_i = 0$) then we have slipping in the opposite direction of $\mathbf{R}_t$:

$$\mathbf{W}_{ti} = -\lambda_i \mathbf{R}_{ti}, \quad \lambda_i \geq 0, \quad i = 1, .., k,$$

where λ_i is the ratio of the magnitude of the local displacement.

Now for given normal contact force, the local friction contions are:

$$\mathbf{W}_{ti} = -\lambda_i \mathbf{R}_{ti}, \quad \varphi_i \geq 0, \quad \lambda_i \geq 0, \quad \varphi_i \lambda_i = 0, \quad i = 1, .., k,$$

and the tangential problem is equivalent to the following convex nonlinear programming problem:

$$\text{Minimize :} \quad J_t(\mathbf{R}_t) = \frac{1}{2}\mathbf{R}_t^T Q_{tt} \mathbf{R}_t + \mathbf{R}_t^T \mathbf{F}_t \quad \text{with} \quad \varphi_i \geq 0, \quad i = 1, .., k. \tag{22}$$

• Computational steps (1) and (2) are repeated until the non-dimensional norm of tangential force is less than a prescribed tolerance. The non-dimensional norm is defined as follows:

$$\frac{|| \mathbf{R}_t^{present} - \mathbf{R}_t^{previous} ||_2}{|| \mathbf{R}_t^{present} ||_2} < \epsilon$$

and typically a value of ϵ is 10^{-5} used.

In this study, the normal problem is solved by the Penalty Method, while the tangential problem is solved by the Lagrangian multiplier technique.

5.2 The preconditioner construction

In this section we present preconditioning techniques for the iterative solver. The introduced $S_1^{-1}+S_2^{-1}$ preconditioner constructions are based on the spectral equivalence relations [13, 14, 15]

$$c_{11}\cdot \mid v_c \mid_{H_{00}^{1/2}(\Gamma_c)} \;\leq\; (S_l V_c, V_c) \;\leq\; c_{12}\cdot \log^2\left(h^{-1}\right)\cdot \mid v_c \mid_{H_{00}^{1/2}(\Gamma_c)}, \tag{23}$$

and

$$c_{21}\cdot \mid v_c \mid_{H_{00}^{1/2}(\Gamma_c)} \;\leq\; \mid \tilde{v}_c^l \mid_{H^{1/2}\left(\partial\Omega^l\right)} \;\leq\; c_{22}\cdot \mid v_c \mid_{H_{00}^{1/2}(\Gamma_c)}, \tag{24}$$

$\forall v \in V_h$ $(l = 1, 2)$, where V_h denotes the finite element approximation of V with the parameter of the discretization h, v_c is the restriction of v to Γ_c, V_c is the vector representation of v_c at the grid points of Γ_c, $\tilde{v}_c^l$ is the extension of v_c from Γ_c to $\partial\Omega^l$ by zero, and c_{ij} $(i, j = 1, 2)$ denote positive constants independent of h.

In the two-dimensional case we use the preconditioning matrices

$$P_{2,1} = E_1^T \cdot Q_1 \cdot E_1 + E_2^T \cdot Q_2 \cdot E_2, \tag{25}$$

and

$$P_{2,2} = \left(E_1^T \cdot L_1^{-1} \cdot Q_1 \cdot E_1 + E_2^T \cdot L_2^{-1} \cdot Q_2 \cdot E_2\right)^{-1}, \tag{26}$$

where E_l denotes the matrix representation of the discrete extension from Γ_c to $\partial\Omega^l$ by zero, Q^l, L^l are the sparse circulant matrix representations of the $H^{1/2}$ seminorm and H^1 norm [5, 6] , respectively. Introducing the notation

$$C = C(c_0, c_1, \ldots, c_{m-1}, c_m, c_{m-1}, \ldots, c_1)$$

$$= \begin{pmatrix} c_0 & c_1 & c_2 & \ldots & c_1 \\ c_1 & c_0 & c_1 & \ldots & c_2 \\ c_2 & c_1 & c_0 & \ldots & c_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_1 & c_2 & c_3 & \ldots & c_0 \end{pmatrix}$$

