Paper 97

©Civil-Comp Press, 2012
Proceedings of the Eighth International Conference
on Engineering Computational Technology,
B.H.V. Topping, (Editor),
Civil-Comp Press, Stirlingshire, Scotland

Transient Wave Analysis for an Inhomogeneous Elastic Thick Plate subjected to Inplane Impact Loading

K. Miura
Department of Mechanical Engineering
Faculty of Engineering and Resource Science
Akita University, Japan

Abstract

The inplane transient problem for a linear inhomogeneous elastic thick plate with the weak inhomogeneity is handled. We assume the linear inhomogeneity in which the phase velocity linearly changes along the thickness direction of the plate. First, using the Cagniard method which analytically caries out the Fourier-Laplace double inversions, the analytic solutions of the transient response for the inhomogeneous half space are deduced, and the numerical results are obtained for stress components σ_{yy}, τ_{yx}, σ_{xx}, strain energy U, displacement components u, v. Furthermore, considering redistribution of the reflected energy for P and S waves and the method of images obtaining the reflection of a couple of P and S waves at the boundary, the reflection responses for the homogeneous elastic thick plate are calculated. The transient response for the linear inhomogeneous thick plate can be accurately obtained by the layered model of the homogenous layer. Finally, relating these results to the Rayleigh wave and the Head wave, the whole aspect of the transient wave propagation for the inhomogeneity is clarified.

Keywords: transient analysis, exact solution, inhomogeneity, elastic plate, Lamb's problem, inplane loading, reflection response, Cagniard method, method of images.

1 Introduction

When we observe the global environment such as ocean, ground and atmosphere, and the typical materials such as shell, bamboo and fang, we understand that these media naturally and significantly consist of the inhomogeneous structures of the density and rigidity. Since there are many requests from the industrial side, the inhomogeneous structure is generally studied by many researchers of the fields of biomaterials, optics and other engineering with the purpose of relaxation of thermal stress and energy transformation. Furthermore, these researches advance, it's been possible to create the industrial product which over two contradictory elastic properties are continuously and functionally built in one material. The thin layered functional graded material made by the PVD and CVD technologies has been generally used. Most of more advanced materials such as the artificial bone and the spacecraft cladding will become inhomogeneous in the future. Therefore, the establishment of the theoretical and experimental strength evaluation technologies of these materials is desired. When the elastic body made up of the inhomogeneous structure is taken up as an object of the transient boundary value problem, it is rational to catch for us as a composition structure in which the phase velocity of the material continuously changes in one direction. Here, we consider the transient problem for the inhomogeneous elastic media.

As we historically survey the research of the wave propagation for the elastic body, the classical analysis was given by Lamb [1] who treated the problem applied the surface normal line source for the elastic half space. The exact solution of this problem was developed by Cagniard [2] and the substantial simplification was provided by de Hoop [3]. From the viewpoint of the earthquake research, Ewing et al. [4] and Brekhovskikh [5] studied the wave propagation for the elastic layered media. Miklowitz [6] and Pao [7] gave the excellent insights for the transient analysis of the elastic plate. The papers of the transient response for the inhomogeneous elastic body can be found in Refs [8,9,10,11]. When we consider the inhomogeneous structure of the elastic body, it's rational for the mathematical model that the phase velocity of SH wave continuously changes along the unidirection of the material. Author [12,13] handled the exact analysis of the SH transient problem for these inhomogeneous half space.

In this report we exactly analyse the transient responses of the stress and displacement components for the inhomogeneous elastic thick plate subjected to the inplane impact load on the free surface. The phase velocity of P and S wave linearly changes along the thickness direction of the plate. Initially, we consider the Lamb's problem for an inhomogeneous elastic half-space with the weak inhomogeneity. In order to solve this problem, it is necessary to define a pair of the potentials for the separation of the wave equation. Introducing the scalar and vector potentials ϕ, ψ which coincides with the infinitesimal deformation of the inhomogeneous wave field, we can deduce the separate two wave equations represented by these potentials. For this elastic half space, we assume that the Lame's constants of the material have an inhomogeneous characteristics. The closed form exact solutions is obtained by the application of the Cagniard inversion to the Fourier-Laplace double transformed solutions deduced from the constitutive equations and the boundary conditions at the free surface. Therefore, we can get the stress components σ_{yy}, σ_{xx}, τ_{yx}, the strain energy U, and the displacement components u, v represented an integral form of the stress, as the analytic solutions for the inhomogeneous half-space.

For the inhomogeneous elastic plate, we examine the transient wave reflection problem from two free surfaces of the plate on the basis of these analytic solutions, and we apply the method of images [14] considered the boundary conditions at the reflection interfaces. The numerical calculation is carried out by the superposition of the analytic solutions. Finally, we examine the energy redistribution [15] to the P and S reflected waves which coincide with the reflection responses deduced from the Snell's law, the coupling of the Head wave, P and S wave at the reflection wave interface, and the contribution to the wave field of the Rayleigh surface wave, the

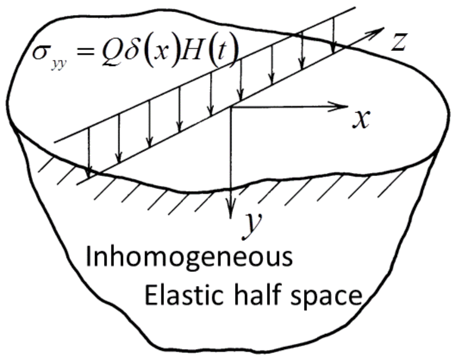


Figure 1 : Inhomogeneous elastic half space subjected to inplane impact line load

Head and P, S wave fronts. It is arranged as the animations of contour mapping representations through the full transient response field. The aspect of the transient responses is clarified with the consistency which satisfies boundary conditions at the top and bottom free surfaces.

2 Analysis

The inplane transient problem for the linear inhomogeneous elastic thick plate is investigated. In this plate, the phase velocity linearly changes along the thickness direction. Since it's difficult to solve directly this problem, after analytically solving the inplane transient problem for the inhomogeneous half space by the Cagniard method, we numerically calculate the reflection response of the transient wave for the plate by using the method of images coupled P and S wave at the boundary.

2.1 Formulation and basic equation

We consider the transient problem for the linear inhomogeneous half space subjected to the impact line load on the free surface $y = 0$ at time $t = 0$ as shown in Figure 1. Therefore, the deformation is plane strain and the displacements u and v are not depend on the z coordinate.
The displacement-potential relations are

$$u = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \quad , \quad v = \eta^2 \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi}{\partial y} \tag{1),(2}$$

where,

$$\eta = 1 + by \tag{3}$$

b is the inhomogeneous parameter, u and v are the displacements in the x and y directions, and Φ , Ψ are the scalar and vector potentials, respectively.
The stress components are

$$\sigma_{xx} = \lambda_x \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu_x \frac{\partial v}{\partial y} \tag{4}$$

$$\sigma_{yy} = \lambda_y \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + 2\mu_y \frac{\partial v}{\partial y} \tag{5}$$

$$\tau_{yx} = \mu_x \frac{\partial v}{\partial x} + \mu_y \frac{\partial u}{\partial y} \tag{6}$$

Lame's constants of x and y directions are considered to have a linear inhomogeneous anisotropic structure, so that

$$\lambda_x = \lambda_0 \quad , \quad \lambda_y = \lambda_0 \eta^2 \tag{7),(8}$$

$$\mu_x = \mu_0 \quad , \quad \mu_y = \mu_0 \eta^2 \tag{9),(10}$$

where λ_0, μ_0 are Lame's constants at the free surface $y = 0$ of the elastic half space. The equations of motion are

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \tag{11}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} \tag{12}$$

where ρ is the mass density of the material.
Substituting Eqs. (1)~(10) in Eqs. (11),(12) give

$$\eta^2 \frac{\partial^2 \Phi}{\partial \eta^2} + 2\eta \frac{\partial \Phi}{\partial \eta} + \frac{1}{b} \frac{\partial^2 \Phi}{\partial x^2} = \frac{1}{b^2 c_L^2} \frac{\partial^2 \Phi}{\partial t^2} \tag{13}$$

$$\eta^2 \frac{\partial^2 \Psi}{\partial \eta^2} + 2\eta \frac{\partial \Psi}{\partial \eta} + \frac{1}{b^2} \frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{b^2 c_T^2} \frac{\partial^2 \Psi}{\partial t^2} \tag{14}$$

where c_L, c_T are the phase velocities of the dilatational wave and the shear wave on the free surface of the inhomogeneous half space, respectively.
Furthermore, we assume that

$$\phi = \eta^{1/2} \Phi \ , \ \psi = \eta^{1/2} \Psi \tag{15),(16}$$

$$q = \frac{1}{b} \ln \eta \tag{17}$$

For the inhomogeneous half space Eqs. (13),(14) reduce to the wave equations

$$\frac{\partial^2 (\phi, \psi)}{\partial q^2} - \frac{b^2}{4} (\phi, \psi) + \frac{\partial^2 (\phi, \psi)}{\partial x^2} = \frac{1}{(c_L^2, c_T^2)} \frac{\partial^2 (\phi, \psi)}{\partial t^2} \tag{18}$$

By how to choose the b value, the inhomogeneity varies from the weak side to the strong side. Here, considering the case of $b \ll 1$, we neglect the second term of the left side of Eq. (18). Therefore, y coordinate variable and the inhomogeneous parameter b appear only in the inhomogeneous variable q. Furthermore, in Eqs. (3),(17), $q = 0$ at $y = 0$, $q \to \infty$ for $y \to \infty$, and the q value monotonously increase for the monotonous increase of the y value. Here, considering the weak inhomogeneity and replacing the (x,y) Cartesian coordinate with the (x,q) Cartesian coordinate, we analyse this transient problem for the inhomogeneous elastic half space

2.2 Transient analysis for inhomogeneous elastic half space

We consider the transient analysis for the inplane deformation problem which the impulsive load acts on the free surface of the inhomogeneous elastic half space as shown in Figure 1.
The wave equations represented by the potentials are

$$\nabla^2\phi = \frac{1}{c_L^2}\frac{\partial^2\phi}{\partial t^2} \tag{19}$$

$$\nabla^2\psi = \frac{1}{c_T^2}\frac{\partial^2\psi}{\partial t^2} \tag{20}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial q^2} \tag{21}$$

The boundary conditions are

$$\sigma_{qq}(x,0,t) = \sigma_0\delta(x)H(t) \tag{22}$$

$$\tau_{qx}(x,0,t) = 0 \tag{23}$$

The initial condition are

$$\phi(x,q,0) = \frac{\partial\phi}{\partial t}(x,q,0) = 0 \tag{24}$$

$$\psi(x,q,0) = \frac{\partial\psi}{\partial t}(x,q,0) = 0 \tag{25}$$

Two pairs of the integral tranforms are defined as follows
The Fourier integral transform is

$$\mathcal{F}[f(x)] \equiv \overline{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-is\xi s}dx \tag{26}$$

$$\mathcal{F}^{-1}\left[\overline{f}(\xi)\right] \equiv f(x) = \frac{s}{2\pi}\int_{-\infty}^{\infty}\overline{f}(\xi)e^{is\xi s}d\xi \tag{27}$$

The Laplace integral transform is

$$\mathcal{L}[f(t)] \equiv f^*(\xi) = \int_0^{\infty} f(t)e^{-st}dt \tag{28}$$

$$\mathcal{L}^{-1}[f^*(s)] \equiv f(t) = \frac{1}{2\pi i}\int_{B_r} f^*(s)e^{st}ds \tag{29}$$

where Br of Eq. (29) stands for the integral of Bromwich-Wagner.
Applying the Fourier Laplace double transforms to Eqs. (1)~(21), we obtain the transformed solutions

$$\overline{\phi}^*(\xi,q,s) = A(\xi,s)e^{-s\gamma_1(\xi)q} \tag{30}$$

$$\overline{\psi}^*(\xi,q,s) = B(\xi,s)e^{-s\gamma_2(\xi)q} \tag{31}$$

where

$$\gamma_1(\xi) = \sqrt{\xi^2 + v_1^2}\ ,\quad \gamma_2(\xi) = \sqrt{\xi^2 + v_2^2}\ ,\quad (\mathrm{Re}\,\gamma_i \geq 0) \tag{32),(33}$$

$$v_1 = \frac{1}{c_L},\ v_2 = \frac{1}{c_T} \tag{34),(35}$$

where v_1, v_2 represent the slowness of the dilatational wave and shear wave, respectively. Considering the boundary conditions Eqs. (22),(23) and the initial conditions (24),(25), we can get the unknown parameters A,B

$$A = \frac{\sigma_0}{\mu s^3}\left(2\xi^2 + v_2^2\right)/\Delta \tag{36}$$

$$B = \frac{\sigma_0}{\mu s^3} 2i\xi\gamma_1 \ /\Delta \tag{37}$$

where

$$\Delta = \left(2\xi^2 + v_2^2\right)^2 - 4\xi^2\gamma_1\gamma_2 \tag{38}$$

$\Delta = 0$ represents the Rayleigh equation.