these matrices can be expressed as

$$Q_l = C(q_0, q_1, \ldots, q_{m-1}, q_m, q_{m-1}, \ldots, q_1), \tag{27}$$

where

$$q_j = \begin{cases} 2\sum_{k=1}^{2^{[log_2(m)]}-1} \frac{1}{k^2} & \text{if } j=0 \\ -\sum_{k=2^{i-1}}^{2^i-1} \frac{1}{k^2} & \text{if } 0<j<m \\ & \text{and } j=2^i-1 \\ 0 & \text{otherwise} \end{cases},$$

and

$$L_l = C(l_0, l_1, \dots, l_{m-1}, l_m, l_{m-1}, \dots, l_1), \tag{28}$$

where

$$l_j = \begin{cases} \frac{2}{h}+h & \text{if } j=0 \\ -\frac{1}{h} & \text{if } j=1 \\ 0 & \text{otherwise} \end{cases},$$

In the three-dimensional case we used the preconditioning matrix

$$P_{3,1} = h \cdot E_1^T \cdot \sum_{p\in\{x,y,z\}} Q_{1,p} \cdot E_1 + h \cdot E_2^T \cdot \sum_{p\in\{x,y,z\}} Q_{2,p} \cdot E_2, \tag{29}$$

where the matrices $Q_{l,p}$ denote a sum of one-dimensional sparse circulant seminorm representations Q_l in the direction p. Here we applied the separability property of the $H^{1/2}$ seminorm [7, 9, 10].

The separability property means that the $H^{1/2}$ seminorm can be represented as the sum

$$\sum_{p\in\{x,y,z\}} \mid \tilde{v}_c^l \mid^2_{H_p^{1/2}(\partial\Omega^l)}, \tag{30}$$

of 'partial' seminorms. The 'partial' seminorm in the direction x is defined by the expression

$$\mid \tilde{v}_c^l \mid^2_{H_x^{1/2}(\partial\Omega^l)} = \tag{31}$$

$$\int_{x_{min}}^{x_{max}} \int_{\partial\Omega_x^l} \int_{\partial\Omega_x^l} \frac{\mid \tilde{v}_c^l(x,y_1,z_1) - \tilde{v}_c^l(x,y_2,z_2) \mid^2}{\mid (y_1-y_2)^2 + (z_1-z_2)^2 \mid} ds_x(y_2,z_2) ds_x(y_1,z_1) dx,$$

where

$$x_{min} = \min_{(x,y,z)\in\partial\Omega^l} x, \quad x_{max} = \max_{(x,y,z)\in\partial\Omega^l} x,$$

$$\partial\Omega_x^l = \left\{ (\tilde{x},\tilde{y},\tilde{z}) \in \partial\Omega^l \mid \tilde{x} = x \right\}$$

and $s_x(.)$ is the arclength on $\partial\Omega_x^l$. The seminorms $\mid \tilde{v}_c^l \mid^2_{H_y^{1/2}(\partial\Omega^l)}$ and $\mid \tilde{v}_c^l \mid^2_{H_z^{1/2}(\partial\Omega^l)}$ are defined analogously.

The spectral equivalence relations ensure that the defined P_{ij} $((i,j) = (2,1), (2,2), (3,2))$ preconditioning matrices are almost optimals and the spectral condition number of a $P_{i,j} \cdot (S_1^{-1} + S_2^{-1})$ matrix is $O(\log^2(N))$, where N denotes the number of grid points on Γ. The matrices $P_{2,1}$ and $P_{2,2}$ have very simple structures, and a multiplication by these matrices requires only $O(N\log(N))$ arithmetic operations. The cost of a matrix multiplication by $P_{2,2}$ is much more expensive. However $P_{2,2}$ theoretically is better preconditioning matrix than $P_{2,1}$.

6 Numerical Results

In this section, some examples and numerical results related to the proposed algorithms are presented.