The Cagniard method [16] is applied in order to obtain the exact solutions from the Fourier Laplace double transformed solutions. Then we finally obtain the following exact solutions in the (x,y) coordinate system for the stress components and the displacement components

$$\begin{aligned}\sigma_{yy} = \frac{\sigma_0}{2\pi}\Big\langle & H(t - v_1 R)\mathrm{Re}\left[M_{yy1}(\xi_{1+})\frac{\partial \xi_{1+}}{\partial t} - M_{yy1}(\xi_{1-})\frac{\partial \xi_{1-}}{\partial t}\right] \\ & + H(t - v_2 R)\mathrm{Re}\left[M_{yy2}(\xi_{2+})\frac{\partial \xi_{2+}}{\partial t} - M_{yy2}(\xi_{2-})\frac{\partial \xi_{2-}}{\partial t}\right] \\ & + H\left\{t - v_1 x - \frac{1}{b}\sqrt{v_2^2 - v_1^2}\ln(1 + by)\right\}H(v_2 R - t)\left[M_{yy\Delta\Gamma_2}(r_-)\frac{\partial r_-}{\partial t}\right]\Big\rangle\end{aligned} \tag{39}$$

$$\begin{aligned}\tau_{yx} = \frac{\sigma_0}{\pi}\frac{1}{\eta}\Big\langle & H(t - v_1 R)\mathrm{Im}\left[M_{yx1}(\xi_{1+})\frac{\partial \xi_{1+}}{\partial t} - M_{yx1}(\xi_{1-})\frac{\partial \xi_{1-}}{\partial t}\right] \\ & + H(t - v_2 R)\mathrm{Im}\left[M_{yx2}(\xi_{2+})\frac{\partial \xi_{2+}}{\partial t} - M_{yx2}(\xi_{2-})\frac{\partial \xi_{2}}{\partial t}\right] \\ & + H\left\{t - v_1 x - \frac{1}{b}\sqrt{v_2^2 - v_1^2}\ln(1 + by)\right\}H(v_2 R - t)\left[M_{yx\Delta\Gamma_2}(r_-)\frac{\partial r_-}{\partial t}\right]\Big\rangle\end{aligned} \tag{40}$$

$$\begin{aligned}\sigma_{xx} = \frac{\sigma_0}{2\pi}\frac{1}{\eta^2}\Big\langle & H(t - v_1 R)\mathrm{Re}\left[M_{xx_1}(\xi_{1+})\frac{\partial \xi_{1+}}{\partial t} - M_{xx_1}(\xi_{1-})\frac{\partial \xi_{1-}}{\partial t}\right] \\ & + H(t - v_2 R)\mathrm{Re}\left[M_{xx2}(\xi_{2+})\frac{\partial \xi_{2+}}{\partial t} - M_{xx2}(\xi_{2-})\frac{\partial \xi_{2-}}{\partial t}\right] \\ & + H\left\{t - v_1 x - \frac{1}{b}\sqrt{v_2^2 - v_1^2}\ln(1 + by)\right\}H(v_2 R - t)\left[M_{xx\Delta\Gamma_2}(r_-)\frac{\partial r_-}{\partial t}\right]\Big\rangle\end{aligned} \tag{41}$$

$$u=-\frac{\sigma_0}{2\pi\mu_0}\frac{1}{\eta^2}\left\langle\int_{v_1R}^{t}\operatorname{Im}\left[u_P\left(\xi_{1+}\right)\frac{\partial\xi_{1+}}{\partial\tau}-u_P\left(\xi_{1-}\right)\frac{\partial\xi_{1-}}{\partial\tau}\right]d\tau\right.$$

$$+\int_{v_2R}^{t}\operatorname{Im}\left[u_S\left(\xi_{2+}\right)\frac{\partial\xi_{2+}}{\partial\tau}-u_S\left(\xi_{2-}\right)\frac{\partial\xi_{2-}}{\partial\tau}\right]d\tau$$

$$\left.+\int_{v_1x+\frac{1}{b}\sqrt{v_2^2-v_1^2}\ln(1+by)}^{v_2R}\left[u_{\Delta\Gamma_2}\left(r_-\right)\frac{\partial r_-}{\partial\tau}\right]d\tau\right\rangle \tag{42}$$

$$v=-\frac{\sigma_0}{2\pi\mu_0}\frac{1}{\eta}\left\langle\int_{v_1R}^{t}\operatorname{Re}\left[v_P\left(\xi_{1+}\right)\frac{\partial\xi_{1+}}{\partial\tau}-v_P\left(\xi_{1-}\right)\frac{\partial\xi_{1-}}{\partial\tau}\right]d\tau\right.$$

$$+\int_{v_2R}^{t}\operatorname{Re}\left[v_S\left(\xi_{2+}\right)\frac{\partial\xi_{2+}}{\partial\tau}-v_S\left(\xi_{2-}\right)\frac{\partial\xi_{2-}}{\partial\tau}\right]d\tau$$

$$\left.+\int_{v_1x+\frac{1}{b}\sqrt{v_2{}^2-v_1{}^2}\ln(1+by)}^{v_2R}\left[v_{\Delta\Gamma_2}\left(r_-\right)\frac{\partial r_-}{\partial\tau}\right]d\tau\right\rangle \tag{43}$$

where

$$R=\sqrt{x^2+\frac{1}{b^2}\ln^2(1+by)} \tag{44}$$

$$\xi_{1\pm}=\left\{\pm\frac{1}{b}\ln(1+by)\sqrt{t^2-v_1^2R^2}+ixt\right\}\Big/R^2 \tag{45}$$

$$\xi_{2\pm}=\left\{\pm\frac{1}{b}\ln(1+by)\sqrt{t^2-v_2^2R^2}+ixt\right\}\Big/R^2 \tag{46}$$

$$r_-=\left\{xt-\frac{1}{b}\ln(1+by)\sqrt{v_2^2R^2-t^2}\right\}\Big/R^2 \tag{47}$$

$$M_{yy1}\left(\xi\right)=\frac{\left(2\xi^2+v_2^2\right)^2}{\left(2\xi^2+v_2^2\right)^2-4\xi^2\sqrt{\xi^2+v_1^2}\sqrt{\xi^2+v_2^2}} \tag{48}$$

$$M_{yy2}\left(\xi\right)=\frac{-4\xi^2\sqrt{\xi^2+v_1^2}\sqrt{\xi^2+v_2^2}}{\left(2\xi^2+v_2^2\right)^2-4\xi^2\sqrt{\xi^2+v_1^2}\sqrt{\xi^2+v_2^2}} \tag{49}$$

$$M_{yy\Delta\Gamma_2}(r) = \frac{-8r^2\left(v_2^2 - 2r^2\right)^2 \sqrt{r^2 - v_1^2}\sqrt{v_2^2 - r^2}}{\left(2r^2 - v_2^2\right)^4 + 16r^4\left(r^2 - v_1^2\right)\left(v_2^2 - r^2\right)} \tag{50}$$

$$M_{yx\Delta\Gamma_2}(r) = \frac{-4r\left(2r^2 - v_2^2\right)^3 \sqrt{r^2 - v_1^2}}{\left(2r^2 - v_2^2\right)^4 + 16r^4\left(r^2 - v_1^2\right)\left(v_2^2 - r^2\right)} \tag{51}$$

$$M_{yx1}(\xi) = -M_{yx2}(\xi) = \frac{2\xi\left(2\xi^2 + v_2^2\right)^2 \sqrt{\xi^2 + v_1^2}}{\left(2\xi^2 + v_2^2\right)^2 - 4\xi^2\sqrt{\xi^2 + v_1^2\sqrt{\xi^2 + v_2^2}}} \tag{52}$$

$$M_{xx1}(\xi) = \frac{\left(2\xi^2 + v_2^2\right)\left(v_2^2 - 2v_1^2 - 2\xi^2\right)}{\left(2\xi^2 + v_2^2\right)^2 - 4\xi^2\sqrt{\xi^2 + v_1^2\sqrt{\xi^2 + v_2^2}}} \tag{53}$$

$$M_{xx2}(\xi) = \frac{4\xi^2\sqrt{\xi^2 + v_1^2}\sqrt{\xi^2 + v_2^2}}{\left(2\xi^2 + v_2^2\right)^2 - 4\xi^2\sqrt{\xi^2 + v_1^2}\sqrt{\xi^2 + v_2^2}} \tag{54}$$

$$M_{xx\Delta\Gamma_2}(r) = \frac{8r^2\left(v_2^2 - 2r^2\right)^2 \sqrt{r^2 - v_1^2}\sqrt{v_2^2 - r^2}}{\left(2r^2 - v_2^2\right)^4 + 16r^4\left(r^2 - v_1^2\right)\left(v_2^2 - r^2\right)} \tag{55}$$

$$u_P(\xi) = \frac{\xi(\tau)\left\{2\xi^2(\tau) + v_2^2\right\}}{\left\{2\xi^2(\tau) + v_2^2\right\}^2 - 4\xi^2(\tau)\sqrt{\xi^2(\tau) + v_1^2}\sqrt{\xi^2(\tau) + v_2^2}} \tag{56}$$

$$u_S(\xi) = \frac{-2\xi(\tau)\sqrt{\xi^2(\tau) + v_1^2}\sqrt{\xi^2(\tau) + v_2^2}}{\left\{2\xi^2(\tau) + v_2^2\right\}^2 - 4\xi^2(\tau)\sqrt{\xi^2(\tau) + v_1^2}\sqrt{\xi^2(\tau) + v_2^2}} \tag{57}$$

$$u_{\Delta\Gamma_2}(r) = \frac{-4r(\tau)\left\{v_2^2 - 2r^2(\tau)\right\}^2 \sqrt{r^2(\tau) - v_1^2}\sqrt{v_2^2 - r^2(\tau)}}{\left\{2r^2(\tau) - v_2^2\right\}^4 + 16r^4(\tau)\left\{r^2(\tau) - v_1^2\right\}\left\{v_2^2 - r^2(\tau)\right\}} \tag{58}$$

$$v_P(\xi) = \frac{\left\{2\xi^2(\tau) + v_2^2\right\}\sqrt{\xi^2(\tau) + v_1^2}}{\left\{2\xi^2(\tau) + v_2^2\right\}^2 - 4\xi^2(\tau)\sqrt{\xi^2(\tau) + v_1^2}\sqrt{\xi^2(\tau) + v_2^2}} \tag{59}$$

$$v_S(\xi) = \frac{-2\xi^2(\tau)\sqrt{\xi^2(\tau) + v_1^2}}{\left\{2\xi^2(\tau) + v_2^2\right\}^2 - 4\xi^2(\tau)\sqrt{\xi^2(\tau) + v_1^2}\sqrt{\xi^2(\tau) + v_2^2}} \tag{60}$$

$$v_{\Delta\Gamma_2}(r)=\frac{-4r^2(\tau)\{v_2^2-2r^2(\tau)\}^2\sqrt{r^2(\tau)-v_1^2}}{\{2r^2(\tau)-v_2^2\}^4+16r^4(\tau)\{r^2(\tau)-v_1^2\}\{v_2^2-r^2(\tau)\}} \tag{61}$$

The strain energy is

$$U=\int_0^{e_{ij}}\sigma_{ij}de_{ij} \tag{62}$$

Since the analytic solutions for the stress components are obtained, the expression for the strain energy U is

$$U=\frac{1}{E}\left[\frac{1-3\nu^2}{2(1-\nu^2)}\left(\sigma_{xx}^2+\sigma_{yy}^2\right)+\frac{2\nu^3}{1-\nu^2}\sigma_{xx}\sigma_{yy}+\frac{1+\nu}{2}\sigma_{yx}^2\right] \tag{63}$$

Since $b \ll 1$, we consider that the poisson ratio $\nu=\text{const.}$, the elastic moduli $E=\text{const.}$ and the mass density $\rho=\text{const.}$ through the material.

2.3 Transient analysis for homogeneous elastic thick plate

We consider the inplane transient analysis for the homogeneous elastic thick plate as shown in Figure 2. Since the coordinate q used for the inhomogeneous analysis represents the logarithm of the coordinate y, the obtained transient responses are not correct except for on x and y axes even if $b \ll 1$. Since the relationship q and y is nonlinear, the radius from the source to the calculation point is represented in the smaller radius in case of $b>0$, and in the larger radius in case of $b<0$. In the reflection analysis of the transient wave for the elastic plate, two responses of the incident wave and the reflected wave accurately do not agree at the reflection boundary. Then, we first try to calculate accurately the reflection response for the homogeneous elastic thick plate by using the method of images.

When the inplane transient wave impinges the free boundary of the elastic body, the incident wave is divided into P wave and S wave. The wave reflection by the Snell's law is shown in Figure 3, and the relationship between the angle θ_p of the reflected P wave and the angle θ_s of the reflected S wave is

$$\cos\left(\frac{\pi}{2}-\theta_s\right)=\frac{1}{\kappa}\cos\left(\frac{\pi}{2}-\theta_p\right) \tag{64}$$

where

$$\kappa=c_L/c_T \tag{65}$$

When y value is known, the arrival time t_{sp} and the arrival coordinate position x_s of the reflected wave ray is easily obtained

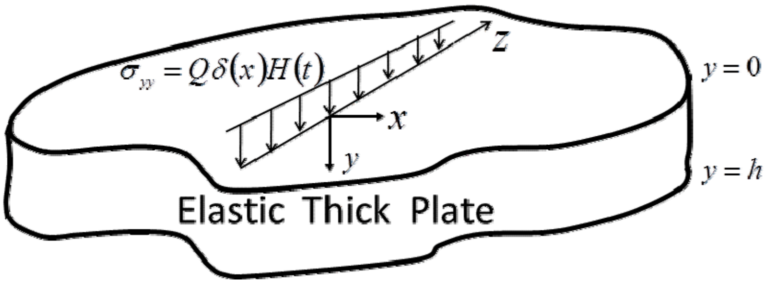


Figure 2 : Elastic thick plate subjected to inplane impact line load

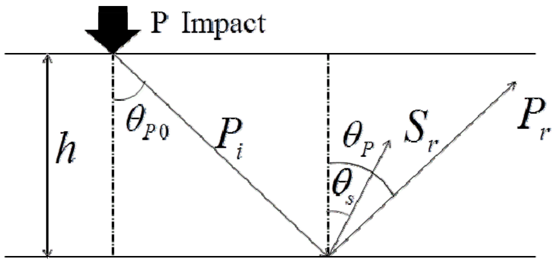


Figure 3 : Reflection of inplane transient wave

For the reflected S wave by the incident P wave

$$t_{sp} = h\operatorname{cosec}\left(\frac{\pi}{2} - \theta_p\right) + \kappa(h-y)\operatorname{cosec}\left(\frac{\pi}{2} - \theta_s\right) \tag{66}$$

$$x_s = h\cot\left(\frac{\pi}{2} - \theta_p\right) + (h-y)\cot\left(\frac{\pi}{2} - \theta_s\right) \tag{67}$$