6.1 First examples

In this example, we consider the problem of a flat punch pressed on an elastic foundation. The foundation is characterized by a Young modulus E_F = 5MPa and Poisson 's ratio v_F = 0.3, while the elastic punch is constructed of a material with E_P = 5MPa and v_P = 0.3. The problem is solved for two cases: two-dimensional and three-dimensional formulations. Both cases we assume the coefficient of friction is $\nu = 0.5$, and the applied normal force at the top of the punch $P = 100N$. Fig 1. shows the computed deformed configurations and the corresponding Von Mises stress distribution. The contact pressures are plotted in Figure 3.(a).

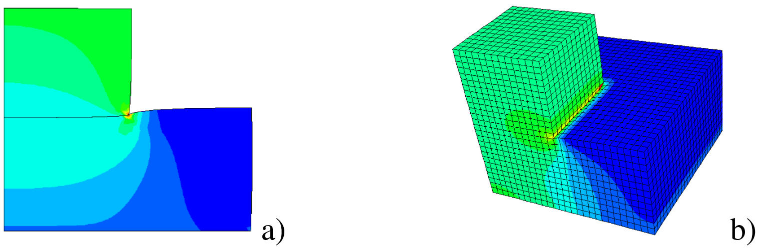

Figure 1: The deformed configurations (a)two-dimensional and (b)three-dimensional.

6.2 Sphere on an elastic foundation

We consider the contact between a half sphere and an elastic foundation. The foundation is characterized by a Young modulus E_F = 5MPa and Poisson's ratio v_F = 0.3, while the elastic sphere is constructed of a material with E_P = 5MPa and v_P = 0.3. Using the symmetry with respect to the vertical axis through the center of the sphere, the study can be reduced to a half-plane with axisymmetrical co-ordinates. In this case, we assume the coefficient of friction is $\nu = 0.5$, and the applied normal force at the top of the sphere $P = 100N$. Fig 2. shows the computed deformed configurations and the corresponding Von Mises stress distribution. The contact pressure is plotted in Figure 3.(b).

7 Conclusion

To simulate the non-linear frictional interaction between two-body, a non-overlapping domain decomposition method based $H^{\frac{1}{2}}$ seminorm preconditioners is proposed. Numerical examples are carried out and our results are in agreement with those obtained

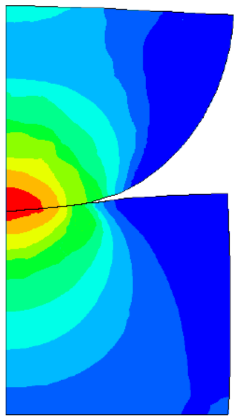

Figure 2: The deformed configuration.

in [1, 12]. This shows that the present scheme leads to an accurate numerical solution to the frictional contact problem.

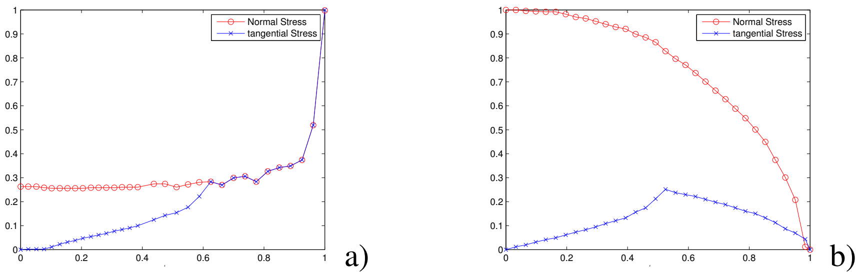

Figure 3: Normalized contact pressures (a) two-dimensional and(b) axisymmetric.

The proposed preconditioning techniques for the iterative solver are based on the spectral equivalence relations which ensure that the defined preconditioning matrices are almost optimals and the condition numbers of the preconditioned matrices are uniformly bounded. It is important to note that the algorithm converges to the solution with only a few iterations, which is especially important in non-linear problems, and a significant reduction in the computational cost can be achieved. The greatest advantage of the proposed preconditioned matrices is its simplicity, stability and can be easily integrated into the standard finite element codes.

The proposed approach can be extended to more contact problems with complex geometry and practical engineering problems of the real world such as metal forming and frictional contact problem in large deformation.

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