For the reflected P wave by the incident S wave

$$t_{ps} = (h-y)\operatorname{cosec}\left(\frac{\pi}{2} - \theta_p\right) + \kappa(h-y)\operatorname{cosec}\left(\frac{\pi}{2} - \theta_s\right) \tag{68}$$

$$x_p = (h-y)\cot\left(\frac{\pi}{2} - \theta_p\right) + h\cot\left(\frac{\pi}{2} - \theta_s\right) \tag{69}$$

From these reflection property, when the wave front comes back the free surface y=0 after the first reflection at the boundary, we have

$$t_{sp} = h\operatorname{cosec}\left(\frac{\pi}{2} - \theta_p\right) + \kappa h\operatorname{cosec}\left(\frac{\pi}{2} - \theta_s\right) = t_{ps} \tag{70}$$

$$x_s = h\cot\left(\frac{\pi}{2}-\theta_p\right)+h\cot\left(\frac{\pi}{2}-\theta_s\right)=x_p \tag{71}$$

After n times reflection, the arrival times which the same type waves reach the y coordinate position are

$$t_{pn}=\sqrt{x^2+(nh-y_n)^2}\quad,\quad t_{sn}=\kappa\sqrt{x^2+(nh-y_n)^2} \qquad (72),(73)$$

After the reflection of the i times as the P wave and the j times as the S wave, finally the arrival time and the arrival x coordinate of P wave and S wave are t_{pij}, x_{pij} and t_{sij}, x_{sij}, respectively

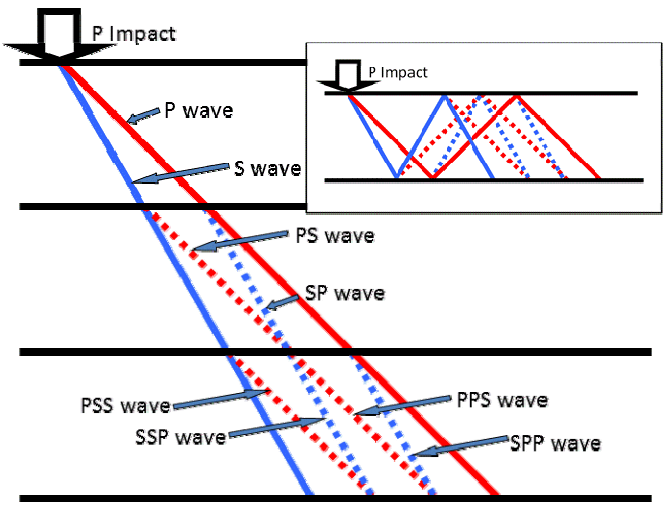


Figure 4 : Reflection of transient wave in elastic thick plate

$$t_{pij}=(ih+y_n)t_p+jht_s\quad,\quad x_{pij}=(ih+y_n)x_p+jhx_s \qquad (74),(75)$$

$$t_{sij}=iht_p+(jh+y_n)t_s\ ,\ x_{sij}=ihx_p+(jh+y_n)x_s \qquad (76),(77)$$

where

$$t_p=\mathrm{cosec}\left(\frac{\pi}{2}-\theta_p\right)\quad,\quad t_s=\kappa\,\mathrm{cosec}\left(\frac{\pi}{2}-\theta_s\right) \qquad (78),(79)$$

$$x_p=\cot\left(\frac{\pi}{2}-\theta_p\right)\quad,\quad x_s=\cot\left(\frac{\pi}{2}-\theta_s\right) \qquad (80),(81)$$

$$n=i+j\quad,\ y_n=\frac{1}{2}h+(-1)^n\left(y-\frac{1}{2}h\right) \qquad (82),(83)$$

We can show the aspect of these reflections as shown in Figure 4, and the wave ray increases two at one reflection. For the first layer from the top the wave rays are the direct waves from the source, for the second layer the wave rays are the 1 time reflection waves and for the third layer the wave rays are the 2 times reflection waves. Folding on each horizontal line, the transient response for the elastic thick plate is represented as shown in the upper figure on the right. On the actual impulsive reflection response, the relationship between the incident wave and the reflected wave is based on the Snell's law, the correspondence of the reflected wave to the incident wave is established at the calculation point.

Next, we examine how the energy of the incident wave will be distributed the reflected P and S waves. The incident angles for P wave and S wave are θ_{P0}, θ_{S0}, respectively. P and S waves reflected from either incident wave are formulated by the Snell's law, we can obtain as follows

For the incident P wave

$$\frac{c_L}{\sin\theta_{P0}} = \frac{c_L}{\sin\theta_P} = \frac{c_T}{\sin\theta_S} \tag{84}$$

For the incident S wave

$$\frac{c_T}{\sin\theta_{S0}} = \frac{c_T}{\sin\theta_S} = \frac{c_L}{\sin\theta_P} \tag{85}$$

The critical angle θ_c which the incident S wave does not produce the reflected P wave is given

$$\theta_c = \sin^{-1}\left(1/\kappa\right) \tag{86}$$

Next, we consider the amplitude ratio for the incident P and S wave

In case of an incident P wave

For the reflected P wave

$$\frac{A_P}{A_{P0}} = \left(\sin2\theta_{P0}\sin2\theta_S - \kappa^2\cos^2 2\theta_S\right)/\Delta_P \tag{87}$$

For the reflected S wave

$$\frac{A_S}{A_{P0}} = 2\kappa\sin2\theta_{P0}\cos2\theta_S/\Delta_P \tag{88}$$

where

$$\Delta_P = \sin2\theta_{P0}\sin2\theta_S + \kappa^2\cos^2 2\theta_S \tag{89}$$

A_{P0}, A_P, A_S are the incident P wave amplitude, the reflected P wave amplitude and the reflected S wave amplitude, respectively.

In case of an incident S wave

For the reflected P wave

$$\frac{A_P}{A_{S0}} = -\kappa\sin4\theta_{S0}/\Delta_S \tag{90}$$

For the reflected S wave

$$\frac{A_S}{A_{S0}} = \left(\sin2\theta_{S0}\sin2\theta_{P1} - \kappa^2\cos^2 2\theta_{S0}\right)/\Delta_S \tag{91}$$

where

$$\Delta_S = \sin2\theta_{S0}\sin2\theta_P + \kappa^2\cos^2 2\theta_{S0} \tag{92}$$

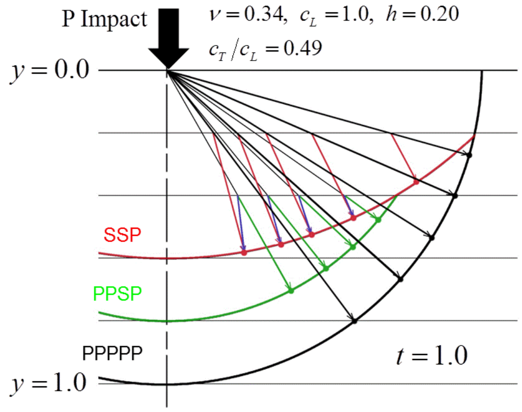


Figure 5 : Aspect of the reflected wave fronts at the plate boundaries

A_{S0} is the incident S wave amplitude.
From the energy balance on the free surface of the half space
In case of an incidence P wave

$$\left(\frac{A_P}{A_{P0}}\right)^2 + \left(\frac{A_S}{A_{P0}}\right)^2 \frac{\sin 2\theta_S}{\sin 2\theta_{P0}} = 1 \tag{93}$$

In case of the incidence S wave

$$\left(\frac{A_P}{A_{S0}}\right)^2 \frac{\sin 2\theta_P}{\sin 2\theta_{S0}} + \left(\frac{A_S}{A_{S0}}\right)^2 = 1 \tag{94}$$

Reflecting one incident wave ray at the boundary, it's divided into P and S wave, and the total transient response can be expressed by the superposition of these waves. The propagation aspect for several wave fronts is shown in Figure 5.

2.4 Reflection analysis for inhomogeneous elastic thick plate

As shown in the previous section, since the wave front of the transient response for the inhomogeneous elastic half space is not accurately depicted except for on the x axis or the axis. Therefore, the discontinuity occurs at the boundary and it becomes difficult to express the transient response for the inhomogeneous elastic plate with the reflection boundary. If we consider the inhomogeneity which is expressed as the thin homogeneous layered laminate model, this discontinuity of the response in the numerical calculation does not occur, but it becomes difficult to satisfy the Snell's law at the boundary. But, we can obtain the response for the inhomogeneous plate by the numerical processing scaled up or down the y coordinate of the response for the homogeneous plate. The shape of the wave front is determined by the b value in the linear inhomogeneous plate, here, we define the inhomogeneous coefficient ς_i as the more direct representation as follows

$$\zeta_i = 1 + (-1)^i \left(1 - {y_{ar}}/{x_{ar}}\right),\ (i = 1.2) \tag{95}$$

where, y_{ar} , x_{ar} are the arrival positions on y axis and x axis, respectively. ς_i represents the shape of the wave front which is transformed from the semicircle in the homogeneous case to the ellipse in the inhomogeneous case. $i=1$ is the incident, reflected wave from the wave source side of the plate, $i=2$ is the reflected wave from the lower boundary of the plate. In case of $\zeta_1=1.2$, $y_{ar}/x_{ar}=1.2$, in case of $\zeta_2=0.8$, $y_{ar}/x_{ar}=0.8$. In case of $\zeta_1=1.2$, the transient wave ray returns in $\zeta_2=0.8$ from the lower boundary.
Therefore

$$\varsigma_1+\varsigma_2=2.0 \tag{96}$$

The propagation time of the transient waves which go and return through the plate is constant regardless of the inhomogeneity. The accurate configuration of the wave front for the inhomogeneous half space can be calculated by the homogeneous layered model as shown in Figure 6.
Figure 7 is comparing the analytic model with the 100 layered laminate model. In case of $\zeta_1=1.05$, the response of the analytic model almost correct, but in case of

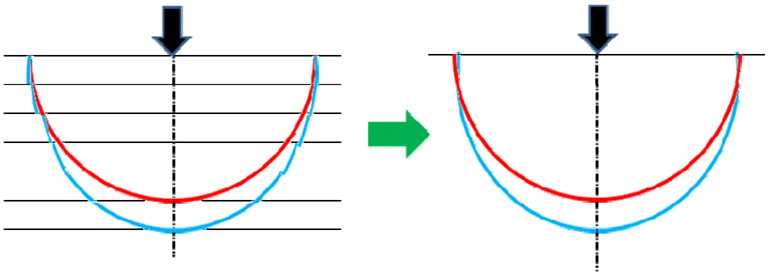


Figure 6 : Wave front of the inhomogeneous half space by the layered model (blue line)

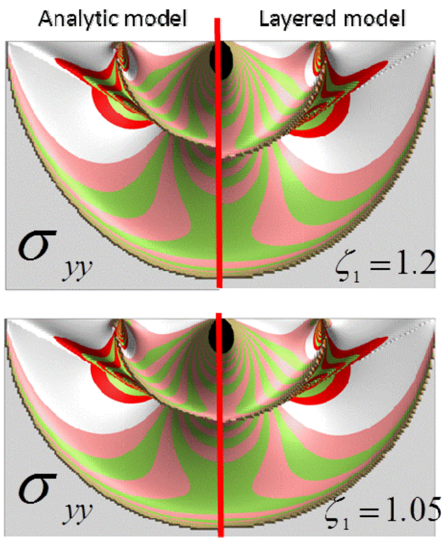


Figure 7 : Different response for analytic model from an accurate one for layered model

$\zeta_1 = 1.2$, the shape of the S wave expressed by the analytic model is smaller than the accurate one for the layered model. Therefore, the numerical calculation for the inhomogeneous elastic plate is carried out by the 100 homogeneous layered approximation.

3 Numerical calculation

In the analysis of the linear inhomogeneous half space, the material composition of the free surface is the fundamental values, we use Al 5052 〔17〕 and we consider that the P wave phase velocity is $c_L = 6{,}500\,{}^{\mathrm{m}}\!/\!_{\mathrm{sec}}$ and Poisson ratio is $\nu = 0.34$. Here, we assume that the phase velocity ratio of the P and S wave is constant and does not change for any inhomogeneity, and the wave front reaches x coordinate position $x = 1.0$ on the free surface of the half space at time $t = 1.0$. By this assumption, the generality is not lost.

Figure 8 is the decibel representation for the response of the strain energy U at $\zeta_1 = 1.0$, $t = 1.0$, 4 waves of P wave, S wave, Head wave and Rayleigh wave are very clearly expressed.

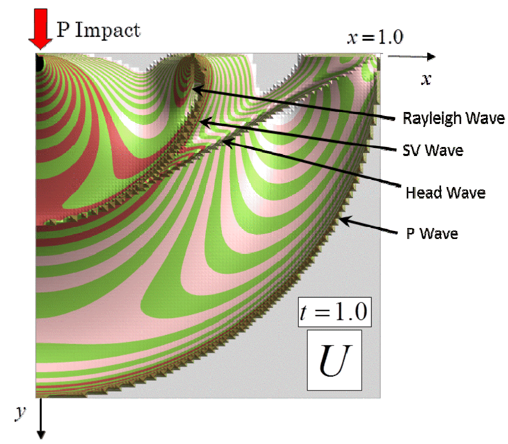
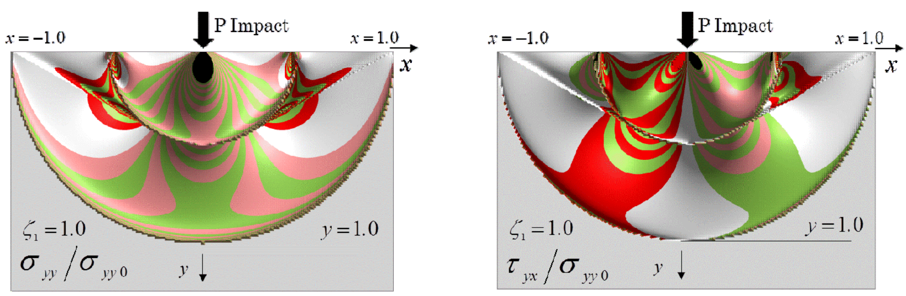
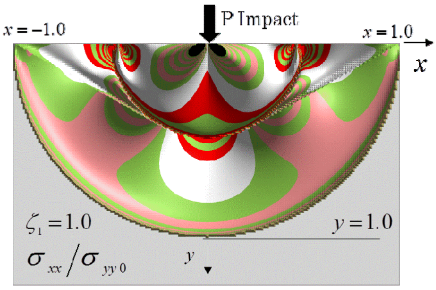


Figure 8 : 4 waves produced by the inplane impact loading

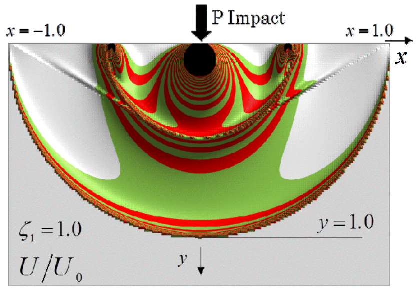


(a) Stress σ_{yy} (b) Stress τ_{yx}

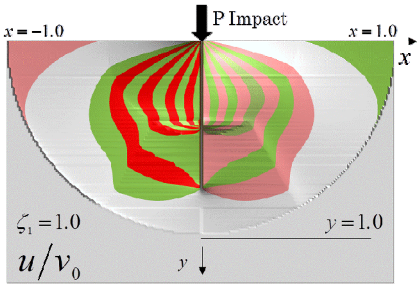
Figure 9 : Contour mapping representation of transient responses for homogeneous half space at t=1.0



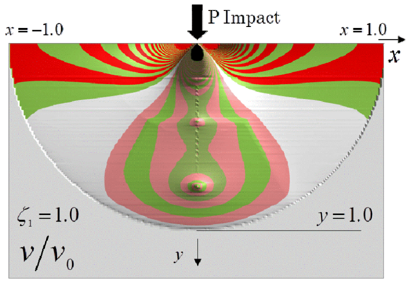
(c) Stress σ_{xx}



(d) Strain energy U

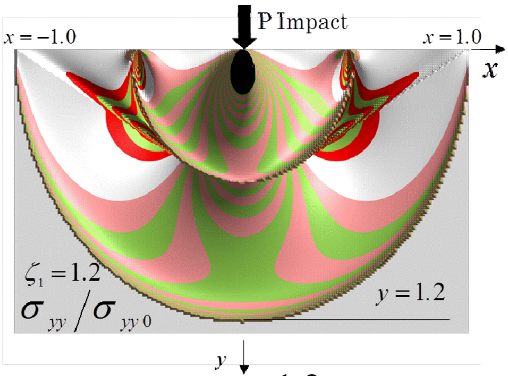


(e) Displacement u

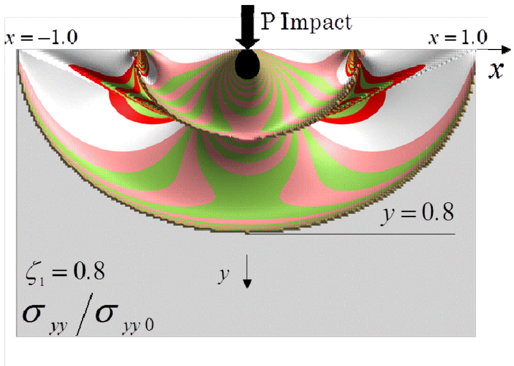


(f) Displacement v

Figure 9 (continued): Contour mapping representation of transient responses for homogeneous half space at t=1.0

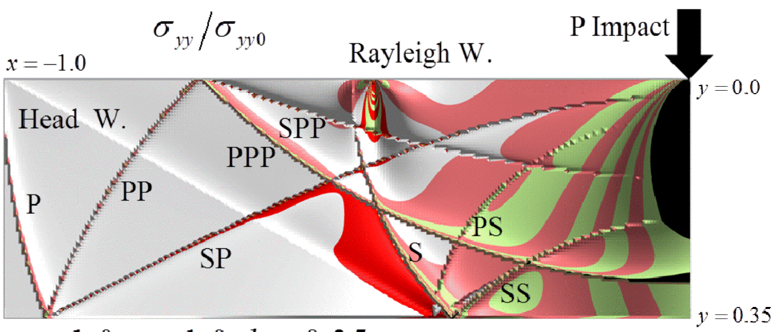


(a) $\varsigma_1=1.2$



(b) $\varsigma_1=0.8$

Figure 10 : Transient response of stress σ_{yy} for inhomogeneous half space at t=1.0



$\varsigma_1=1.0,\ t=1.0,\ h=0.35$

Figure 11 : Transient response of stress σ_{yy} for homogeneous thick plate of h=0.35

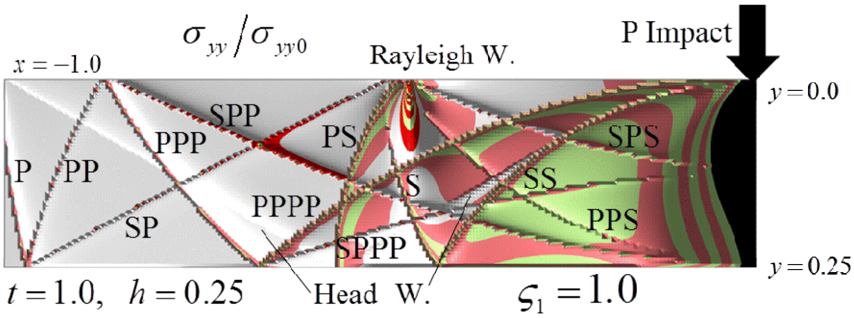


Figure 12 : Transient response of stress σ_{yy} for homogeneous thick plate of h=0.25

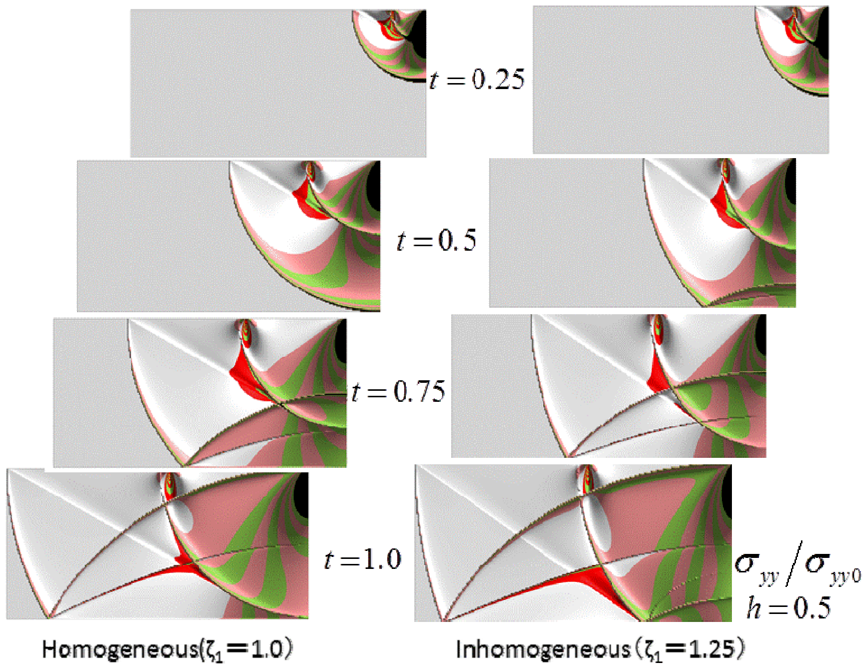


Figure 13 : Transient response of stress σ_{yy} for inhomogeneous thick plate

Figure 9 is the transient responses of the homogeneous half space, and this is in case of the inhomogeneous parameter $b=0.000001$, that is, the inhomogeneous coefficient $\zeta_1 \approx 1.0$. Each figure of Figure 9(a)$\frown$9(f) is the transient response of stress components σ_{yy}/σ_{yy0}, τ_{yx}/σ_{yy0}, σ_{xx}/σ_{yy0}, the strain energy U/U_0, the displacement components u/v_0, v/v_0, respectively, and each response has been expressed by the contour mapping representation. 100 point Gauss-Legendre numerical integration was used for the calculation of the displacement components. In case of σ_{yy}/σ_{yy0}, the colors were alternately changed in every 0.01 for $-0.2 \le \sigma_{yy}/\sigma_{yy0} \le 0.2$. σ_{yy0}, U_0, v_0 are the response values of σ_{yy}, U, v at the coordinate $(0,0.01)$. For y axis, σ_{yy}, σ_{xx}, U, v become the symmetric responses and

τ_{yx}, u become the anti-symmetric responses. Figure 10 is the response for the inhomogeneous half space of the stress σ_{yy} expressed in 100 layered laminate model. Figure 10(a) is in case of $\varsigma_1 = 1.2$ and Figure 10(b) is in case of $\varsigma_1 = 0.8$. Figures 11 represents the transient response of stress σ_{yy} of the homogeneous elastic thick plate. Plate thickness is $h = 0.35$. The aspect of the coupled reflection of P and S waves and the contribution to the whole wave field of Rayleigh wave and Head wave are clearly expressed. In this figure, the response is magnified 100 times in order to make the visible wave fronts. Figures 12 is in case of $h = 0.25$, the reflected wave lays drastically increases in comparison with the previous figure. The boundary condition is satisfied except for the vicinity of the impact point. Figures 13 is the transient response of the inhomogeneous elastic thick plate for the stress σ_{yy} in case of $\varsigma_1 = 1.25, h = 0.5$. Regardless of the inhomogeneity, the propagation time of the transient waves which go and return through the plate is constant.

4 Conclusion

The Cagniard method established as a technique of the exact analysis of the inplane transient problem for the homogeneous elastic half space, was applied to the problem for the inhomogeneous elastic half space with the weak inhomogeneity. Deciding the potentials which expressed the displacement for the linear inhomogeneous elastic material, we obtained the exact solutions for the transient responses. The numerical result of the stress components σ_{yy}, τ_{yx}, σ_{xx}, the strain energy U and the displacement components u, v were obtained from these deduced analytic solutions. In the case of the weak inhomogeneity, the calculated response for the analytic solutions are almost correct, even if in the case of the strong inhomogeneity, we could accurately calculate the transient response by using the 100 homogeneous layered laminate model. Furthermore, we incorporated the method of images, using Snell's law which clarified the coupled reflection of P and S waves at the boundary of the plate, into the reflection response calculation algorithm and discussed the wave propagation including Head wave and Rayleigh Wave. Finally as a result of these wave propagation researches for the inhomogeneity, we could show that the transient response of the inhomogeneous thick plate was obtained from the transient response of the homogeneous thick plate.

References

[1] H. Lamb, "On the propagation of tremors over the surface if an elastic solid", Philosophical Trans. Royal Society (London) , A203, 1-42, 1904.

[2] L. Cagniard, "Reflexion et Refraction des Ondes Seismiques Progressives", Cauthiers-Villars, Paris, 1939.

[3] A.T. de Hoop, "A modification of Cagniard's method for solving seismic pulse problems", Appl. Scientific Res. 8, 349-356, 1960.

[4] W.M. Ewing, W. Jardetzky and F. Press, "Elastic Wave in Layered Media", McGraw-Hill, New York, 1957.

[5] L.M. Brekhovskikh, "Wave in Layered Media", Academic Press, 1960.

[6] Y.H. Pao and R.R. Gajewski, "The generalized ray theory and transient responses of layered elastic solids", Physical Acoustic 13,183-265, Academic Press, New York, 1977.

[7] J. Miklowitz, "The theory of elastic waves and waveguides", North-Holland, New York, 1978.

[8] J. Virieux, "P-SV wave propagation in heterogeneous media : Velocity stress finite-difference method", Geophysics, Vol.51, No.4,889-901, 1986.

[9] G.R. Liu, J. Tani. T. Ohyoshi and K Watanabe, "Transient waves in anisotropic laminated plates, Part 1 : Theory", Transaction of the ASME, Journal of Applied Mechanics, Vol.113, 230-234, 1991.

[10] G.R. Liu, J. Tani. T. Ohyoshi and K Watanabe, "Transient waves in anisotropic laminated plates, Part 2 : Application", Transaction of the ASME, Journal of Applied Mechanics, Vol.113, 235-239, 1991.

[11] A. Chkraborty and S. Gopalakrishnan, "Wave propagation in inhomogeneous layered media : solution of forward and inverse problems", Acta Mechanica, 169, 153-185, 2004.

[12] K. Miura, "Simple calculation for SH transient wave front propagation through a linear gradient inhomogeneous half-space", Transaction of the Japan Society of Mechanical Engineers, Series A, Vol.67, No.663, 1742-1746, 2001.

[13] K. Miura, "SH transient response for the graded inhomogeneous half-space with elastic principal axis with tilted for free surface", Transaction of the Japan Society of Mechanical Engineers, Series A, Vol.69, No.678, 480-486, 2003.

[14] K.F. Graff, "Wave Motion in Elastic Solids", Clarendon Press, 1975.

[15] J.D. Achenbach, "Wave propagation inelastic Solids", North-Holland, 1973.

[16] K. Miura, "Analytical study relative to elastic waves and response in homogeneous and gradient inhomogeneous media", Doctoral thesis of Akita University, 1997.

[17] The Japan Soc. Of Mechanical Engineers edition, "Engineering data. The elastic modulus of metallic material", 159-165, 1980.